# SELF-AVOIDING WALK IS SUB-BALLISTIC

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ABSTRACT. We prove that self-avoiding walk on $\mathbb{Z}^d$ is sub-ballistic in any dimension $d \geq 2$. That is, writing $||u||$ for the Euclidean norm of $u \in \mathbb{Z}^d$, and $\mathsf{P}_{\mathsf{SAW}_n}$ for the uniform measure on self-avoiding walks $\gamma : \{0, \dots, n\} \to \mathbb{Z}^d$ for which $\gamma_0 = 0$, we show that, for each $v > 0$, there exists $\varepsilon > 0$ such that, for each $n \in \mathbb{N}$, $\mathsf{P}_{\mathsf{SAW}_n}\left( \max \left\{ ||\gamma_k|| : 0 \leq k \leq n \right\} \geq vn \right) \leq e^{-\varepsilon n}$.

## 1. INTRODUCTION

Flory and Orr [12, 25] introduced self-avoiding walk as a simple model of a polymer. It quickly became clear that the model exhibits rich behaviour the rigorous understanding of which poses a real challenge to mathematicians. Originally, Flory was interested in the typical behaviour of self-avoiding walk, and in particular in the mean squared displacement of its endpoint. Since then, much effort has been invested in the study of typical geometric properties of the model.

1.1. **The model and the results.** Let $d \geq 2$. For $u \in \mathbb{R}^d$, let $||u||$ denote the Euclidean norm of $u$. Let $E(\mathbb{Z}^d)$ denote the set of nearest-neighbour bonds of the integer lattice $\mathbb{Z}^d$. A *walk* of length $n \in \mathbb{N}$ is a map $\gamma : \{0, \dots, n\} \to \mathbb{Z}^d$ such that $(\gamma_i, \gamma_{i+1}) \in E(\mathbb{Z}^d)$ for each $i \in \{0, \dots, n-1\}$. An injective walk is called *self-avoiding*. Let $\mathrm{SAW}_n$ denote the set of self-avoiding walks of length $n$ that start at the origin and let $\mathsf{P}_{\mathsf{SAW}_n}$ denote the uniform law on $\mathrm{SAW}_n$. This article is devoted to proving that self-avoiding walk is sub-ballistic:

**Theorem 1.1.** *Let $v > 0$. There exists $\varepsilon > 0$ such that, for each $n \in \mathbb{N}$,*

$$\mathsf{P}_{\mathsf{SAW}_n}\left( \max \left\{ ||\gamma_k|| : 0 \leq k \leq n \right\} \geq vn \right) \leq e^{-\varepsilon n} \, .$$

The theorem has the following immediate consequence.

**Corollary 1.2.** *Set $\langle ||\gamma_n||^2 \rangle = \frac{1}{|\mathrm{SAW}_n|} \sum_{\gamma \in \mathrm{SAW}_n} ||\gamma_n||^2$. Then*

$$\lim_{n \to \infty} n^{-2} \langle ||\gamma_n||^2 \rangle = 0 \, .$$

1.2. **Conjectures on the mean-square displacement.** Numerical computations and non-rigorous theory predict the precise behaviour of the mean-squared displacement of the walk's endpoint. It is conjectured (see [21, Section 1.4] for details) that

$$\langle ||\gamma_n||^2 \rangle = n^{2\nu + o(1)} \text{ where } \nu = \begin{cases} 1 & d = 1 \\ 3/4 & d = 2 \\ \approx 0.59 \cdots & d = 3 \\ 1/2 & d = 4 \\ 1/2 & d \geq 5 \,. \end{cases}$$

The behaviour predicted by this conjecture should be compared to that of simple random walk. In general, the displacement of the endpoint of self-avoiding walk is expected to exceed that of simple random walk. In dimensions two and three, this difference is manifested in a strong form, with the value of $\nu$ in the self-avoiding case exceeding its counterpart for the simple one (which is $1/2$). Dimension four is known as the *upper critical dimension*: here, the two values of $\nu$ coincide, with self-avoiding walk experiencing further displacement in the form of a poly-logarithmic correction. In dimensions five and higher, the two values coincide and no logarithmic corrections occur; indeed, the scaling limit of each process is Brownian motion. However, the diffusion rates of the Brownian motions in the simple random walk and self-avoiding walk scaling limits differ, with the rate being higher in the self-avoiding case.

