

Lectures 2 and 3 : Probability

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In this lecture we will learn about

- why we need to learn about probability
- what probability is
- how to assign probabilities
- how to manipulate probabilities and calculate probabilities of complex events

1 Why do we need to learn about probability?

In Lecture 1 we discussed why we need to study statistics and we saw that statistics plays a crucial role in the scientific process (see figure 1). We saw that we use a sample from the population in order to test our hypothesis. There will usually be a very large number of possible samples we could have taken and the conclusions of the statistical test we use will depend on the exact sample we take. It might happen that the sample we take leads us to make the wrong conclusion about the population. Thus we need to know what the chances are of this happening. Probability can be thought of as the study of chance. That is why we study probability!

Example 1

Consider the Baby-boom dataset we saw in Lecture 2. Suppose we have a hypothesis that in the population boys weigh more than girls at birth. We can use our sample of boys and girls to examine this hypothesis. One intuitive way of doing this would be to calculate the mean weights of the boys and girls in the sample and compare the difference between these two means

$$\text{Sample mean of boys weights} = \bar{x}_{\text{boys}} = 3375.308$$

$$\text{Sample mean of girls weights} = \bar{x}_{\text{girls}} = 3132.444$$

$$\Rightarrow D = \bar{x}_{\text{boys}} - \bar{x}_{\text{girls}} = 3375.30 - 3132.444 = 242.8632$$

Does this allow us to conclude that in the population boys are born heavier than girls? On what scale do we assess the size of D ? Maybe boys and girls weigh the same at birth and we obtained a sample with heavier boys just by chance. To be able to conclude confidently that boys in the population are heavier than

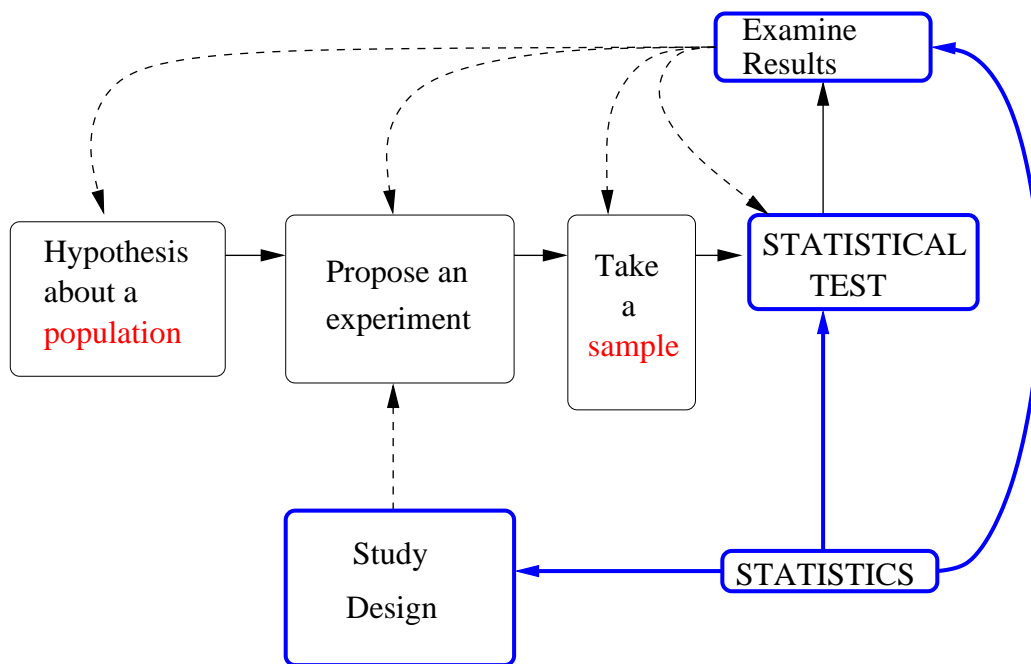


Figure 1: The scientific process and role of statistics in this process.

girls we need to know what the chances are of obtaining a difference between the means that is 242.8632 or greater, i.e. we need to know the probability of getting such a large value of D . If the chances are small then we can be confident that in the population boys are heavier than girls on average at birth. Before we can calculate this probability we need to understand the basics of probability

2 What is probability?

Example 2

To answer this question lets consider a simple **experiment** of tossing a fair die.

When we toss the die there are six possible outcomes i.e. 1, 2, 3, 4, 5 and 6. We say that the **sample space** of our experiment is the set $S = \{1, 2, 3, 4, 5, 6\}$.

The outcome "the top face shows a three" is the **sample point** 3.

The **event** A_1 , that the die shows an even number is the subset $A_1 = \{2, 4, 6\}$ of the sample space.

The **event** A_2 that the die shows a number larger than 4 is the subset $A_2 = \{5, 6\}$ of S_2 .

2.1 Definitions

The example above introduced some terminology that we will use repeatedly when we talk about probabilities.

An **experiment** is some activity with an observable outcome.

The set of all possible outcomes of the experiment is called the **sample space**.

A particular outcome is called a **sample point**.

A collection of possible outcomes is called an **event**.

