Lecture 4 : The Binomial Distribution

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Permutations and Combinations (Recap)

Consider 7 students applying to a college for 3 places

Abi Ben Claire Dave Emma Frank Gail

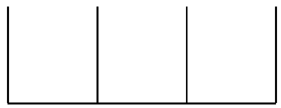
How many ways are there of choosing 3 students from 7 when

(i) order is important (permutations)
i.e. (Abi, Dave) $\neq$ (Dave, Abi)

(ii) order isn't important (combinations)
i.e. (Abi, Dave) = (Dave, Abi)

Permutations

We have to fill 3 places at the college



There will be $7 \times 6 \times 5$ possible permutations.

$$7 \times 6 \times 5 = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} = \frac{7!}{4!} = {}^{7}P_{3}$$

In general, the no. of permutations of r objects from n is

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

Combinations

For every combination of 3 students there will be 3! permutations
e.g. the combination DBA gives rise to the permutations

DBA DAB ABD ADB BAD BDA

$$^7P_3 = 3!\,^7C_3 \qquad \Rightarrow {}^7C_3 = \frac{^7P_3}{3!}$$

In general, the no. of combinations of r objects from n is

$$^nC_r = \frac{^nP_r}{r!} = \frac{n!}{(n-r)!r!}$$

An example of the Binomial distribution

An unfair coin P(Head) = 2/3 P(Tail) = 1/3

Let X = No. of heads observed in 5 coin tosses

X can take on any of the values 0, 1, 2, 3, 4, 5

X is a **discrete random variable**

Some values of X will be more likely to occur than others. Each value of X will have a probability of occurring. What are these probabilities?

What is P(X = 1) ?

One possible way of obtaining one head is if we observe the pattern HTTTT. The probability of obtaining this pattern is

$$P(HTTTT) = \frac{2}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3}$$

There are 32 possible patterns of heads and tails we might observe. 5 of the patterns contain just one head

HHHHH	THHHH	HTHHH	HHTHH	HHHTH	HHHHT
TTHHH	THTHH	THHTH	THHHT	HTTHH	HTHTH
HTHHT	HHTTH	HHTHT	HHHTT	TTTHH	TTHTH
TTHHT	THTTH	THTHT	THHTT	HTTTH	HTTHT
HTHTT	HHTTT	HTTTT	THTTT	TTHTT	TTTHT
TTTTH	TTTTT				

The other 5 possible combinations all have the same probability so the probability of obtaining one head in 5 coin tosses is

$$P(X = 1) = 5 \times \left(\frac{2}{3} \times (\frac{1}{3})^4\right) = 0.0412 \text{ (to 4dp)}$$

What about P(X = 2)?

