Actuarial Science Hilary Term Lecture 5

"No arbitrage" assumption

Arbitrage in financial mathematics is generally defined as a risk-free trading profit. This can arise in 2 ways:

> An investor arranges a deal which yields an immediate profit with no risk of any future loss.
>
> An investor arranges a deal with no immediate cost, no risk of any future loss AND a positive probability of future profit.

2 examples of arbitrage opportunities:

2 securities A and B each with term 1 year. Prices at time 0 are $P_0(A)$ and $P_0(B)$. At time 1, the payout from each security depends on whether the stock market has risen or fallen. If it goes up, the payouts are $P_1(A,u)$ and $P_1(B,u)$; if it goes down, they are $P_1(A,d)$ and $P_1(B,d)$.

Investors can either buy or sell each security. If they buy a security, they pay the time 0 price and receive the time 1 payout; if they sell, they receive the time 0 price and pay the time 1 payout.

Consider the following price/payout schedule:

Security S	Time 0 price P_0	Market rises $P_1(S,u)$	Market falls $P_1(S,d)$
A	6	7	5
B	11	14	10

This produces an "immediate profit" arbitrage opportunity:

> Buy 1 unit of security B
> Sell 2 units of security A

This delivers the following income and outgo:

	Time 0	Time 1 Market rises	Time 1 Market falls
Income	12	14	10
Outgo	11	14	10
Total	1	0	0

A profit of 1 is made at time 0 with no prospect of loss (or profit) at time 1 – whatever happens to the stock market. Clearly security A is much less attractive than security B. Market pressures will act to ensure that equilibrium exists when:

$$P_0(A) = P_0(B)/2$$

Or take a different example:

Security S	Time 0 price P_0	Market rises $P_1(S,u)$	Market falls $P_1(S,d)$
A	6	7	5
B	6	7	4

which produces a "no loss" arbitrage possibility:

Buy 1 unit of security A
Sell 1 unit of security B

	Time 0	Time 1 Market rises	Time 1 Market falls
Income	6	7	5
Outgo	6	7	4
Total	0	0	1

Clearly investors will have a strong preference for security A and market pressures will work to increase the price of A. The arbitrage opportunity disappears as soon as:

$$P_0(A) > P_0(B)$$

Modern financial mathematics is based on the "no arbitrage assumption". In other words, in a developed financial market place, market pressures will ensure that arbitrage opportunities do not exist. A consequence of the "no arbitrage assumption" is the "Law of One Price" which states that any two securities or combinations of securities that have an identical payment schedule must have the same price.

The "no arbitrage assumption" as developed into the "Law of One Price" enables us to calculate the price of complex financial instruments by "replicating" the payment schedules. In other words, if we can discover a portfolio of "simpler" assets that have exactly the same payment schedules as the complex instrument that we wish to value, then the value (price) of the instrument must equal the value (price) of the sum of the "simpler" assets. The aggregation of the simpler assets is called the "replicating portfolio".

Forward Contracts

A forward contract is an agreement at time 0 between 2 parties whereby one agrees to buy from the other a specified amount of an asset at a specified price at a specified FUTURE date. The buyer is said to hold a "long forward position" and the seller a "short forward position". In general, the future price of the asset is not known. For example, you might enter into a forward contract to buy a particular number of ordinary shares in BP in six months' time. The BP share price is known today but not in 6 months. Indeed the price will vary continuously for the next 6 months.

Let: S_r = the price of the asset (e.g. the BP share) at time r
K = the price agreed at time 0 to be paid at time T
δ = the force of interest available on a risk free investment from time 0 to T.

K is called the "forward price"; T is the maturity date of the forward contract and δ is known as the risk free force of interest.

At time 0 (when the contract is agreed), no money changes hands. K is so chosen that the present value of the forward contract equals zero. In general there will be a profit/loss at the maturity of the contract since it is extremely unlikely that K will equal S_T. The buyer of the contract will pay K and the seller will deliver S_T.

Replicating Portfolios

We will apply the "Law of One Price" to the valuation of forward contracts by finding "replicating portfolios".

1. Consider a simple forward contract for an asset where there are no intervening income payments (i.e. no dividends or coupons are payable before the maturity of the contract).

Look at 2 portfolios:

Portfolio A
Enter a forward contract to buy 1 unit of asset S at forward price K at time T
Invest $Ke^{-\delta T}$ in the risk free investment

Portfolio B
Buy 1 unit of asset S at the current price S_0

At time 0 the value of Portfolio A is $Ke^{-\delta T}$ (bear in mind that the price of a forward contract at inception is zero).

The value of Portfolio B is S_0

At time T the cash flow of Portfolio A is three fold:

Risk free investment yields $Ke^{-\delta T}.e^{\delta T} = K$
Payment for forward contract K
Receipt from forward contract S_T

Thus total amount is S_T (Risk free investment accumulates to contract payment)

At time T the cash flow of Portfolio B is a payout of S_T

In other words at time T the payout from both portfolios is identical. Applying the "Law of One Price" this means that the value (or price) at time 0 must be the same.

