Problem Sheet 6 - Part B Actuarial Science I - Oxford MT 2006

For some of the exercises on this sheet, you will need values of the standard normal distribution function. If $Z \sim N(0,1)$, then:

z	-3	-2.3263	-2.4778	-1.8349	-0.039	0.233
$P(Z \leq z)$	0.00135	0.01000	0.00661	0.03326	0.48445	0.59211

1. An annuity is payable continuously at a rate of $\rho(t)$ per annum at time t provided the holder, who is aged x at time 0, is still alive. T_x is a random variable which models the remaining lifetime in years of a person aged x.

   (a) Write down an expression, in terms of T_x, for the (random) present value at time 0 of this cash-flow, at a constant force of interest δ p.a., and show that the expected present value at time 0 of the cash flow is equal to
   $$\int_0^\infty e^{-\delta s}\rho(s)P(T_x > s)ds.$$

   (b) An annuity is payable continuously during the lifetime of a person now aged 30, but for at most 10 years. The rate of payment at all times t during the first 5 years is £5,000 p.a., and thereafter £10,000 p.a. The force of mortality to which this person is subject is assumed to be 0.01 p.a. at all ages between 30 and 35, and 0.02 p.a. between 35 and 40. Find the expected present value of this annuity at a force of interest of 0.05 p.a.

2. An insurer issues n identical policies. Let Y_j be the claim amount from the jth policy, and suppose that the random variables Y_j, $j = 1, \ldots, n$ are i.i.d. with mean $\mu > 0$ and variance σ^2. The insurer charges a premium of A for each policy.

   (a) Show that if $A = \mu + 10\sigma n^{-1/2}$, then the probability that total claims exceed total premiums is no more than 1%, for any value of n.

   (b) Use the Central Limit Theorem to show that if instead $A = \mu + 3\sigma n^{-1/2}$, then this probability is still less than 1%, provided n is large enough.

3. Suppose that the interest rate for the next six months is known to be 5.5%, while the rate for the six months after that is unknown and assumed to be uniformly distributed on the interval $(4\%, 6\%)$. Under this assumption, find the expectations of:

   (a) the accumulated value after one year of £100 invested now;

   (b) the discounted present value of a payment of £100 in a year's time.

4. Let I_j denote the effective rate of interest in the year $j-1$ to j. Suppose that, for $j \geq 1$
   $$I_{j+1} = \begin{cases} I_j + 0.02 & \\ I_j & \text{with probability} \\ I_j - 0.02 & \end{cases} \quad \begin{matrix} 0.25 \\ 0.5 \\ 0.25 \end{matrix}$$
   Given that $I_1 \equiv 0.06$, calculate the probability that an investment of 1 at time 0 accumulates to more than 1.2 at time 3.

5. Let $1+I$ be a lognormal random variable with parameters μ and σ^2, mean $1+j$ and variance s^2. Show that

$$\sigma^2 = \log\left(1+\left(\frac{s}{1+j}\right)^2\right) \qquad \text{and} \qquad \mu = \log\left(\frac{1+j}{\sqrt{1+\left(\frac{s}{1+j}\right)^2}}\right)$$

6. The rate of return on an investment in a given year is denoted by Y. Suppose $1+Y$ is lognormally distributed. The expected value of the rate of return is 5% and its standard deviation is 11%.

   (a) Calculate the parameters of the lognormal distribution of $1+Y$.

   (b) Calculate the probability that the rate of return for the year lies between 4% and 7%.

7. A company is adopting a particular investment strategy such that the expected annual effective rate of return from investments is 7% and the standard deviation of annual returns is 9%. Annual returns are independent and $(1+I_j)$ is lognormally distributed, where I_j is the return in the jth year.

   (a) Calculate the expected value and standard deviation of an investment of 1 unit over 10 years, deriving all formulae that you use.

   (b) Calculate the probability that the accumulation of such an investment will be less than 50% of its expected value in ten years' time.

   (c) The company has an outstanding debt and must make a payment of £140,000 in 10 years time. Calculate the probability that an investment of £120,000 now will provide sufficient funds to meet this liability.

Optional question for further practice:

8. Suppose the force of interest Δ_j during the year from $j-1$ to j is given by

$$\Delta_j = \mu + \frac{1}{\sqrt{2}}\,\epsilon_{j-1} + \frac{1}{\sqrt{2}}\,\epsilon_j,$$

   where ϵ_0, ϵ_1, $\epsilon_2, \ldots$ are i.i.d. random variables with distribution $N(0,\sigma^2)$.

   (a) Show that $\Delta_j \sim N(\mu,\sigma^2)$ for all j.

   (b) Write an expression for $\Delta_1 + \Delta_2 + \cdots + \Delta_n$ in terms of the random variables ϵ_j. Hence show that the accumulated value at time n of 1 unit invested at time 0 has a lognormal distribution, and find its parameters.

Course webpage: `http://www.stats.ox.ac.uk/~martin/BS4a.html`