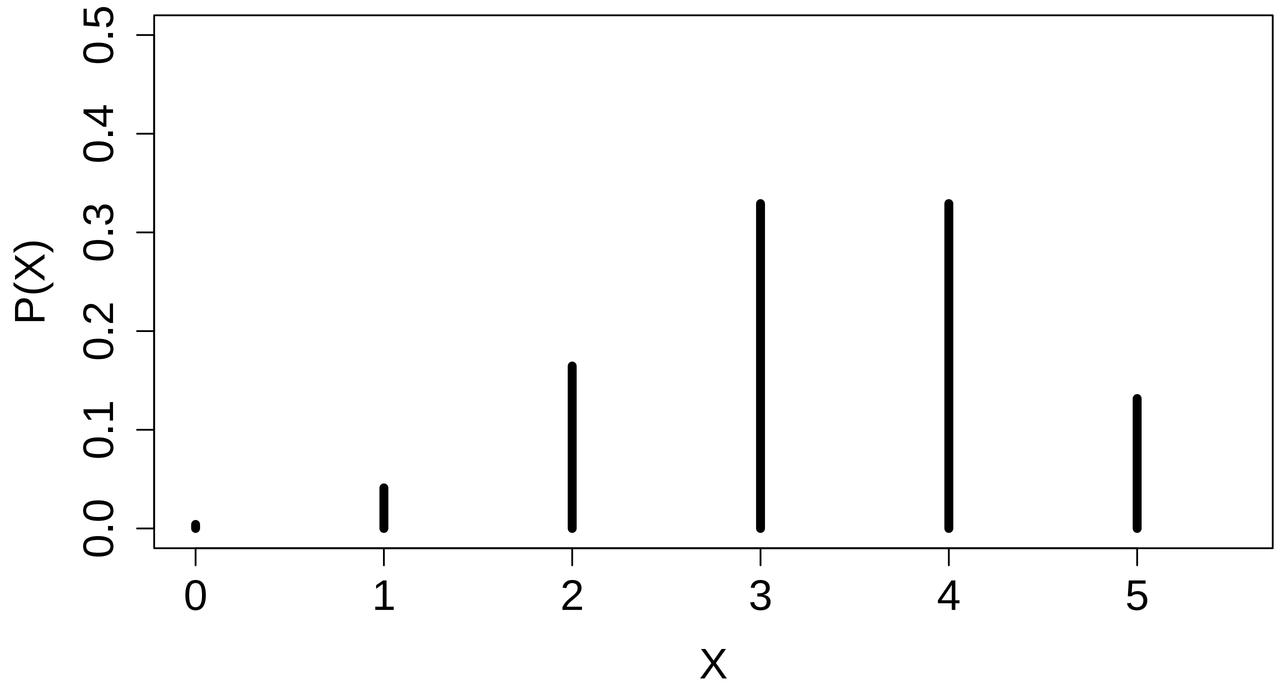
This probability can be written as

$$
\begin{aligned}
P(X=2) &= \text{No. of patterns} \times \text{Probability of pattern} \\
&= {}^5C_2 \times \left(\frac{2}{3}\right)^2 \times \left(\frac{1}{3}\right)^3 \\
&= 10 \times \frac{4}{243} \\
&= 0.165
\end{aligned}
$$

Similarly

$$
\begin{aligned}
P(X = 0) &= {}^5C_0 \times \left(\tfrac{2}{3}\right)^0 \times \left(\tfrac{1}{3}\right)^5 = 0.0041 \\
P(X = 1) &= {}^5C_1 \times \left(\tfrac{2}{3}\right)^1 \times \left(\tfrac{1}{3}\right)^4 = 0.0412 \\
P(X = 2) &= {}^5C_2 \times \left(\tfrac{2}{3}\right)^2 \times \left(\tfrac{1}{3}\right)^3 = 0.1646 \\
P(X = 3) &= {}^5C_3 \times \left(\tfrac{2}{3}\right)^3 \times \left(\tfrac{1}{3}\right)^2 = 0.3292 \\
P(X = 4) &= {}^5C_4 \times \left(\tfrac{2}{3}\right)^4 \times \left(\tfrac{1}{3}\right)^1 = 0.3292 \\
P(X = 5) &= {}^5C_5 \times \left(\tfrac{2}{3}\right)^5 \times \left(\tfrac{1}{3}\right)^0 = 0.1317 \\
\hline
\text{Sum} &= 1.0000
\end{aligned}
$$

Plotting these probabilities we obtain a **distribution**. This situation is a specific example of a Binomial distribution.



The Binomial Distribution

In general a Binomial distribution arises when we have the following 4 conditions

- n identical trials, e.g. 5 coin tosses
- 2 possible outcomes for each trial "success" and "failure", e.g. Heads or Tails
- Trials are independent, e.g. each coin toss doesn't affect the others
- P("success") = p and P("failure") = q = 1- p are the same for each trial, e.g. P(Head) = 2/3 P(Tail) = 1/3 are the same for each trial

If we have the above 4 conditions then if we let

X = No. of "successes"

$\Rightarrow$ the probability of observing x successes out of n trials is

$$P(X = x) = {}^{n}C_{x} p^{x} q^{(n-x)} \qquad x = 0, 1, \ldots, n$$

We write $X \sim \text{Bin}(n, p)$

n and p are called the **parameters** of the distribution.

Example 1

Suppose X ∼ Bin(10, 0.4), what is P(X = 7)?

$$
\begin{aligned}
P(X = 7) &= {}^{10}C_7 (0.4)^7 (1 - 0.4)^{(10-7)} \\
&= (120)(0.4)^7 (0.6)^3 \\
&= 0.0425
\end{aligned}
$$

Example 2

Suppose Y $\sim$ Bin(8, 0.15), what is P(Y $<$ 3)?

