Linear Programming **Trinity Term 2006**

Sheet 1 `http://www.stats.ox.ac.uk/~martin/lp.html`

1. A manufacturer wishes to produce an alloy that is, by weight, 30% metal A and 70% metal B. Five alloys are available at various prices indicated below:

Alloy	1	2	3	4	5
% A	10	25	50	75	95
% B	90	75	50	25	5
Price/kg	£5	£4	£3	£2	£1.50

The desired alloy will be produced by combining some of the other alloys. The manufacturer wishes to find the amounts of the various alloys needed and to determine the least expensive combination. Write a linear programming formulation of this problem.

2. Solve the following linear programming problem by drawing a diagram:

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 2x_2 \\ \text{subject to} \quad & -x_1 + 3x_2 \leqslant 12 \\ & x_1 + x_2 \leqslant 8 \\ & 2x_1 - x_2 \leqslant 10 \\ & x_1, x_2 \geqslant 0. \end{aligned}$$

3. Show how to introduce slack variables to put the problem in the previous question into the standard form

$$\text{maximize } \mathbf{c}^T\mathbf{x} \quad \text{subject to } A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geqslant \mathbf{0}.$$

and solve it using the simplex method. You should use the obvious initial solution with the slack variables positive and the other variables zero. Try both choices of pivot column on the first step. Compare the various tableau with your diagram—will the number of pivots be minimized by pivoting on the first column or the second column at the first step?

4. Solve the following problem using the simplex algorithm:

$$\begin{aligned} \text{maximize} \quad & 3x_1 + x_2 + 3x_3 \\ \text{subject to} \quad & 2x_1 + x_2 + x_3 \leqslant 2 \\ & x_1 + 2x_2 + 3x_3 \leqslant 5 \\ & 2x_1 + 2x_2 + x_3 \leqslant 6 \\ & x_1, x_2, x_3 \geqslant 0. \end{aligned}$$

5. Consider applying the simplex algorithm to the problem

$$\begin{aligned} \text{maximize} \quad & 2x_1 + 3x_2 \\ \text{subject to} \quad & x_1 - x_2 \leqslant 2 \\ & -2x_1 + x_2 \leqslant 1 \\ & x_1, x_2 \geqslant 0. \end{aligned}$$

Explain what happens using a diagram.

6. Consider the problem

$$\begin{array}{lrcl} \text{maximize} & 2x_1 + 3x_2 + \ x_3 & & \\ \text{subject to} & x_1 + 2x_2 + 3x_3 + x_4 & & = 6 \\ & 2x_1 + \ x_2 + 2x_3 & + x_5 & = 4 \\ & x_1, \ldots, x_5 \geqslant 0 & & \end{array}$$

with initial tableau:

1	2	3	1	0	6
2	1	2	0	1	4
2	3	1	0	0	0

and final tableau:

·	·	·	$\frac{2}{3}$	$-\frac{1}{3}$	·
·	·	·	$-\frac{1}{3}$	$\frac{2}{3}$	·
·	·	·	$-\frac{4}{3}$	$-\frac{1}{3}$	·

Without going through the simplex algorithm, fill in the other numbers in the final tableau. Suppose that you now wish to change the problem by adding a term $+2x_6$ to the objective function and a term $+x_6$ to the left-hand side of the first constraint. Show how to add an extra column to the final tableau (again without using the simplex algorithm) to accommodate this change. Complete the solution of the new problem using the simplex algorithm.

7. Use the two-phase simplex algorithm to solve:

$$\text{maximize} \quad 13x_1 + 5x_2 - 12x_3 \quad \text{subject to} \quad \begin{array}{rcl} 2x_1 + x_2 + 2x_3 & \leqslant & 5 \\ 3x_2 + 3x_2 + x_3 & \geqslant & 7 \\ x_1 + 5x_2 + 4x_3 & = & 10 \\ x_1, x_2, x_3 \geqslant 0 & & \end{array}$$

8. Use the two-phase simplex method to consider the problem: maximize $x_1 + x_2 + x_3 + x_4$ subject to $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 1 & 1 & -2 & 1 \\ 3 & 1 & -2 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.$$

9. Define $P(\epsilon)$ to be the problem obtained by replacing the vector $\mathbf{b} = (12, 8, 10)^T$ in question 1 by the perturbed vector $\mathbf{b}(\epsilon) = (12 + \epsilon_1, 8 + \epsilon_2, 10 + \epsilon_3)^T$. Give a formula, in terms of $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$, for the optimal value for $P(\epsilon)$ when the ϵ_i are small. If $\epsilon_1 = \epsilon_2 = 0$, for what range of ϵ_3 values does the formula hold?