In dimensions five and above, the conjecture was proved for a version of self-avoiding walk with weak repulsion in Brydges and Spencer [7]. Hara and Slade [14, 15] proved the conjecture for self-avoiding walk in these dimensions, and also established that this process has Brownian motion as a scaling limit.

Rigorous analysis in dimension four is much more subtle. Recently some impressive results have been achieved using a supersymmetric renormalization group approach. These results concern continuous-time weakly self-avoiding walk: see [5, 6, 2] and references within.

As far as we know, no significant physical predictions have been made regarding the scaling limit of self-avoiding walk in three dimensions. In dimension two, the Coulomb gas formalism [23, 24] provides the prediction that $\nu = 3/4$, with this prediction later being made [10, 11] using conformal field theory. The scaling limit of the model in this dimension has been identified [19] subject to certain assumptions principal among which is conformal invariance: given that the scaling limit may be expected to be supported on simple curves and to enjoy a natural restriction property, the assumption of a certain conformal covariance forces the limit to be given by Schramm-Loewner Evolution with parameter $\kappa = 8/3$.
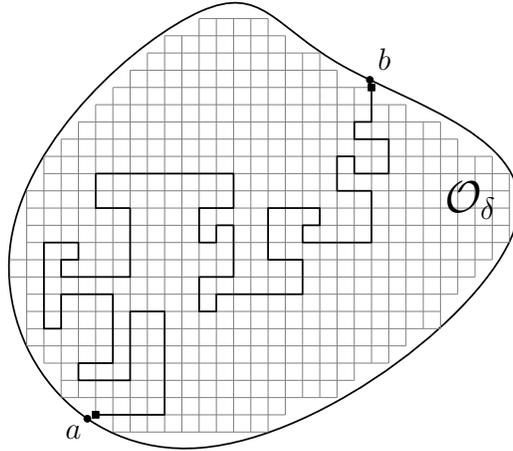
FIGURE 1. The path $\gamma_\delta$ in $(\mathcal{O}_\delta, a_\delta, b_\delta)$.

There are however very few rigorous unconditional results regarding dimensions two and three. The question of a non-trivial upper bound on mean-squared endpoint displacement is raised in the introduction of [21]. That the intuitively very natural assertion that self-avoiding walk in dimensions two and three is sub-ballistic has remained unresolved for a long time is one of several archetypical examples which bear witness to the theoretical difficulty that this model presents in the low-dimensional case.

1.3. **Self-avoiding walk between two points of a domain.** The self-avoiding walk model exhibits a phase transition when defined slightly differently. Let $\mathcal{O}$ be a simply connected smooth domain in $\mathbb{R}^d$ with two points $a, b$ on the boundary. For $\delta > 0$, let $\mathcal{O}_\delta$ be the largest connected component of $\mathcal{O} \cap \delta \mathbb{Z}^d$ and let $a_\delta$, $b_\delta$ be the two sites of $\mathcal{O}_\delta$ closest to $a$ and $b$ respectively. We think of $(\mathcal{O}_\delta, a_\delta, b_\delta)$ as being an approximation of $(\mathcal{O}, a, b)$; see Figure 1. Let $z > 0$. On $(\mathcal{O}_\delta, a_\delta, b_\delta)$, define a probability measure on the finite set of self-avoiding walks in $\mathcal{O}_\delta$ from $a_\delta$ to $b_\delta$ by the formula

$$(1.1) \qquad \mathbb{P}_{(\mathcal{O}_\delta, a_\delta, b_\delta, z)}(\gamma_\delta) \;=\; \frac{z^{|\gamma_\delta|}}{Z_{(\mathcal{O}_\delta, a_\delta, b_\delta)}(z)}\,,$$

where $|\gamma_\delta|$ is the length of $\gamma_\delta$, and $Z_{(\mathcal{O}_\delta, a_\delta, b_\delta)}(z)$ is a normalizing factor.