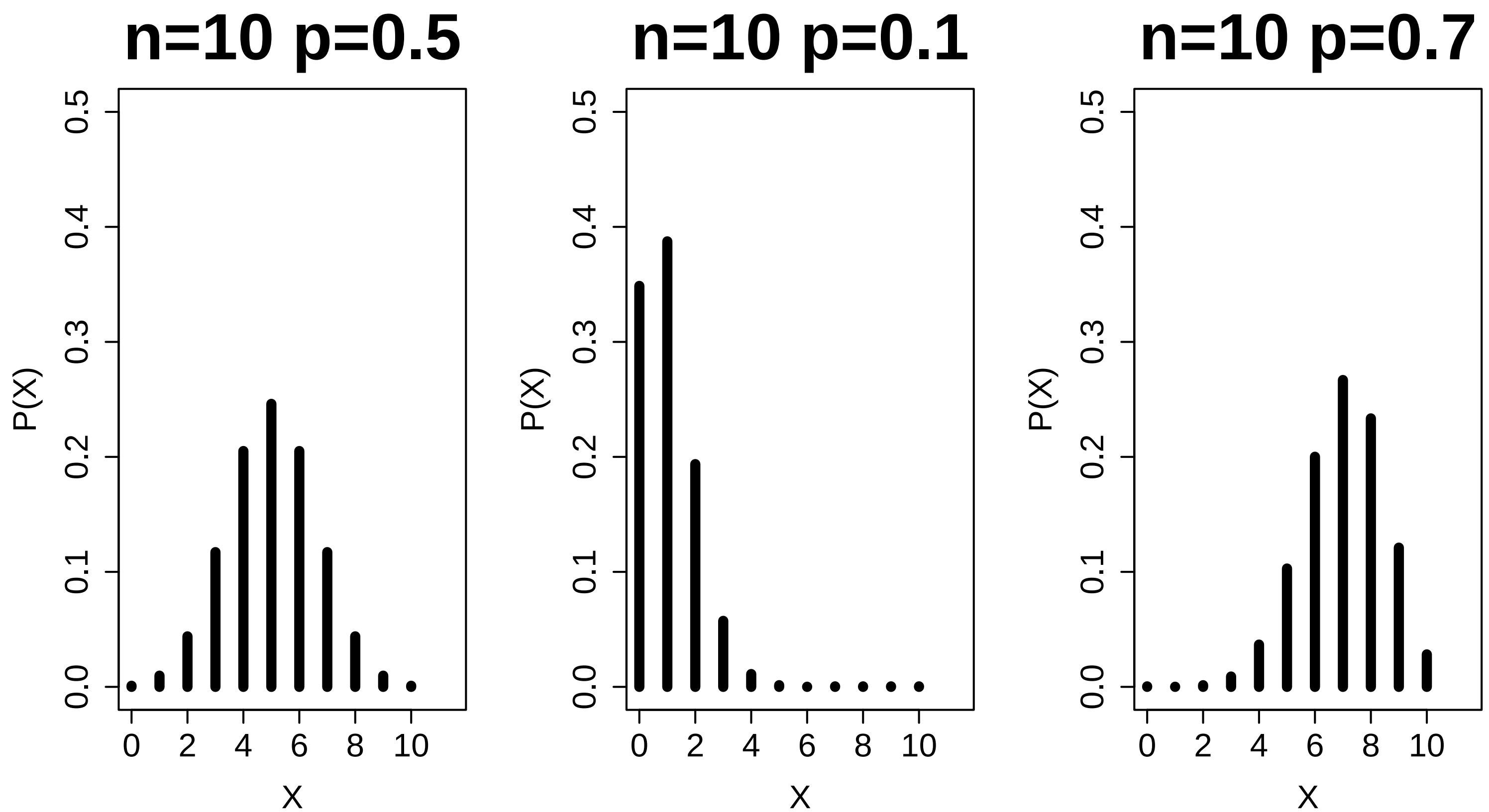
$$
\begin{aligned}
P(Y < 3) &= P(Y = 0) + P(Y = 1) + P(Y = 2) \\
&= {}^8C_0(0.15)^0(0.85)^8 \\
&\qquad + {}^8C_1(0.15)^1(0.85)^7 \\
&\qquad\qquad + {}^8C_2(0.15)^2(0.85)^6 \\
&= 0.2725 + 0.3847 + 0.2376 \\
&= 0.8948
\end{aligned}
$$

Example 3

Suppose W $\sim$ Bin(50, 0.12), what is P(W $>$ 2)?

