DMAC 2485
DMSU 2485

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C11.1
Honour School of Mathematics and Statistics Part C: Paper C11.1

GRAPH THEORY and PROBABILISTIC COMBINATORICS

Trinity Term 2007

Friday 1 June 2007, 9.30am to 12.30pm

You may attempt as many questions as you like but **only your two best answers from each section of the paper will count.**

You must start a new booklet for each question you attempt. Group together the booklets for each section, and indicate on the front page of each group, the questions attempted. Attach all groups of answer booklets together with the treasury tag provided. At least one booklet must be handed in for each section.

Do not turn this page until you are told that you may do so

A. Graph Theory

1. (a) State and prove Hall's Theorem on matchings in bipartite graphs.

(b) Let G be a bipartite graph with vertex classes V_1 and V_2, where $|V_1| = |V_2| = n$. Prove that if $e(G) > n(n-1)$ then G contains a complete matching from V_1 to V_2. Give an example showing that $e(G) \geqslant n(n-1)$ is not enough to guarantee a complete matching.

(c) Let G be a graph of order $n \geqslant 3$ such that $\delta(G) \geqslant n/2$. Prove that G contains a Hamilton cycle.

2. (a) Define the *chromatic polynomial* $p_G(x)$ of a graph G.

(b) Let G be a graph of order n. Prove that

$$p_G(x) = \sum_{i=0}^{n-1} (-1)^i a_i x^{n-i},$$

where $a_i \geqslant 0$ for every i. [*You may use a relationship between* p_G, p_{G-e} *and* $p_{G/e}$, *where* e *is any edge of* G, *provided that you state it clearly.*]

(c) Prove that, for $n \geqslant 3$, the cycle C_n has chromatic polynomial

$$p_{C_n}(x) = (x-1)^n + (-1)^n(x-1).$$

(d) For $n \geqslant 3$, the *wheel* W_{n+1} is the graph on $n+1$ vertices obtained from C_n by adding a new vertex v and joining it to every other vertex. What is the chromatic polynomial $p_{W_{n+1}}(x)$?

3. (a) Define the *Ramsey number* $R(k,l)$. Prove that, for $k, l \geqslant 2$,

$$R(k,l) \leqslant \binom{k+l-2}{k-1}.$$

(b) For graphs G, H, define the *graph Ramsey number* $R(G,H)$.

(c) Prove that if T is a tree on t vertices then $R(K_s, T) = (s-1)(t-1)+1$. [*You may assume that every graph with chromatic number* t *contains a copy of* T.]

(d) Prove that if G is a graph with $\Delta(G) \geqslant R(t,t)$ then G contains either a copy of K_{t+1} or an *induced* copy of $K_{1,t}$.

Show that if $t \geqslant 2$ is an integer, then every *connected* graph with sufficiently many vertices contains either a copy of K_{t+1}, or an *induced* copy of $K_{1,t}$ or an *induced* path of length t.

4. (a) Define the quantity $\mathrm{ex}(n; H)$, where H is a graph and $n \geqslant 1$ is an integer. State the Erdős–Stone theorem.

(b) For $k \geqslant 3$, determine the value of

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n; C_k)}{\binom{n}{2}}.$$

(c) Let G be a graph, and let X and Y be disjoint subsets of $V(G)$. Define what it means to say that (X, Y) is an *ϵ-uniform* pair.

(d) Suppose that $0 < \epsilon < d/2$, and (X, Y) is an ϵ-uniform pair with density $d(X, Y) \geqslant d$. Prove that if $|X| = |Y| = n$ then there are at most ϵn vertices $x \in X$ such that $|\Gamma(x) \cap Y| < d|Y|/2$.

(e) Now suppose that X_1, X_2, X_3 are (nonempty) disjoint sets of n vertices, and each pair (X_i, X_j) is ϵ-uniform with density at least d. Prove that G contains a triangle.

B. Probabilistic Combinatorics

5. (a) State the general form of the Lovász Local Lemma. State the symmetric form of the Lovász Local Lemma and show how it implies the symmetric form.

