Part C Probabilistic Combinatorics - Oxford 2007 - Practice questions

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1. (a) Prove that if $k, n \in \mathbb{N}$ with $1 \le k \le n$,

$$\binom{n}{k} \le \left(\frac{en}{k}\right)^k.$$

   (b) Define the Ramsey number $R(k,l)$. Prove that for any k, l and n, and any $p \in [0,1]$,

$$R(k,l) > n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{l} (1-p)^{\binom{l}{2}}.$$

   Hence show that for some constant $c > 0$, $R(4,k) \ge c\left(\frac{k}{\ln k}\right)^2$ for each k.

2. (a) State and prove the Lovász Local Lemma.

   (b) Let $\mathcal{H}$ be a hypergraph in which each edge has size at least 3, and each edge intersects with at most a_i edges of length i (not including itself), where

$$\prod_{i=3}^{\infty} \left(1 - 2^{-i+2}\right)^{a_i} \ge \frac{1}{2}.$$

   Show that $\mathcal{H}$ is two-colourable (i.e. there is a colouring of the vertices with two colours such that no edge is monochromatic).

   [*Hint: choose the weight of an event in the Local Lemma to be twice its probability*]

3. (a) Let S have Binomial$(n, \frac{1}{2})$ distribution.

   (i) Show that for $0 < h < 1/2$,

$$\mathbb{P}\left(S \ge n\left(\frac{1}{2} + h\right)\right) \le e^{-un\left(\frac{1}{2}+h\right)} \left(\frac{1}{2} + \frac{1}{2}e^u\right)^n.$$

   (ii) Deduce that

$$\mathbb{P}\left(S \ge n\left(\frac{1}{2} + h\right)\right) \le \exp\left(nf(h)\right),$$

   where $f(h) = -\left(\frac{1}{2} + h\right)\ln(1+2h) - \left(\frac{1}{2} - h\right)\ln(1-2h)$.

   (iii) Using Taylor's theorem or otherwise, show that $f(h) \le -2h^2$, and deduce a form of Chernoff's bound for $\mathbb{P}\Big(S \ge n\left(\frac{1}{2} + h\right)\Big)$.

   (b) Let G be a random graph with distribution $G(n, 1/2)$. Show that the probability that G contains a bipartite subgraph with $n^2/8 + n^{3/2}$ edges tends to 0 as $n \to \infty$.

4. (a) (i) Explain what is meant by the random graph model $G(n,p)$, where $n \in \mathbb{N}$ and $p \in [0,1]$.

(ii) Suppose $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ are independent and both have distribution $G(n,p)$. Describe the distributions of $G_+ = (V, E_1 \cup E_2)$ and $G_- = (V, E_1 \cap E_2)$.

(iii) Suppose $\mathcal{Q}$ is an increasing property of a graph, and $0 \le p \le 1/2$. Show that if

$$\mathbb{P}\big(G(n,p) \text{ has property } \mathcal{Q}\big) = \alpha$$

then

$$\mathbb{P}\big(G(n,2p) \text{ has property } \mathcal{Q}\big) \ge 1 - (1-\alpha)^2.$$

(iv) Show that if the sequence $\hat{p}(n)$ satisfies

$$\mathbb{P}\big(G(n,\hat{p}(n)) \text{ has property } \mathcal{Q}\big) = 1/2$$

for all n, then $\hat{p}(n)$ is a threshold function for $\mathcal{Q}$.

(b) Consider the following model of a random 3-partite graph $G((n,n,n),p)$: the vertex set consists of 3 classes each containing n vertices; each of the $3n^2$ possible edges which connect vertices in different classes is present with probability p and absent with probability $1-p$, independently.

Find the expectation and the variance of the number of cycles of length 3 in a graph with distribution $G((n,n,n),p)$. Hence or otherwise, find a threshold function for the property that the graph contains a cycle of length 3.

[*You may use standard moment bounds without proof if you state them clearly.*]