2.2 Calculating simple probabilities

Simply speaking, the probability of an event is a number between 0 and 1, inclusive, that indicates how likely the event is to occur.

In some settings (like the example of the fair die considered above) it is natural to assume that all the sample points are equally likely.

In this case, we can calculate the probability of an event A as

$$P(A) = \frac{|A|}{|S|},$$

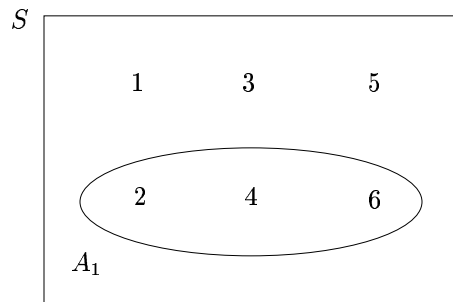
where $|A|$ denotes the number of sample points in the event A .

2.3 Example 2 (Continued)

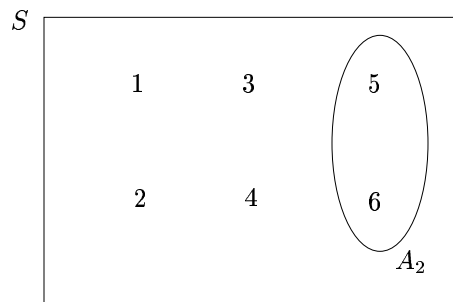
It is often useful in simple examples like this to draw a diagram (know specifically as a venn diagram) that represents the sample space and then label specific events in the diagram by grouping together individual sample points.

$$S = \{1, 2, 3, 4, 5, 6\} \quad A_1 = \{2, 4, 6\} \quad A_2 = \{5, 6\}$$

$$P(A_1) = \frac{|A_1|}{|S|} = \frac{3}{6} = \frac{1}{2}$$



$$P(A_2) = \frac{|A_2|}{|S|} = \frac{2}{6} = \frac{1}{3}$$



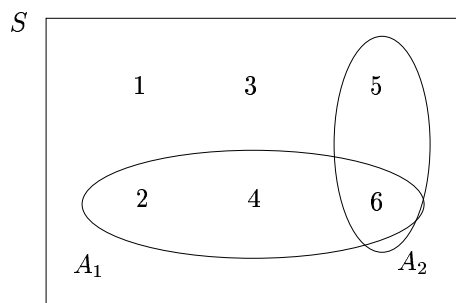
2.4 Intersection

What about $P(\text{face is even, and larger than 4})$?

We can write this event in set notation as $A_1 \cap A_2$.

This is the **intersection** of the two events, A_1 and A_2
i.e the set of elements which belong to both A_1 and A_2 .

$$A_1 \cap A_2 = \{6\} \quad \Rightarrow \quad P(A_1 \cap A_2) = \frac{|A_1 \cap A_2|}{|S|} = \frac{1}{6}$$



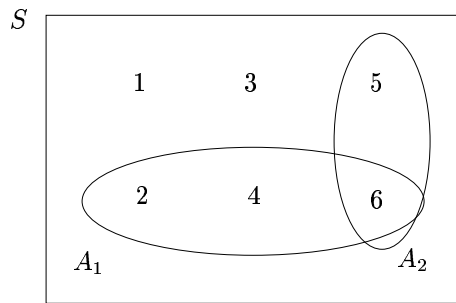
2.5 Union

What about $P(\text{face is even, or larger than 4})$?

We can write this event in set notation as $A_1 \cup A_2$.

This is the **union** of the two events, A_1 and A_2
i.e the set of elements which belong either A_1 and A_2 or both.

$$A_1 \cup A_2 = \{2, 4, 5, 6\} \Rightarrow P(A_1 \cup A_2) = \frac{|A_1 \cup A_2|}{|S|} = \frac{4}{6} = \frac{2}{3}$$



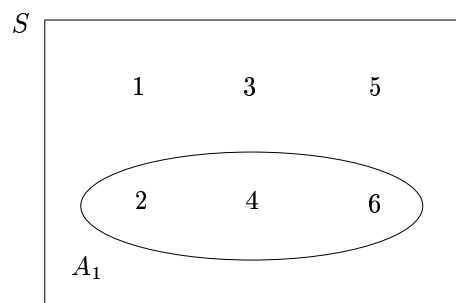
2.6 Complement

What about $P(\text{face is not even})$?

We can write this event in set notation as A_1^c .

This is the **complement** of the event, A_1
i.e the set of elements which do not belong to A_1 .

$$A_1^c = \{1, 3, 5\} \Rightarrow P(A_1^c) = \frac{|A_1^c|}{|S|} = \frac{3}{6} = \frac{1}{2}$$



3 Probability in more general settings

In many settings, either the sample space is infinite or all possible outcomes of the experiment are not equally likely. We still wish to associate probabilities with events of interest. Luckily, there are some rules/laws that allow us to calculate and manipulate such probabilities with ease.

3.1 Probability Axioms (Building Blocks)

Before we consider the probability rules we need to know about the axioms (or mathematical building blocks) upon which these rules are built. There are three axioms which we need in order to develop our laws

- I. $0 \leq P(A) \leq 1$ for any event A .

