Comparing with the effective duration we see that

$$\tau = (1+i)\nu$$

Remembering that $\quad v^t = e^{-\delta t} \longrightarrow \dfrac{dA}{d\delta} = -\sum_1^n t_k C_{t_k} v^{t_k}$

$$\therefore \quad \tau = -\frac{1}{A}\frac{dA}{d\delta} = \frac{di}{d\delta}\nu$$

$$\text{But } i = e^{\delta} - 1 \qquad \therefore \frac{di}{d\delta} = e^{\delta} \qquad \therefore \tau = e^{\delta}\nu = (1+i)\nu$$

Example

Macauley duration for an n year bond with redemption price R and coupon D is

$$\tau = \frac{D \times (\bar{I}a)_{\overline{n}|} + Rnv^n}{D \times a_{\overline{n}|} + Rv^n}$$

Macauley duration for a zero coupon bond of term n is n.

Convexity

Going for the second differential the convexity of the cash flow C is defined as:

$$c = \frac{1}{A}\frac{d^2A}{di^2} = \frac{A''}{A} = \frac{1}{A}\left\{\sum_1^n C_{t_k} t_k(t_k+1) v^{t_k+2}\right\}$$

For small changes in interest rates we have:

$$\frac{A(i+\varepsilon) - A(i)}{A} = \frac{1}{A}\frac{\partial A}{\partial i}\varepsilon + \frac{1}{2A}\frac{\partial^2 A}{\partial i^2}\varepsilon^2 + \cdots$$

$$\approx -\varepsilon\nu + \varepsilon^2\frac{c}{2}$$

Positive convexity means that if interest rates fall by a small amount, liabilities rise by a greater amount than they fall for an equivalent rise in interest rate.

Frank Redington

Frank Redington was the outstanding British actuary of his generation. He became Chief Actuary of the Prudential at the age of 46, President of the Institute of Actuaries and received the Institute's Gold Medal. In 1952 he presented a paper to the Institute entitled "A ramble through the actuarial countryside". In it he introduced the subject of immunisation.