Problem Sheet 1 - Part C Probabilistic Combinatorics - Oxford HT 2008

Estimates and asymptotics:

1. Prove the following inequalities:
   (i) $\binom{n}{k} \le \frac{n^k}{k!} \le \left(\frac{en}{k}\right)^k$ for $1 \le k \le n$.
   (ii) $1 + x \le e^x$ for all $x \in \mathbb{R}$.
   (iii) $(1+a)^m \le e^{am}$ for $a > -1$, $m > 0$.

2. Use Stirling's Formula to show that if $n, k \to \infty$ with $k^2 = o(n)$, then

$$\binom{n}{k} \sim \frac{1}{\sqrt{2\pi k}} \left(\frac{en}{k}\right)^k.$$

3. For the following functions f, g, decide whether $f = o(g)$ or $g = o(f)$ or $f = \Theta(g)$ as $n \to \infty$:
   (a) $f = \ln n$, $g = n/\ln n$;
   (b) $f = \binom{n}{k}$, $g = n^k$, first for k fixed and then for the case where $k \to \infty$ as $n \to \infty$;
   (c) $f = (\ln n)^{1000}$, $g = n^{1/1000}$.

4. In lectures we showed that the kth diagonal Ramsey number satisfies

$$R(k,k) > N - \binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}-1},$$

   for any integer N. Deduce that

$$R(k,k) > \frac{1}{e}\big(1 + o(1)\big)k2^{k/2}.$$

Union bound and first-moment method:

5. (a) Consider a 3-SAT problem with m clauses. Show that there is an assignment of values (`TRUE` or `FALSE`) to the variables such that at least $7m/8$ clauses are satisfied (i.e. have the value `TRUE`).
   (b) Consider a SAT formula with m clauses in which the ith clause has length c_i. Show that if $\sum_{i=1}^{m} 2^{-c_i} < 1$ then the formula is satisfiable.

6. Let H be an n-uniform hypergraph with fewer than $\frac{4^{n-1}}{3^n}$ edges, where $n \ge 4$. Prove that the vertices of H can be coloured using four colours in such a way that in every edge, all four colours are represented.

7. Let F be a collection of binary strings ("codewords") of finite lengths. Suppose the ith codeword has length c_i. Suppose that no member of F is a prefix of another member (so you can decode any string made up of a sequence of codewords as you go along, without looking ahead). Show that $\sum_i 2^{-c_i} \leq 1$ (the *Kraft inequality* for prefix-free codes).

8. Let G be a bipartite graph with n vertices. Suppose each vertex v has a list $S(v)$ of more than $\log_2 n$ colours associated to it. Show that there is a proper colouring of G under which each vertex v receives a colour from its list $S(v)$.

Course webpage: `http://www.stats.ox.ac.uk/~martin/PC.html`