This axiom says that probabilities must lie between 0 and 1

- II. $P(S) = 1$.

This axiom says that the probability of everything in the sample space is 1. This says that the sample space is complete and that there are no sample points or events that allow outside the sample space that can occur in our experiment.

- III. If A_1, \dots, A_n are **mutually exclusive** events, then

$$P(A_1 \cup \dots \cup A_n) = P(A_1) + \dots + P(A_n).$$

A set of events are **mutually exclusive** if at most one of the events can occur in a given experiment.

This axiom says that to calculate the probability of the union of distinct events we can simply add their individual probabilities.

3.2 Complement Law

Suppose,

A = The event that a randomly selected student from the class has a bike

What is the probability that a student does not have a bike?

This is the **complement** of the event A, denoted A^c (pronounced 'A complement')

$$\boxed{P(A^c) = 1 - P(A)} \quad (\text{Law 1})$$

Example 3 If $P(A) = 0.36$ then $P(A^c) = 1 - 0.36 = 0.64$

3.3 Addition Law (Union)

Suppose,

A = The event that a randomly selected student from the class has brown eyes

B = The event that a randomly selected student from the class has blue eyes

What is the probability that a student has brown eyes **OR** blue eyes?

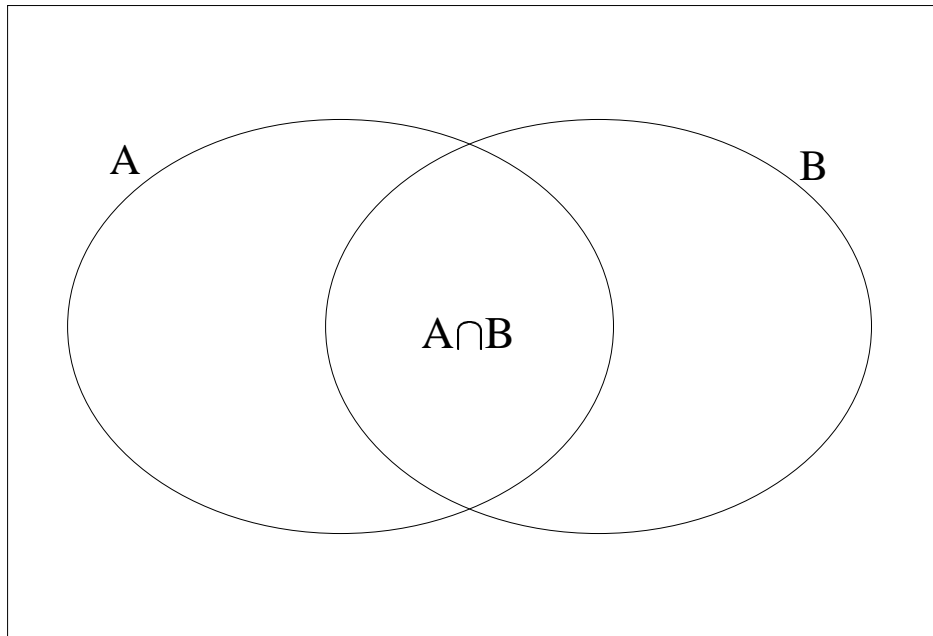
This is the **union** of the two events A and B, denoted $A \cup B$ (pronounced 'A or B')

We want to calculate $P(A \cup B)$.

In general for two events

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad (\text{Addition Law})$$

To understand this law consider a venn diagram of the situation (below) in which we have two events A and B . The event $A \cup B$ is represented in such a diagram by the combined sample points enclosed by A or B . If we simply add $P(A)$ and $P(B)$ we will count the sample points in the intersection $A \cap B$ twice and thus we need to subtract $P(A \cap B)$ from $P(A) + P(B)$ to calculate $P(A \cup B)$.



Example 4 : SNPs Single nucleotide polymorphisms (SNPs) are nucleotide positions in a genome which exhibit variation amongst individuals in a species. In some studies in humans, SNPs are discovered in European populations. Suppose that of such SNPs, 70% also show variation in an African population, 80% show variation in an Asian population and 60% exhibit variation in both the African and Asian population.

Suppose one such SNP is chosen at random, what is the probability that it is variable in either the African or the Asian population?

Write A for the event that the SNP is variable in Africa, and B for the event that it is variable in Asia. We are told

$$\begin{aligned}P(A) &= 0.7 \\P(B) &= 0.8 \\P(A \cap B) &= 0.6.\end{aligned}$$

We require $P(A \cup B)$. From the addition rule:

$$\begin{aligned}P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\&= 0.7 + 0.8 - 0.6 \\&= 0.9.\end{aligned}$$

4 Conditional Probability Laws

Suppose,

A = The event that a randomly selected student from the class has a bike

B = The event that a randomly selected student from the class has blue eyes

and $P(A) = 0.36$, $P(B) = 0.45$ and $P(A \cap B) = 0.13$

What is the probability that a student has a bike **GIVEN** that the student has blue eyes?