A phase transition occurs at the value $\mu_c^{-1}$, where $\mu_c$ is the connective constant (whose definition we will shortly provide). For $z < \mu_c^{-1}$, there exists $C = C(z) > 0$ such that $\mathbb{P}_{(\mathcal{O}_\delta, a_\delta, b_\delta, z)}(|\gamma_\delta| \leq C/\delta) \to 1$ as $\delta \searrow 0$. In fact, $\gamma_\delta$ has Gaussian fluctuation of order $\sqrt{\delta}$ around the geodesic between $a$ and $b$ in $\mathcal{O}$ (assuming the geodesic is unique — otherwise some changes are needed). Broadly, these results may be expected to be a consequence

of the theory developed by Ioffe [16] that treats unrestricted self-avoiding walk in $\mathbb{Z}^d$; however, they have not been rigorously derived to the best of our knowledge. For $z > \mu_c^{-1}$, the converse is true [8] in the sense that $\gamma_\delta$ becomes space-filling.

Viewed in this light, Theorem 1.1 rules out the possibility that the model's critical behaviour coincides with that of the subcritical phase $z < \mu_c^{-1}$:

**Corollary 1.3.** *Let $(\mathcal{O}, a, b)$ be such that $\partial\mathcal{O}$ is smooth in a neighbourhood of $a$ and of $b$. For every $K > 0$, $\mathbb{P}_{(\mathcal{O}_\delta, a_\delta, b_\delta, \mu_c^{-1})}\big(|\gamma_\delta| \leq K/\delta\big) \to 0$ as $\delta \searrow 0$.*

1.4. **More general graphs.** The proof of Theorem 1.1 can be extended (with additional technicalities) to lattices with symmetry. For instance, the case of the hexagonal lattice should follow from the same reasoning. While the question can be asked on any locally-finite infinite transitive graph, the answer will differ drastically depending on the growth of the graph. For instance, it is strongly expected that self-avoiding walk on the Cayley graph of a non-amenable group is ballistic (see the upcoming Problem 5); see [22] for a study of some hyperbolic planar graphs.

In [4] is considered a model where a walk $\gamma \in \mathrm{SAW}_n$ is chosen with weight proportional to $\prod_{i=1}^d z_{i,+}^{N_{i,+}} z_{i,-}^{N_{i,-}}$ where $N_{i,+/-}$ is the number of increasing or decreasing steps made by $\gamma$ in direction $e_i$ and $z_{i,+/-} \in (0, \infty)$ are parameters. In the asymmetric case that $z_{i,+} \neq z_{i,-}$ for some $i \in [1, d]$ (and no $z_{j,+/-} = 0$), [4, Theorem 1.2] is a result to the effect that the walk is typically ballistic.

1.5. **Open problems.** Theorem 1.1 is a first step towards proving the conjecture on mean-squared displacement. An interesting improvement would be a quantitative bound:

**Problem 1.** Show that for some $\varepsilon > 0$, $\lim_{n \to \infty} n^{-(2-\varepsilon)}\langle ||\gamma_n||^2 \rangle = 0$.

Theorem 1.1 raises the prospect of confirming that the walk from the lower-left to upper-right corner of a square sways to macroscopic distance from the diagonal, but it does not resolve this question:

**Problem 2.** Fix $(\mathcal{O}, a, b)$. Taking $z = \mu_c^{-1}$, show that $\gamma_\delta$ does not converge to a geodesic.

The theorem rules out the extreme of fast movement by the walk. What about the other extreme? Does self-avoiding walk move further than simple random walk?

**Problem 3.** Show that $\liminf_{n \to \infty} n^{-1}\langle ||\gamma_n||^2 \rangle > 0$.

In fact, far slower motion by self-avoiding walk has yet to be ruled out. In principle, the walk may be space-filling:

**Problem 4.** Show that $\lim_{n \to \infty} n^{-2/d}\langle ||\gamma_n||^2 \rangle = \infty$.

As we have mentioned, the question of ballisticity may be posed for self-avoiding walk on many transitive graphs, such as Cayley graphs.

