Problem Sheet 4 - Part C Probabilistic Combinatorics - Oxford HT 2008

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1. Let $H = (V, E)$ be a hypergraph. Suppose the vertices are k-coloured uniformly at random; each $v \in V$ receives each colour with probability $1/k$, and the colours of different vertices are independent. For $e \in E$, let A_e be the event that edge e is monochromatic.

   Show that if $|e \cap f| \leq 1$, then A_e and A_f are independent.

   Is it true that A_e is independent of the collection $\{A_f : |e \cap f| \leq 1\}$?

2. (a) Check that $(1 - x) \geq e^{-1.2x}$ whenever $x \leq 1/4$.

   (b) Let $A_1, \ldots, A_n$ be a collection of events. Let $\mathcal{D}_i \subseteq \{1, 2, \ldots, n\}$ for each $i \in \{1, 2, \ldots, n\}$.

   Suppose that for all i:

   (i) A_i is independent of the collection $\{A_j, j \neq i, j \notin \mathcal{D}_i\}$;

   (ii) $\mathbb{P}(A_i) \leq 1/8$;

   (iii) $\sum_{j \in \mathcal{D}_i} \mathbb{P}(A_j) \leq 1/4$.

   Prove that with positive probability, none of the events A_i occur.

3. Let $G = (V, E)$ be a graph with maximum degree Δ, and let $V_1, V_2, \ldots, V_s$ be a collection of disjoint subsets of V, each of size at least $2e\Delta$. Show that G has an independent set which contains a vertex from each set V_i.

4. Let $G = (V, E)$ be a graph and suppose that each $v \in V$ is associated with a list $S(v)$ of colours of size at least $6r$, where r is a positive integer. Suppose also that for each $v \in V$ and each $c \in S(v)$, there are at most r neighbours u of v such that $c \in S(u)$.

   Prove that there is a proper colouring of G under which each vertex v receives a colour from its list $S(v)$.

5. Let $W(k)$ be the smallest integer n such that any two-colouring of the set $\{1, 2, \ldots, n\}$ gives a monochromatic arithmetic progression of length k.

   (a) Use the first-moment method to show that $W(k) \geq 2^{k/2}$.

   (b) Use the Local Lemma to show that $W(k) \geq \frac{2^k}{2ek}\big(1 + o(1)\big)$.

6. (a) Consider a branching process starting from one individual at generation 0. Find the expectation of the number of individuals in the nth generation, in terms of the common family size distribution.

   (b) Find the probability of extinction in a branching process whose mean family size μ is precisely 1. (One way to do this is to extend the approach from lectures for $\mu < 1$ and $\mu > 1$). What is the expectation of the total number of individuals in the process?