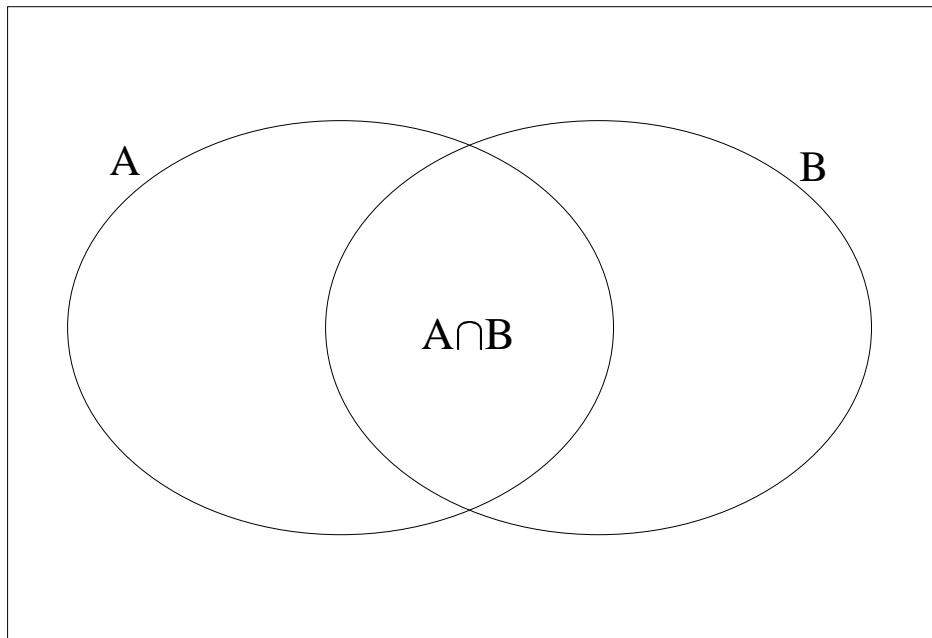
in other words

Considering just students who have blue eyes, what is the probability that a randomly selected student has a bike?

This is a **conditional** probability.

We write this probability as $P(B|A)$ (pronounced 'probability of B given A')

Think of $P(B|A)$ as 'how much of A is taken up by B'.



Then we see that

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \quad (\text{Conditional Probability Law})$$

Example 5 For the SNP example what is the probability that a SNP is variable in the African population given that it is variable in the Asian population?

We have that

$$\begin{aligned}P(A) &= 0.7 \\P(B) &= 0.8 \\P(A \cap B) &= 0.6.\end{aligned}$$

We want

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.6}{0.8} = 0.75$$

We can rearrange the conditional probability law to obtain a general Multiplication Law.

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \Rightarrow P(B|A)P(A) = P(A \cap B)$$

Similarly

$$P(A|B)P(B) = P(A \cap B)$$

$$\Rightarrow \boxed{P(A \cap B) = P(B|A)P(A) = P(A|B)P(B)} \quad (\text{Multiplication Law})$$

Example 6 If $P(B) = 0.2$ and $P(A|B) = 0.36$ what is $P(A \cap B)$?

$$P(A \cap B) = 0.36 \times 0.2 = 0.072$$

4.1 Independence of Events

Definition Two events A and B are said to be *independent* if $P(A \cap B) = P(A)P(B)$.

Note that in this case (provided $P(B) > 0$), if A and B are independent

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A),$$

and similarly $P(B|A) = P(B)$ (provided $P(A) > 0$).

So for independent events, knowledge that one of the events has occurred does not change our assessment of the probability that the other event has occur.

Example 7 : Snails In a population of a particular species of snail, individuals exhibit different forms. It is known that 45% have a pink background colouring, while 55% have a yellow background colouring. In addition, 30% of individuals are striped, and 20% of the population are pink and striped.

1. Is the presence or absence of striping independent of background colour?
2. Given that a snail is pink, what is the probability that it will have stripes.

Define the events: A , B , that a snail has a pink, respectively yellow, background colouring, and S for the event that it has stripes.

Then we are told $P(A) = 0.45$, $P(B) = 0.55$, $P(S) = 0.3$, and $P(A \cap S) = 0.2$.

For part (1), note that

$$0.2 = P(A \cap S) \neq 0.135 = P(A)P(S),$$

so the events A and S are not independent.