**Problem 5.** Is self-avoiding walk ballistic whenever simple random walk is? In particular, is the model ballistic if $G$ is non-amenable?

## 2. Preliminaries

2.1. **Notation.** The symbol $\mathbb{N}$ denotes the set of integers $\{0, 1, 2, 3, \ldots\}$. We set $[a, b] = \{a, a+1, \ldots, b\}$.

For $u \in \mathbb{R}^d$, we write $u = (u_1, \ldots, u_d)$. We also write $x(u) = u_1$ and $y(u) = u_2$. For $k \in [1, d]$, let $e_k$ be the vector of Euclidean norm one whose $k^{\text{th}}$ entry is 1. As we tend to visualise our constructions for the two-dimensional case, we will sometimes refer to the directions $e_1$ and $e_2$ as east and north.

The cardinality of a finite set $A$ is denoted by $|A|$. Abusing slightly this notation, the length of a walk $\gamma$ will be denoted by $|\gamma|$; recall that $|\gamma|$ is the number of edges (rather than the number of vertices) comprising $\gamma$.

For $m, n \in \mathbb{N}$, let $\gamma$ and $\tilde{\gamma}$ be two walks of lengths $m$ and $n$, neither of which need to start at 0. The concatenation $\gamma \circ \tilde{\gamma}$ of $\gamma$ and $\tilde{\gamma}$ is given by

$$
\left(\gamma \circ \tilde{\gamma}\right)_k = \begin{cases} \gamma_k & k \leq m\,, \\ \gamma_m + \left(\tilde{\gamma}_{k-m} - \tilde{\gamma}_0\right) & m+1 \leq k \leq m+n\,. \end{cases}
$$

Let $A$ and $B$ be two sets. Let $\mathfrak{P}(B)$ denote the power-set of a set $B$. A multi-valued map from $A$ to $B$ is a function $f : A \to \mathfrak{P}(B)$. We will use such maps in order to estimate the size of sets. On several occasions, we will use the following basic principle, which we name for future reference:

**Multi-valued map principle.** *Let $f$ be a multi-valued map from $A$ to $B$. Write $f^{-1}(b) := \{a \in A : b \in f(a)\}$. Then*

$$
|A| \leq \frac{\max_{b \in B} |f^{-1}(b)|}{\min_{a \in A} |f(a)|} \cdot |B|\,.
$$

2.2. **Bridges.** Recall the classical definition of a bridge. A self-avoiding walk is called a *self-avoiding bridge* if

- its first element attains uniquely the minimal $y$-coordinate on the walk;
- and its final element attains, not necessarily uniquely, the maximal $y$-coordinate on the walk.

The self-avoiding walk being the object of attention, we will usually omit the term "self-avoiding" in referring to walks and bridges. For $n \in \mathbb{N}$, let $\mathrm{SAB}_n$ and $\mathrm{SAW}_n$ denote the set of bridges and walks starting at the origin and of length $n$. Let SAW and SAB be the corresponding sets where now no condition on the length is applied. Let $\mathsf{P}_{\mathsf{SAB}_n}$ and $\mathsf{P}_{\mathsf{SAW}_n}$ denote the uniform law on $\mathrm{SAB}_n$ and $\mathrm{SAW}_n$.
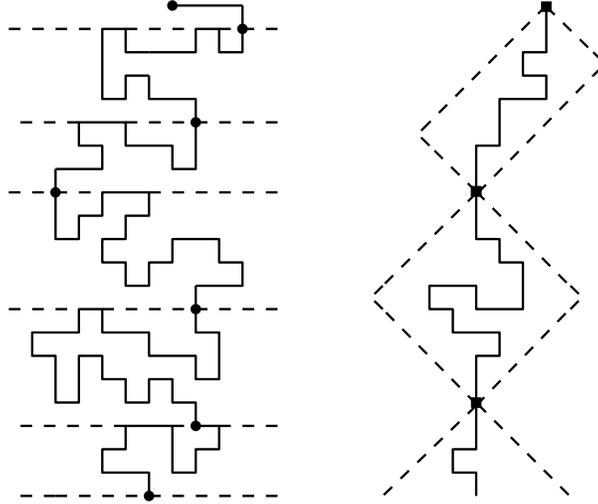


FIGURE 2. On the left, a bridge with its renewal points. On the right, a bridge with its diamond points (which we will shortly discuss).

