Problem Sheet 7 - Part C Probabilistic Combinatorics - Oxford HT 2008

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1. Let g be a random function from $\{1, \ldots, n\}$ to itself, with all n^n possibilities equally likely. Let X be the number of values not in the image of g, i.e. the number of $y \in \{0, 1, \ldots, n\}$ such that $g(x) = y$ has no solution.

   (i) Show that $\mathbb{E}(X) \sim n/e$ as $n \to \infty$.

   (ii) Use the result on concentration of Lipschitz functions to derive a concentration inequality for the deviation of X from its mean.

2. Let B be any normed vector space, and let $v_1, \ldots, v_n \in B$ with $|v_i| \leq 1$ for all i. Let $\epsilon_1, \ldots, \epsilon_n$ be independent, with $\epsilon_i = \pm 1$ with probability $1/2$ each for each i. Let $X = |\epsilon_1 v_1 + \cdots + \epsilon_n v_n|$. Show that for some $c > 0$ (not depending on n or the choice of v_i),
$$\mathbb{P}\left(|X - \mathbb{E}\, X| > \lambda\sqrt{n}\right) \leq 2e^{-c\lambda^2} \text{ for all } \lambda > 0.$$

3. A more general version of Azuma's inequality.

   (i) Show that if Y is a random variable with $|Y| \leq c$ with probability 1 and $\mathbb{E}\, Y = 0$, and $\alpha > 0$, then $\mathbb{E}\left(e^{\alpha Y}\right) \leq e^{\alpha^2 c^2/2}$.

   (ii) Prove that if $X_0, X_1, \ldots, X_m$ is a martingale with the property that $|X_i - X_{i-1}| \leq c_i$ for $i = 1, 2, \ldots, m$, where $c_1, \ldots, c_m$ are constants, then
$$\mathbb{P}[|X_0 - X_m| > t] \leq 2 \exp\left(-\frac{t^2}{2\sum_{i=1}^m c_i^2}\right).$$

4. Which of the following graph theoretic functions are edge-Lipschitz and which are vertex-Lipschitz: (a) the number of components (b) the size of the largest component (c) the size of the largest independent set (d) the number of isolated vertices?

5. Let $G = (V, E)$ be a graph with chromatic number $\chi(G) = 1000$. Let $U \subset V$ be a random subset of V, with all $2^{|V|}$ possibilities equally likely. Let H be the induced subgraph of G on U.

   (i) Show that the expectation of $\chi(H)$ is at least 500.

   (ii) Show that $\mathbb{P}\big(\chi(H) \leq 400\big) \leq e^{-5}$. [*Hint: try to write $\chi(H)$ as a Lipschitz function of a suitable sequence of independent random quantities.*]

6. Consider $A = \{0, 1, \ldots, n\}^2$ as a subset of the square lattice $\mathbb{Z}^2$. With each point $\mathbf{z} \in A$, we associate a random variable $Y(\mathbf{z})$. The collection $\{Y(\mathbf{z}), \mathbf{z} \in A\}$ is i.i.d. and each $Y(\mathbf{z})$ takes value 1 with probability p and 0 with probability $1 - p$.

   Consider *directed paths* starting $(0, 0)$ at (n, n). Each step of the path consists of increasing one of the two coordinates by 1. Thus each such path has $2n+1$ vertices, and there are $\binom{2n}{n}$ such paths. Let Π_n be the set of such paths.

For each path $\pi \in \Pi_n$, define $W(\pi)$, the weight of π, to be the sum of $Y(\mathbf{z})$ over all the vertices $\mathbf{z}$ included in π.

Finally let $X_n = \max_{\pi \in \Pi_n} W(\pi)$ be the maximum weight of a directed path between $(0,0)$ and (n,n).

Let p be fixed. Show that the expectation and the median of X_n are $\Theta(n)$.

Use (i) Azuma's inequality and the corollary on the concentration of Lipschitz functions, and (ii) Talagrand's inequality, to derive concentration inequalities for X_n, and compare them.

7. Consider the bond percolation model on $\mathbb{Z}^2$ (as in the last lecture). The vertices of the graph are the points of $\mathbb{Z}^2$. Each edge between nearest neighbour vertices is present with probability p and absent with probability $1-p$, independently (so every vertex has between 0 and 4 neighbours).

   By considering an exploration process and comparing to a branching process, or otherwise, find a $\hat{p} > 0$ such that if $p \leq \hat{p}$, the probability that the component containing the origin is finite is 1.

Course webpage: `http://www.stats.ox.ac.uk/~martin/PC.html`