For part (2),

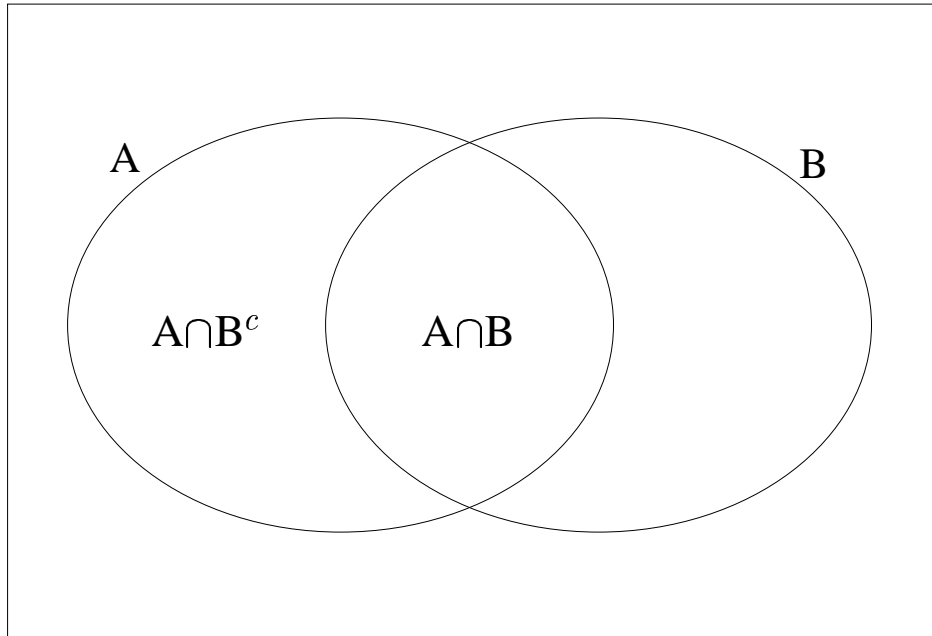
$$P(S|A) = \frac{P(S \cap A)}{P(A)} = \frac{0.2}{0.45} = 0.44.$$

Thus, knowledge that a snail has a pink background colouring increases the probability that it is striped. (That $P(S|A) \neq P(S)$ also establishes that background colouring and the presence of stripes, are not independent.)

4.2 The Partition law

The partition law is a very useful rule that allows us to calculate the probability of an event by spitting it up into a number of mutually exclusive events. For example, suppose we know that $P(A \cap B) = 0.52$ and $P(A \cap B^c) = 0.14$ what is $p(A)$?

$P(A)$ is made up of two parts (i) the part of A contained in B (ii) the part of A contained in B^c .



So we have the rule

$$P(A) = P(A \cap B) + P(A \cap B^c)$$

and $P(A) = P(A \cap B) + P(A \cap B^c) = 0.52 + 0.14 = 0.66$

More generally, if E_1, \dots, E_n are a set of *mutually exclusive* events then

$$P(A) = \sum_{i=1}^n P(A \cap E_i) = \sum_{i=1}^n P(A | E_i)P(E_i)$$

A set of events are *mutually exclusive* if at most one of the events can occur in a given experiment.

Example 8 : Mendelian segregation At a particular locus in humans, there are two alleles A and B , and it is known that the population frequencies of the genotypes AA , AB , and BB , are 0.49, 0.42, and 0.09, respectively. An AA man has a child with a woman whose genotype is unknown.

What is the probability that the child will have genotype AB ?

We assume that as far as her genotype at this locus is concerned the woman is chosen randomly from the population.

Use the partition rule, where the partition corresponds to the three possible genotypes for the woman. Then

$$\begin{aligned} P(\text{child } AB) &= P(\text{child } AB \text{ and mother } AA) \\ &\quad + P(\text{child } AB \text{ and mother } AB) \\ &\quad + P(\text{child } AB \text{ and mother } BB) \\ &= P(\text{mother } AA)P(\text{child } AB|\text{mother } AA) \\ &\quad + P(\text{mother } AB)P(\text{child } AB|\text{mother } AB) \\ &\quad + P(\text{mother } BB)P(\text{child } AB|\text{mother } BB) \\ &= 0.49 \times 0 + 0.42 \times 0.5 + 0.09 \times 1 \\ &= 0.3, \end{aligned}$$

5 Bayes Rule

Bayes Rule is one of the most powerful probability laws we have. In short Bayes Rule allows us to calculate the conditional probabilities we are often interested in for a specific problem.

Example 9

In a medical setting we might want to calculate the probability that a person has a disease D given they have a specific symptom S , i.e. we want to calculate $P(D|S)$. This is a hard probability to assign as we would need to take a random sample of people from the population with the symptom.

A probability that is much easier to calculate is $P(S|D)$, i.e. the probability that a person with the disease has the symptom. This probability can be assigned much more easily as medical records for people with serious diseases are kept.

The power of Bayes Rule is its ability to take $P(S|D)$ and calculate $P(D|S)$.

We have actually already seen a version of Bayes Rule before

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Using the Multiplication Law we can re-write this as

$$\boxed{P(B|A) = \frac{P(A|B)P(B)}{P(A)}} \quad (\text{Bayes Rule})$$

Example 9 (cont.)

Suppose $P(S|D) = 0.12$, $P(D) = 0.01$ and $P(S) = 0.03$ then

$$P(D|S) = \frac{0.12 \times 0.01}{0.03} = 0.04$$

Example 10 A gene has two possible types A_1 and A_2 . 75% of the population have A_1 . B is a disease that has 3 forms B_1 (mild), B_2 (severe) and B_3 (lethal). A_1 is a protective gene, with the probabilities of having the three forms given A_1 as 0.9, 0.1 and 0 respectively. People with A_2 are unprotected and have the three forms with probabilities 0, 0.5 and 0.5 respectively.

What is the probability that a person has gene A_1 given they have the severe disease?