We begin by recording a straightforward statement concerning the number of bridges of length $n$. Recall from [13] that a standard subadditive argument assures the existence of the connective constant $\mu_c$, given by $\mu_c = \lim_n \left| \mathrm{SAW}_n \right|^{1/n} \in (0, \infty)$. (See [9] for a rigorous identification of $\mu_c$'s value for the hexagonal lattice.)

**Proposition 2.1.** *The generating functions of walks and bridges satisfy*

$$(2.1) \qquad \sum_{\gamma \in \mathrm{SAW}} z^{|\gamma|} \leq z^{-1} \exp\left( 2 \sum_{\gamma \in \mathrm{SAB}} z^{|\gamma|} - 2 \right).$$

*Furthermore, there exists $C > 0$ such that, for each $n \in \mathbb{N}$,*

$$(2.2) \qquad e^{-C\sqrt{n}}|\mathrm{SAW}_n| \leq |\mathrm{SAB}_n| \leq |\mathrm{SAW}_n| \, .$$

In a classic argument of Hammersley and Welsh [13], (2.2) is demonstrated by the construction of a map from $\mathrm{SAW}_n$ and $\mathrm{SAB}_n$ enjoying the property that each preimage has cardinality at most $e^{C\sqrt{n}}$; formally, then, the multi-valued map principle (applied in this case to single-valued maps) yields (2.2). In fact, the same techniques imply (2.1): see [21, (3.1.13)].

The notion of cutting an object into irreducible pieces has been much used, largely because it is crucial for developing a renewal theory. This theory was developed for bridges in [17, 18]. The set $\mathsf{R}_\gamma$ of *renewal points* of $\gamma \in \mathrm{SAB}_n$ is the set of points $\gamma_i$, with $i \in [0, n]$, for which $\gamma[0, i]$ and $\gamma[i, n]$ are bridges. See Figure 2. A bridge $\gamma \in \mathrm{SAB}_n$ with $n \geq 1$ is said to be *irreducible* if $\gamma_k$ is not a renewal point for any $k \in [1, n-1]$.

Let iSAB be the set of irreducible bridges of arbitrary length whose initial point is 0. Every bridge is the concatenation of a finite number of irreducible elements; the decomposition is unique and the set $\mathsf{R}_\gamma$ is the union of the initial and terminal points of the bridges that comprise this decomposition. This leads to the following result due to Kesten.

**Lemma 2.2** (Kesten). *We have that* $\displaystyle\sum_{\gamma \in \mathrm{iSAB}} \mu_c^{-|\gamma|} = 1.$

**Proof.** Since the argument is short, we provide it here. The decomposition of walks into irreducible bridges yields

$$\sum_{\gamma \in \mathrm{SAB}} z^{|\gamma|} = \frac{1}{1 - \sum_{\gamma \in \mathrm{iSAB}} z^{|\gamma|}}$$

for every $z$ such that the two series converge. Proposition 2.1 shows that the radius of convergence of the series on the left-hand side is $\mu_c^{-1}$. The generating function does not blow up for $z < \mu_c^{-1}$, and thus $\sum_{\gamma \in \mathrm{iSAB}} z^{|\gamma|} < 1$ for every $z < \mu_c^{-1}$, whence $\sum_{\gamma \in \mathrm{iSAB}} \mu_c^{-|\gamma|} \leq 1$ follows by means of the monotone convergence theorem.