$$
\begin{aligned}
P(W > 2) &= P(W = 3) + P(W = 4) + \ldots + P(W = 50) \\
&= 1 - P(W \leq 2) \\
&= 1 - \Big(P(W = 0) + P(W = 1) + P(W = 2) \Big) \\
&= 1 - \Big({}^{50}C_0 (0.12)^0 (0.88)^{50} \\
&\qquad + {}^{50}C_1 (0.12)^1 (0.88)^{49} \\
&\qquad\qquad + {}^{50}C_2 (0.12)^2 (0.88)^{48} \Big) \\
&= 1 - \Big(0.00168 + 0.01142 + 0.03817 \Big) \\
&= 0.94874
\end{aligned}
$$

Different values of n and p lead to different distributions with different shapes.



Bin(5, 2/3)

x	0	1	2	3	4	5
P(X = x)	0.004	0.041	0.165	0.329	0.329	0.132

The expected mean value of the distribution, denoted μ can be calculated as

$$\begin{aligned}\mu &= 0 \times (0.004) + 1 \times (0.041) + 2 \times (0.165) \\ &\quad + 3 \times (0.329) + 4 \times (0.329) + 5 \times (0.132) \\ &= 3.333\end{aligned}$$

In general, there is a formula for the mean of a Binomial distribution. There is also a formula for the standard deviation, σ.

If X $\sim$ Bin(n, p) then

$$\mu = np$$

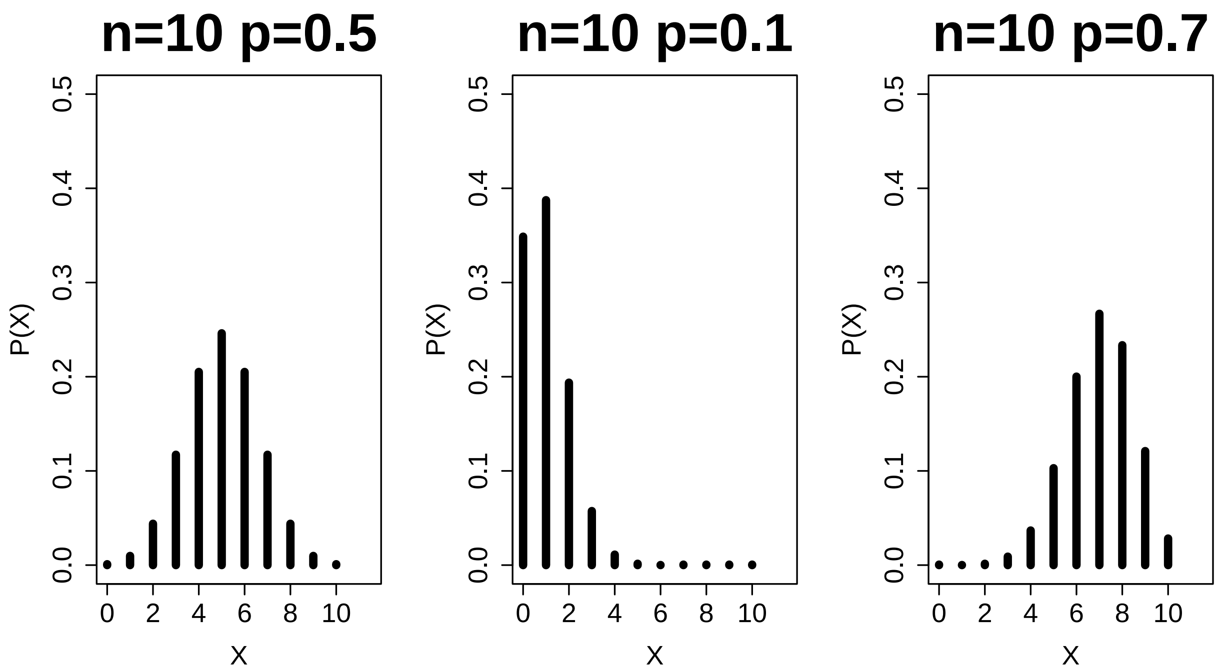
$$\sigma = \sqrt{npq} \quad \text{where } q = 1 - p$$