Therefore: $Ke^{-\delta T} = S_0 \Rightarrow K = S_0 .e^{\delta T}$

So the "no arbitrage assumption" means that we have derived a price for the forward contract without any model of how the asset price S will move during the period $0<t<T$.

2. Now assume that there is a fixed payment c due on the asset S at time t between 0 and T (for instance a coupon if the asset is Government stock). Again consider 2 portfolios:

Portfolio A
Enter a forward contract to buy 1 unit of asset S at forward price K at time T
Invest $Ke^{-\delta T} + ce^{-\delta t}$ in the risk free investment

Portfolio B
Buy 1 unit of asset S at the current price S_0
At time t invest the income of c in the risk free investment

At time T the cash flow of Portfolio A is three fold:

Risk free investment yields $K + ce^{\delta(T-t)}$
Payment for forward contract K
Receipt from forward contract S_T

Thus total amount is $S_T + c\, e^{\delta(T-t)}$

At time T the cash flow of Portfolio B is a payout of $S_T + ce^{\delta(T-t)}$

As before, using the "Law of One Price" :

$$Ke^{-\delta T} + ce^{-\delta t} = S_0 \Rightarrow K = S_0 e^{\delta T} - ce^{\delta(T-t)}$$

This can be extended to a series of coupon payments. Let the present value (at time 0) of the payments be I. Then $K = (S_0 - I)\, e^{\delta T}$

3. Finally assume there is a known dividend yield (D per annum) which is received continuously – and is immediately reinvested in the underlying security S. Starting at time 0 a unit investment accumulates to e^{DT} at time T.

Again consider 2 portfolios:

Portfolio A
Enter a forward contract to buy 1 unit of asset S at forward price K at time T
Invest $Ke^{-\delta T}$ in the risk free investment

Portfolio B
Buy e^{-DT} units of asset S at price S_0
Reinvest dividend income in asset S on receipt

At time T the cash flow of Portfolio A is three fold:

Risk free investment yields $Ke^{-\delta T}.e^{\delta T} = K$
Payment for forward contract K
Receipt from forward contract S_T

The payout from Portfolio B is $e^{-DT} e^{DT} S_T = S_T$ (equal to Portfolio A)

As before, using the "Law of One Price":

$Ke^{-\delta T} = S_0 e^{-DT} \Rightarrow K = S_0 e^{(\delta - D)T}$

The distinction between the last 2 examples is the basis on which the income is calculated. If the income is a fixed amount (e.g. a guaranteed coupon) regardless of the price of the underlying asset, then it is important to assume it is invested in a risk free asset. If it is expressed as a proportion of the price of the underlying asset, then it should be assumed to be reinvested in the underlying asset. In this way one can compute the accumulated amount at time T without knowing the intermediate performance of the price of the underlying asset.

Valuation of a Forward Contract

Consider a forward contract entered into at time 0 for 1 unit of security S with maturity at time T. Let the forward price be K_0.

What is the value of the long forward contract at time r where $0<r<T$?

Again consider 2 portfolios – both purchased at time r.

Portfolio A
Buy the existing forward contract at price V_l
Invest $K_0e^{-\delta(T-r)}$ in the risk free asset

Portfolio
Buy a new long forward contract with maturity at T, forward price $K_r=S_r.e^{\delta(T-r)}$
Invest $K_re^{-\delta(T-r)}$ in the risk free asset

Price of portfolios at time r is:

A: $V_l + K_0e^{-\delta(T-r)}$

B: $K_re^{-\delta(T-r)}$

Payout at time T is:

A: S_T ; $+K_0$; $-K_0$ Total S_T

B: S_T ; $+K_r$; $-K_r$ Total S_T

Again using the "Law of One Price" we have:

$V_l + K_0 e^{-\delta(T-r)} = K_r e^{-\delta(T-r)} \Rightarrow V_l = (K_r - K_0)\, e^{-\delta(T-r)}$

Substituting for K gives $V_l = S_r - S_0\, e^{\delta r}$

By general reasoning the value of the short forward contract V_s is $- V_l$.

Hedging

This is the general term which describes the use of financial instruments (from straightforward bonds to highly sophisticated derivative products) which reduce or eliminate the future risk of loss.

For instance an investor may enter into a forward contract to sell an asset (S) at a future date (T) for price K. He need not be holding the asset at the start of the contract but he must have it at the maturity date. To hedge the risk he could borrow an amount $Ke^{-\delta T}$ at the risk free rate and buy asset S at price S_0. At time T the asset is available to sell to the counterparty. The payment of K (the forward price) by the counterparty in settlement exactly matches the amount required to repay the loan.

In this way the investor has avoided any possibility of loss on the contract (but also any possibility of profit!).

This is a "static hedge" because, once put in place, it remains untouched until the maturity of the contract. The more complicated the financial instruments, the more necessity there is for continual re-balancing – a "dynamic hedge".