Furthermore, this sum will equal one if $\sum_{\gamma \in \mathrm{SAB}} z^{|\gamma|}$ diverges as $z$ tends to $\mu_c^{-1}$. But (2.1) shows that $\sum_{\gamma \in \mathrm{SAB}} z^{|\gamma|}$ diverges as $z \nearrow \mu_c^{-1}$ if $\sum_{\gamma \in \mathrm{SAW}} z^{|\gamma|}$ diverges in this limit. Divergence of the latter sum follows from $|\mathrm{SAW}_n| \geq \mu_c^n$, which is a consequence of the submultiplicative bound $|\mathrm{SAW}_{n+m}| \leq |\mathrm{SAW}_n||\mathrm{SAW}_m|$. $\qquad \square$

By means of Lemma 2.2, we define the probability measure $\mathbb{P}_{\mathrm{iSAB}}$ on iSAB by setting $\mathbb{P}_{\mathrm{iSAB}}(\gamma) = \mu_c^{-|\gamma|}$.

2.3. **Statements and sketches of the proofs of the main elements.**
We will prove Theorem 1.1 by contradiction.

We first work with bridges. We record the assumption whose negation we
seek to prove by contradiction; when a result requires the assumption, we
will say so in its statement.

**Ballistic assumption.** *Suppose that for some $v > 0$,*

$$\limsup_{n \to \infty} n^{-1} \log \mathsf{P}_{\mathsf{SAB}n}\Big(y(\gamma_n) \geq vn\Big) = 0 \,.$$

We first prove that, under the ballistic assumption, the probability that a
bridge possesses a positive density of renewal points decays subexponentially.
More precisely:

**Theorem 2.3.** *Suppose that the ballistic assumption holds. Then there exists
$\delta > 0$ such that*

$$\limsup_{n \to \infty} n^{-1} \log \mathsf{P}_{\mathsf{SAB}n}\big(|\mathsf{R}_\gamma| \geq \delta n\big) = 0 \,.$$

While its statement is very intuitive, the proof of this result contains some
of the central ideas of this paper. The fact that self-avoiding walk is not
Markovian makes the study delicate. The argument begins by noting that,
if a bridge travels ballistically in the northerly direction, many hyperplanes
of the form $\{u \in \mathbb{Z}^d : y(u) = h\}$ are visited on at most a few occasions. We
then use unfoldings in order to prove that, by paying a subexponential cost,
many of these sets are in fact visited only once.

We will see that the macroscopic presence of such renewal points at subex-
ponential cost implies the stronger statement that bridges typically have a
positive density of renewal points, or, in other words, that the average size
of an irreducible bridge is finite.

**Corollary 2.4.** *If the ballistic assumption holds, then $\mathbb{E}_{\mathrm{iSAB}}(|\gamma|) < \infty$.*

An independent argument rules out this possibility.

**Theorem 2.5.** *We have that $\mathbb{E}_{\mathrm{iSAB}}(|\gamma|) = \infty$.*

We now explain roughly how we will prove Theorem 2.5. Seeking a con-
tradiction, we assume that $\mathbb{E}_{\mathrm{iSAB}}(|\gamma|) < \infty$.

Let $\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}$ denote the law on semi-infinite walks $\gamma : \mathbb{N} \to \mathbb{Z}^d$ formed by the
concatenation of infinitely many independent samples of $\mathbb{P}_{\mathrm{iSAB}}$. This measure
may be viewed heuristically as the law of a semi-infinite self-avoiding walk
conditioned to remain in the upper-half plane. See [21, Section 8.3] for further
discussion about $\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}$.

Our assumption that the irreducible renewal block has finite mean length
means that we may invoke the law of large numbers to conclude that, under

the law $\mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{N}}$, the walk $\gamma$ proceeds northwards at a constant rate $\nu \in (0,1)$, so that $y(\gamma_n) = \nu n\big(1 + o_{n\to\infty}(1)\big)$ as $n \to \infty$. However, for a typical sample $\gamma$ of $\mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{N}}$, the interval $[0, \nu n]$ is populated to positive density by the $y$-coordinates of renewal points; there are thus an order of $n^2$ pairs of renewal points such that each element of the pair has $y$-coordinate in this interval and where the distance between the two points is at least a small constant multiple of $n$. Writing $(r, r')$ for the indices of such a pair of renewal points, we make a deformation of $\gamma$, rotating $\gamma_{[r,\infty)}$ clockwise about $\gamma_r$ by a right-angle, advancing along the new curve to the image of $\gamma_{r'}$, and then rotating the subsequent trajectory counterclockwise by a right-angle about the encountered point, so that the altered curve again points in the northerly direction. The new curve thus proceeds northwards, then turns eastward for at least small positive multiple of $n$ steps, before returning northward. To a typical sample of $\mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{N}}$ we thus associate an order of $n^2$ alternative walks. To each alternative walk, there are at most order $n^2$ candidates for the original walk to which surgery was applied: indeed, to perform this reconstruction, it is enough to find both of the renewal point indices $r$ and $r'$, and a factor of at most $n$ is needed to find each of $r$ and $r'$.