The first thing to do with such a question is ‘decode’ the information, i.e. write it down in a compact form we can work with.

$$P(A_1) = 0.75 \quad P(A_2) = 0.25$$

$$P(B_1|A_1) = 0.9 \quad P(B_2|A_1) = 0.1 \quad P(B_3|A_1) = 0$$

$$P(B_1|A_2) = 0 \quad P(B_2|A_2) = 0.5 \quad P(B_3|A_2) = 0.5$$

We want $P(A_1|B_2)$?

From Bayes Rule we know that

$$P(A_1|B_2) = \frac{P(B_2|A_1)P(A_1)}{P(B_2)}$$

We know $P(B_2|A_1)$ and $P(A_1)$ but what is $P(B_2)$?

We can use the Partition Law since A_1 and A_2 are mutually exclusive.

$$\begin{aligned} P(B_2) &= P(B_2 \cap A_1) + P(B_2 \cap A_2) && \text{(Partition Law)} \\ &= P(B_2|A_1)P(A_1) + P(B_2|A_2)P(A_2) && \text{(Multiplication Law)} \\ &= 0.1 \times 0.75 + 0.5 \times 0.25 \\ &= 0.2 \end{aligned}$$

We can use Bayes Rule to calculate $P(A_1|B_2)$.

$$P(A_1|B_2) = \frac{0.1 \times 0.75}{0.2} = 0.375$$

6 Probability Laws

$$\boxed{P(A^c) = 1 - P(A)} \quad (\text{Complement Law})$$

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)} \quad (\text{Addition Law})$$

$$\boxed{P(B|A) = \frac{P(A \cap B)}{P(A)}} \quad (\text{Conditional Probability Law})$$

$$\boxed{P(A \cap B) = P(B|A)P(A) = P(A|B)P(B)} \quad (\text{Multiplication Law})$$

If E_1, \dots, E_n are a set of *mutually exclusive* events then

$$\boxed{P(A) = \sum_{i=1}^n P(A \cap E_i) = \sum_{i=1}^n P(A|E_i)P(E_i)} \quad (\text{Partition Law})$$

$$\boxed{P(B|A) = \frac{P(A|B)P(B)}{P(A)}} \quad (\text{Bayes Rule})$$

7 Permutations and Combinations (Probabilities of patterns)

In some situations we observe a specific pattern from a large number of possible patterns. To calculate the probability of the pattern we need to count the total number of patterns. This is why we need to learn about permutations and combinations.

7.1 Permutations of n objects

Consider 2 objects A B

Q. How many ways can they be arranged? i.e. how many **permutations** are there?

A. 2 ways AB BA

Consider 3 objects A B C

Q. How many ways can they be arranged (permuted)?

A. 6 ways ABC ACB BCA BAC CAB CBA

Consider 4 objects A B C D

Q. How many ways can they be arranged (permuted)?

A. 24 ways

ABCD ABDC ACBD ACDB ADBC ADCB
BACD BADC BCAD BCDA BDAC BDCA
CBAD CBDA CABD CADB CDBA CDAB
DBCA DBAC DCBA DCAB DABC DACB

There is a pattern emerging here.

No. of objects	2	3	4	5	6	...
No. of permutations	2	6	24	120	720	...

Can we find a formula for the number of permutations of n objects?

A good way to think about permutations is to think of putting objects into boxes.

Suppose we have 5 objects. How many different ways can we place them into 5 boxes?



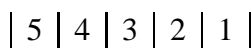
There are 5 choices of object for the first box.



There are now only 4 objects to choose from for the second box.



There are 3 choices for the 3rd box, 2 for the 4th and 1 for the 5th box.



Thus, the number of permutations of 5 objects is $5 \times 4 \times 3 \times 2 \times 1$.

In general, the number of permutations of n objects is

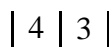
$$n(n-1)(n-2)\dots(3)(2)(1)$$

We write this as $n!$ (pronounced 'n factorial'). There should be a button on your calculator that calculates factorials.

7.2 Permutations of r objects from n

Now suppose we have 4 objects and only 2 boxes. How many permutations of 2 objects when we have 4 to choose from?

There are 4 choices for the first box and 3 choices for the second box



So there are 12 permutations of 2 objects from 4. We write this as

$${}^4P_2 = 12$$

We say there are ${}^n P_r$ permutations of r objects chosen from n .

The formula for ${}^n P_r$ is given by

${}^n P_r = \frac{n!}{(n-r)!}$

To see why this works consider the example above ${}^4 P_2$.

Using the formula we get

$${}^4 P_2 = \frac{4!}{2!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1} = 4 \times 3$$

7.3 Combinations of r objects from n

Now consider the number of ways of choosing 2 objects from 4 when the order doesn't matter. We just want to count the number of possible **combinations**.

We know that there are 12 permutations when choosing 2 objects from 4. These are

AB AC AD BC BD CD
BA CA DA CB DB DC

Notice how the permutations are grouped in 2's which are the same combination of letters. Thus there are $12/2 = 6$ possible combinations.

AB AC AD BC BD CD

We write this as

$${}^4C_2 = 6$$

We say there are nC_r combinations of r objects chosen from n .

The formula for nC_r is given by

$${}^nC_r = \frac{n!}{(n-r)!r!}$$

Another way of writing this formula that makes it clearer is

$${}^nC_r = \frac{{}^nP_r}{r!}$$

Effectively this says we count the number of permutations of r objects from n and then divide by $r!$ because the nP_r permutations will occur in groups of $r!$ that are the same combination.