   (b) An r-regular graph is one in which every vertex has degree r. Show that if r is sufficiently large, then every r-regular graph $G = (V, E)$ has a five-colouring $\chi : V \to \{1, 2, 3, 4, 5\}$ with the following property: for every vertex v, there is a neighbour w of v such that $\chi(w) \equiv \chi(v) + 1 \bmod 5$.

   (c) Deduce that for sufficiently large r, every r-regular graph contains a cycle whose length is a multiple of 5.

6. (a) State and prove Chebyshev's inequality.

   (b) Let $A_1, A_2, \ldots, A_m$ be a collection of events. For $1 \leqslant i \leqslant m$, let $X_i = I(A_i)$, and let $X = \sum_{i=1}^{m} X_i$.
   For $1 \leqslant i, j \leqslant m$, write "$i \sim j$" if $i \neq j$ and A_i and A_j are not independent. Let $D = \sum_{i,j: i \sim j} \mathbb{P}(A_i \cap A_j)$.
   Show that
   $$\mathbb{P}(X = 0) \leqslant \frac{1}{\mathbb{E}X} + \frac{D}{(\mathbb{E}X)^2}.$$

   (c) Consider the random graph model $G(n, p)$. Find, with proof, a threshold for the property that the graph contains a copy of K_4.

   (d) Now consider the random graph *process* $\big(G(n,p)\big)_{p \in [0,1]}$.
   Let A be the event that at some stage in the process, the graph contains K_4 as an *isolated* subgraph (i.e. there exists a component with exactly 4 vertices and 6 edges).
   Show that as $n \to \infty$, $\mathbb{P}(A) \to 0$.
   [*Hint: for given vertices u, v, w, x, which edges can affect the property that $\{u, v, w, x\}$ is a component isomorphic to K_4? What order must these edges arrive in for $\{u, v, w, x\}$ to be an isolated K_4 at some stage in the process?*]

7. Let G be a random graph with distribution $G\left(n, \frac{1}{2n}\right)$.

   (a) Show that for some constant α, the probability that the largest component of G has size more than $\alpha \log n$ tends to 0 as $n \to \infty$.

   (b) Let v be any fixed vertex. Show that there exists a constant K such that the expectation of the size of the component containing v is less than or equal to K for all n.

[*You may use any standard version of the Chernoff bounds without proof, provided you state it clearly. If you need to use any standard facts about branching processes, you may also use them without proof.*]

8. (a) Let $\Omega_1, \ldots, \Omega_m$ be finite sets, and let $\Omega = \Omega_1 \times \cdots \times \Omega_m$. Let f be a function from Ω to $\mathbb{R}$. What does it mean to say that f is *Lipschitz*?

(b) State Azuma's inequality for martingales with increments bounded by 1.

Explain how this can be used to give a concentration inequality for Lipschitz functions (a detailed proof is not required, but a precise statement of the result should be given).

(c) Suppose $Y_1, Y_2, \ldots, Y_n$ are independent random points in $\mathbb{R}^2$, with arbitrary distributions.

Let A be the union of the balls of radius 1 around the n points; that is, for $x \in \mathbb{R}^2$, $x \in A$ if and only if $|x - Y_i| \leqslant 1$ for some i.

Let X be the area of the set A. Prove that for some constant $c > 0$ (not depending on n or on the distributions of the points),

$$\mathbb{P}\big(|X - \mathbb{E}X| \geqslant t\big) \leqslant 2\exp\left(-\frac{ct^2}{n}\right)$$

for all $t \geqslant 0$.

(d) Consider the complete graph K_n on n vertices. Suppose that to each edge e of the graph we associate a random weight X_e, where the weights $\{X_e\}$ are i.i.d. and each has uniform distribution on $[0,1]$.

A *matching* is a subset of the edge-set such that no vertex is contained in more than one edge. The *weight* of a matching M is $\sum_{e \in M} X_e$, the sum of the weights of the edges in the matching.

Let W be the maximum weight of a matching (considered over all possible matchings in the graph). Prove that for all $t \geqslant 0$,

$$\mathbb{P}\left(|W - \mathbb{E}W| \geqslant t\right) \leqslant 2\exp\left(-\frac{t^2}{2(n-1)}\right).$$

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