There exist the same linkages as between δ and i.

So after t years at the discrete spot rate, 1 accumulates to $(1+y_t)^t$. At the spot force of interest (or continuous time spot rate) it accumulates to $e^{Y_t t}$

Thus $y_t = e^{Y_t} - 1$

Continuous time forward rates

Similarly the continuous time forward rate $F_{t,r}$ is the force of interest equivalent to the forward spot rate $f_{t,r}$

At time 0 there is an agreement to invest 1 at time t for r years. This will accumulate at the end of the term to

$$e^{F_{t,r}\, r}$$

As before: $\quad f_{t,r} = e^{F_{t,r}} - 1$

As we did with the discrete rates we can consider the relationship between the spot and forward rates by considering the accumulation of 1 for t years at the spot rate followed by r years at the forward rate. Then we equate that to the accumulation of 1 for t+r years at the relevant spot rate:

$$e^{tY_t} \cdot e^{rF_{t,r}} = e^{(t+r)Y_{t+r}}$$

$$t\,Y_t + r\,F_{t,r} = (t+r)\,Y_{t+r}$$

$$F_{t,r} = \frac{(t+r)\,Y_{t+r} - t\,Y_t}{r}$$

Considering the zero coupon bond price P_n we can see that $Y_n = \frac{-1}{n} \log P_n$

Therefore $F_{t,r} = \frac{1}{r} \log \left\{ \frac{P_t}{P_{t+r}} \right\}$

Instantaneous forward rates

F_t is defined as $\lim\limits_{r \to 0} F_{t,r}$

$$\text{Thus } F_t = \lim_{r \to 0} \frac{1}{r} \log \left\{ \frac{P_t}{P_{t+r}} \right\}$$

$$= \lim_{r \to 0} \frac{-(\log P_{t+r} - \log P_t)}{r}$$

$$= -\frac{d}{dt} \log P_t$$