8 Worked Examples

1. Four letters are chosen at random from the word RANDOMLY. Find the probability that all four letters chosen are consonants.

8 letters, 6 consonants, 2 vowels

$$P(\text{all four are consonants}) = \frac{\# \text{ of ways of choosing 4 consonants}}{\# \text{ of ways of choosing 4 letters}}$$

$$\# \text{ of ways of choosing 4 consonants} = {}^6C_4 = \frac{6!}{4!2!} = 15$$

$$\# \text{ of ways of choosing 4 letters} = {}^8C_4 = \frac{8!}{4!4!} = 70$$

$$\Rightarrow P(\text{all four are consonants}) = \frac{15}{70} = \frac{3}{14}$$

2. A bag contains 8 white counters and 3 black counters. Two counters are drawn, one after the other. Find the probability of drawing one white and one black counter, in any order

- (a) if the first counter is replaced
(b) if the first counter is not replaced

What is the probability that the second counter is black (assume that the first counter is replaced after it is taken)?

A useful way of tackling many probability problems is to draw a 'probability tree'. The branches of the tree represent different possible events. Each branch is labelled with the probability of choosing it given what has occurred before. The probability of a given route through the tree can then be calculated by multiplying all the probabilities along that route (using the Multiplication Rule)

(a) *With replacement*

Let

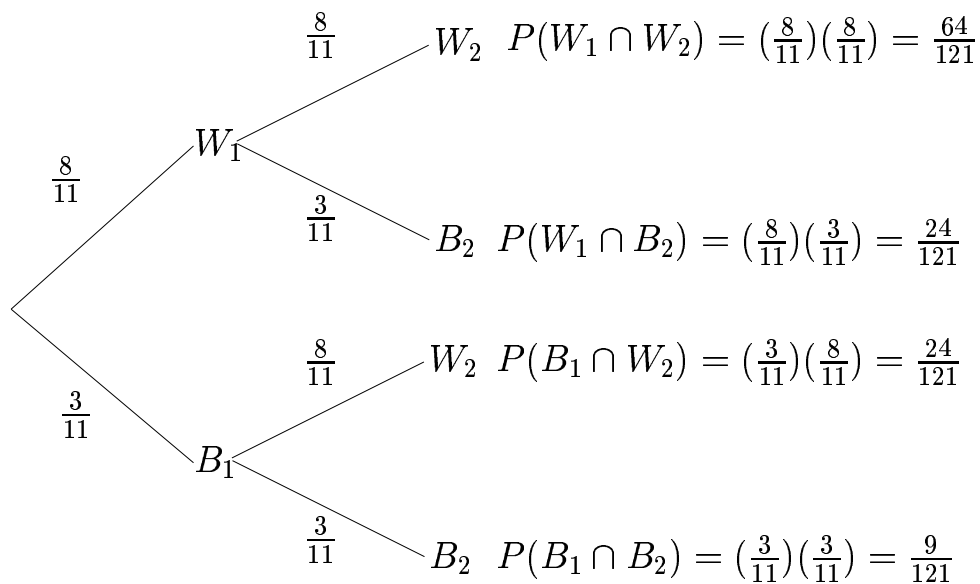
W_1 be the event 'a white counter is drawn first',

W_2 be the event 'a white counter is drawn second',

B_1 be the event 'a black counter is drawn first',

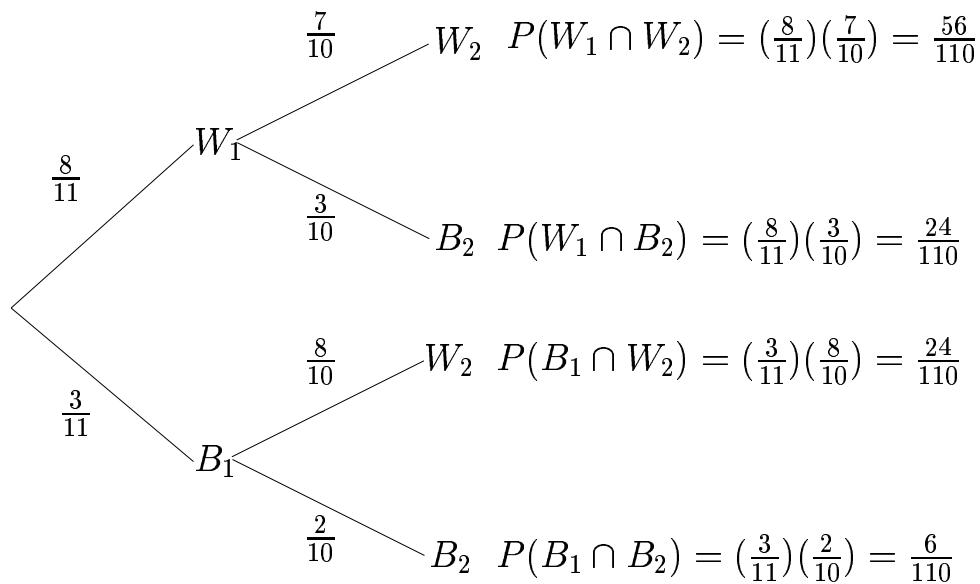
B_2 be the event 'a black counter is drawn second',

$$\begin{aligned} P(\text{one white and one black counter}) &= P(W_1 \cap B_2) + P(B_1 \cap W_2) \\ &= \frac{24}{121} + \frac{24}{121} \\ &= \frac{48}{121} \end{aligned}$$