Thus, by an argument in the spirit of the multi-valued map principle, were all such alternative walks self-avoiding, they would have a $\mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{N}}$-probability which is bounded away from zero uniformly in $n$. However, alternative walks reach $y$-coordinate $\nu n$ after a number of steps which exceeds a multiple of $n$ where the factor is a constant which strictly exceeds one; this is in contradiction to $y(\gamma_n) = \nu n(1 + o_{n\to\infty}(1))$ ensured by the assumption that $\mathbb{E}_{\mathrm{iSAB}}(|\gamma|) < \infty$. This is an outline of an argument to arrive at a contradiction which would prove Theorem 2.5.

The difficulty with the plan is that our notion of renewal point is not sufficient to ensure that right-angle rotation about a renewal point of the path subsequent to that point results in a self-avoiding walk. Thus we introduce a stronger notion of renewal point for which this property is (in essence) valid. The point $\gamma_i$ is a *diamond point* of $\gamma$ if

- for any $j > i$, $(x+y)(\gamma_j) > (x+y)(\gamma_i)$ and $(y-x)(\gamma_j) > (y-x)(\gamma_i)$,
- for any $j > i$, $(x+y)(\gamma_j) < (x+y)(\gamma_i)$ and $(y-x)(\gamma_j) < (y-x)(\gamma_i)$.

Note that a diamond point is indeed a renewal point.

To implement our plan using diamond points in place of renewal points, we will first prove that, if $\mathbb{E}_{\mathrm{iSAB}}(|\gamma|) < \infty$, then a positive proportion of renewal points are in fact diamond points, so that the latter points also populate the $y$-coordinate to positive density. Our operation of turning the curve first clockwise and then counterclockwise again at appropriate diamond points will be called the stickbreak operation and is illustrated in Figure 3. Stickbreak
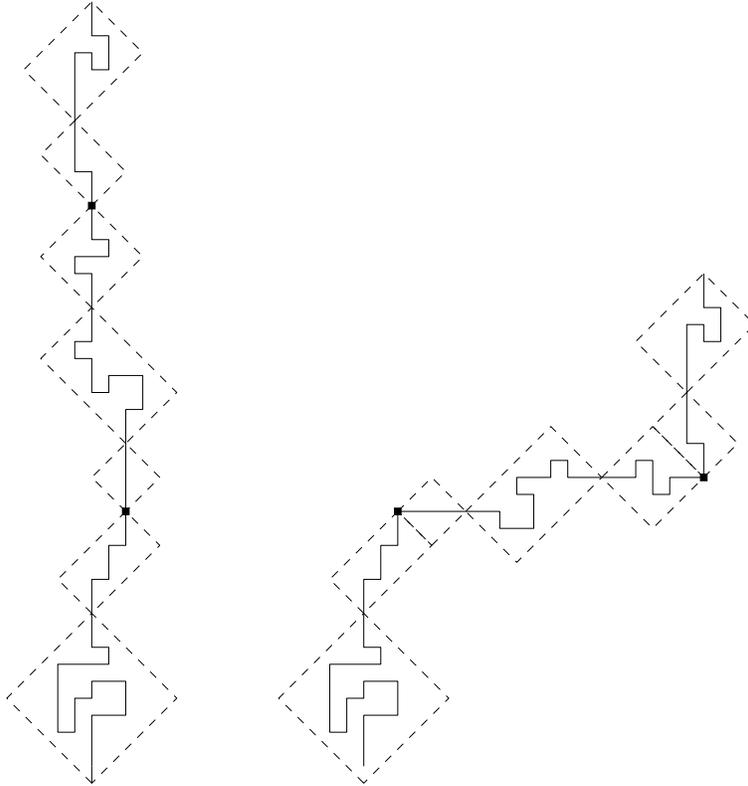
FIGURE 3. The stickbreak operation is depicted.

now produces self-avoiding outputs and this allows us to derive the sought contradiction.