In the example above, X $\sim$ Bin(5, 2/3) and so the mean and standard deviation are given by

$$\mu = np = 5 \times (2/3) = 3.333$$

and

$$\sigma = \sqrt{npq} = 5 \times (2/3) \times (1/3) = 1.111$$

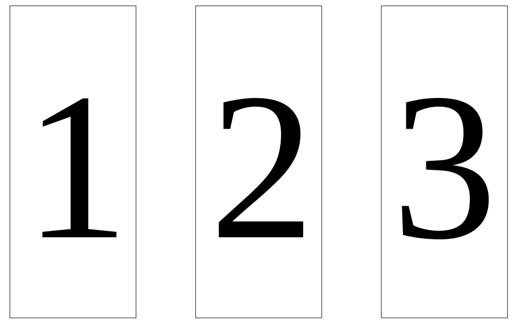


$p < 0.5 \Rightarrow$ POSITIVE SKEW

$p = 0.5 \Rightarrow$ SYMMETRIC

$p > 0.5 \Rightarrow$ NEGATIVE SKEW

The Game Show Problem



You have reached the final of a game show. The host shows you 3 doors and tells you that there is a prize behind one of the doors. You pick a door. The host then opens one of the doors you didn't pick that contains no prize and asks you if you want to change from the door you chose to the other remaining door.

Q. Should you change?

Intuitively, you might think there would be no advantage to changing doors,
i.e. P(Win | Change) = 1/2.

We can test this in a scientific way

The basic idea is to

- posit a **hypothesis**
- design and carry out an **experiment** to collect a **sample** of data
- **test** to see if the sample is consistent with the hypothesis

Hypothesis The probability that you win the prize if you change doors is 1/2.

Experiment To test the hypothesis we could play out the scenario many times and count the number of occasions in which changing your choice would result in you winning the prize.

Sample Last week we carried out the experiment 200 times and observed that on (approximately) 119 occasions it was better to change.

Testing the hypothesis Assuming our hypothesis is true what is the probability that we would have observed such a sample or a sample more extreme, i.e. is our sample quite unlikely to have occurred under the assumptions of our hypothesis?

Assuming our hypothesis is true the experiment satisfies the conditions of the Binomial distribution

- n identical trials, i.e. 200 game shows
- 2 possible outcomes for each trial “success” and “failure”, i.e. ”Changing doors leads to a WIN” or ”Changing doors leads to a LOSS”
- Trials are independent, i.e. each game show is independent
- P(“success”) = p is the same for each trial, i.e. P(Changing doors leads to a WIN) = 1/2 is the same for each trial

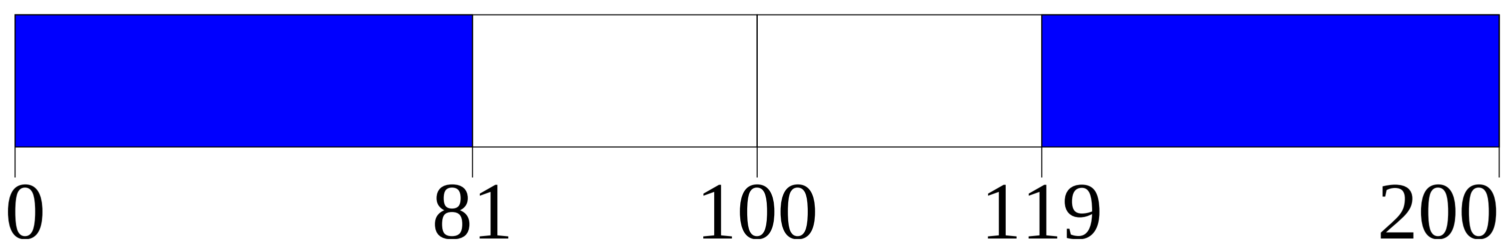
X = No. of game shows in which changing doors lead to a WIN

We observed X = 119. Which samples are more extreme than this?

Under our hypothesis we would expect X = 100

$X \geq 119$ and $X \leq 81$ are the samples as or more extreme than X = 119.

Thus we want $P(X \geq 119 \cup X \leq 81)$



We can calculate each of these probabilities using the Binomial probability formula

$\Rightarrow$

$$\mathrm{P}(\mathrm{X} \geq 119 \cup \mathrm{X} \leq 81) = 0.008722501$$

This is a very small probability. This tells us that if our hypothesis is true then it is very unlikely that we would have observed 119 out of 200 experiments in which changing doors leads to a WIN. In the language of hypothesis testing ‘we say we would reject the hypothesis’.