(b) Without replacement

$$\begin{aligned}
 P(\text{one white and one black counter}) &= P(W_1 \cap B_2) + P(B_1 \cap W_2) \\
 &= \frac{24}{110} + \frac{24}{110} \\
 &= \frac{48}{110}
 \end{aligned}$$



$$\begin{aligned}
 P(\text{second counter is black}) &= P(W_1 \cap B_2) + P(B_1 \cap B_2) \\
 &= \frac{24}{110} + \frac{6}{110} \\
 &= \frac{30}{110}
 \end{aligned}$$

3. **From 2001 TT Prelim Q1** Two drugs that relieve pain are available to treat patients. Drug A has been found to be effective in three-quarters of

all patients who find that they are not in pain one hour after taking this drug. Drug B acts quicker but only works with one half of all patients who benefit from relief of pain after 30 mins. The physician cannot decide which patients should be prescribed which drug so he prescribes randomly. Assuming that there is no variation between patients in the times taken to act for either drug, calculate the probability that:

- (a) a patient is prescribed drug B and is relieved of pain;
- (b) a patient is relieved of pain after one hour;
- (c) a patient who was relieved of pain after one hour took drug A;
- (d) two patients receiving different drugs are both relieved of pain after one hour.
- (e) out of six patients treated with the same drug, three are relieved of pain after one hour and three are not.

Let

R_{30} = The event that a patient is relieved of pain after 30 mins

R_{60} = The event that a patient is relieved of pain after 60 mins

A = Event that a patient takes drug A

B = Event that a patient takes drug B

$$P(R_{60}|A) = 0.75 \quad P(R_{30}|B) = 0.5 \quad P(A) = P(B) = 0.5$$

(a) $P(R_{30}|B) = 0.5$

(b) $P(R_{60}) = P(R_{60}|A)P(A) + P(R_{60}|B)P(B)$ since $R_{30} \Rightarrow R_{60}$
 $P(R_{60}) = 0.75 \times 0.5 + 0.5 \times 0.5 = 0.625$

(c) $P(A|R_{60}) = \frac{P(A \cap R_{60})}{P(R_{60})}$

$$P(A \cap R_{60}) = P(R_{60}|A)P(A) = 0.75 \times 0.5 = 0.375$$

$$\Rightarrow P(A|R_{60}) = \frac{0.375}{0.625} = 0.6$$

(d) We want $P\left((R_{60}|A) \cap (R_{60}|B)\right)$

$$P(R_{60}|A) = 0.75 \quad P(R_{60}|B) = 0.5$$

$$P\left((R_{60}|A) \cap (R_{60}|B)\right) = P(R_{60}|A)P(R_{60}|B) = 0.75 \times 0.5 = 0.375$$

Assuming the events are independent.

(e) $n = 6$ X = no. of patients relieved of pain after 1hr

For A , $p = P(R_{60}|A) = 0.75$

$$P(X = 3|A) = {}^6C_3(0.75)^3(0.25)^3 = 0.1312$$

For B , $p = P(R60|B) = 0.5$

$$P(X = 3|B) = {}^6C_3(0.5)^3(0.5)^3 = 0.3125$$

$$\begin{aligned}\Rightarrow P(X = 3) &= P(X = 3|A)P(A) + (X = 3|B)P(B) \\ &= 0.1312 \times 0.5 + 0.3125 \times 0.5 = 0.2222\end{aligned}$$

4. In the National Lottery you need to choose 6 balls from 49.

What is the probability that I choose all 6 balls correctly?

There are 2 ways of answering this question

(i) using permutations and combinations

(ii) using a tree diagram

Method 1 - using permutations and combinations

$$\begin{aligned}P(6 \text{ correct}) &= \frac{\text{No. of ways of choosing the 6 correct balls}}{\text{No. of ways of choosing 6 balls}} \\ &= \frac{{}^6P_6}{{}^{49}P_6} \\ &= \frac{6!}{\frac{49!}{43!}} \\ &= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{49 \times 48 \times 47 \times 46 \times 45 \times 44} \\ &= 0.0000000715112 \quad (1 \text{ in } 14 \text{ million})\end{aligned}$$

Method 2 - using a tree diagram

Consider the first ball I choose, the probability it is correct is

$$\frac{6}{49}$$

The second ball I choose is correct with probability

$$\frac{5}{48}$$

The third ball I choose is correct with probability

$$\frac{4}{47}$$

and so on.

Thus the probability that I get all 6 balls correct is

$$\frac{6}{49} \frac{5}{48} \frac{4}{47} \frac{3}{46} \frac{2}{45} \frac{1}{44} = 0.0000000715112 \quad (1 \text{ in } 14 \text{ million})$$