**Remark 2.6.** *A proof based on the stickbreak operation introduced in this article has been employed in* [3] *to prove that the critical fugacity for surface adsorption of the hexagonal lattice equals* $1 + \sqrt{2}$.

2.4. **Deducing Theorem 1.1.** Here, we apply Corollary 2.4 and Theorem 2.5 to obtain Theorem 1.1.

**Proof of Theorem 1.1.** By the symmetry of the law $\mathsf{P}_{\mathsf{SAW}_n}$ under permutation of the Euclidean coordinate indices and under reflection in the axial hyperplanes, note that

$$\mathsf{P}_{\mathsf{SAW}_n}\Big( \max \big\{ ||\gamma_k|| : 0 \leq k \leq n \big\} \geq vn \Big)$$
$$\leq \quad 2d\mathsf{P}_{\mathsf{SAW}_n}\Big( \max \big\{ y(\gamma_k) : 0 \leq k \leq n \big\} \geq d^{-1/2}vn \Big).$$

The classical Hammersley-Welsh unfolding operation from walks to bridges, discussed in [21, Section 3.1] and already mentioned in the proof of Proposition 2.1, may be applied to walks $\gamma \in \mathrm{SAW}_n$ satisfying $\max \left\{ y(\gamma_k) : 0 \leq k \leq n \right\} \geq d^{-1/2} vn$. Noting that $\max \left\{ y(\gamma_k) : 0 \leq k \leq n \right\}$ increases at each step of the unfolding operation, and that the outcome of the unfolding, being an element of $\mathrm{SAB}_n$, has the property that this maximum is attained by $k = n$, we learn that

$$\mathsf{P}_{\mathsf{SAB}n} \Big( y(\gamma_n) \geq d^{-1/2} vn \Big) \geq \tfrac{1}{2d} e^{-C\sqrt{n}} \mathsf{P}_{\mathsf{SAW}n} \Big( \max \left\{ ||\gamma_k|| : 0 \leq k \leq n \right\} \geq vn \Big),$$

where the constant $C$ is from (2.2). Therefore, if the conclusion of Theorem 1.1 is violated, the ballistic assumption must hold. Corollary 2.4 then contradicts Theorem 2.5. $\square$

2.5. **Structure of the paper.** Section 3 contains the proofs of Theorem 2.3 and Corollary 2.4. In Section 4, Theorem 2.5 is proved. The final Section 5 is devoted to the proof of the corollaries.

## 3. POSITIVE DENSITY OF RENEWAL POINTS AT SUBEXPONENTIAL COST

In this section, we prove Theorem 2.3. For $v > 0$, set

$$(3.1) \qquad \mathrm{SAB}_{n,v} = \left\{ \gamma \in \mathrm{SAB}_n : y(\gamma_n) \geq vn \right\}.$$

Theorem 2.3 follows from the next result.

**Theorem 3.1.** *Let $v > 0$. Let $\left\{ u_n : n \in \mathbb{N} \right\}$ be a subsequence of $\mathbb{N}$. There exists $\delta > 0$ as well as a subsequence $\left\{ t_n : n \in \mathbb{N} \right\}$ of $\left\{ u_n : n \in \mathbb{N} \right\}$ and a sequence $\left\{ \varepsilon_n : n \in \mathbb{N} \right\}$ with $\varepsilon_n \searrow 0$ such that*

$$\mathsf{P}_{\mathsf{SAB}t_n,v} \Big( |\mathsf{R}_\gamma| \geq \delta t_n \Big) \geq e^{-\varepsilon_n t_n}.$$

For $h \in \mathbb{Z}$, let $E_h$ denote the set of (necessarily north-south) edges $e = (u, u') \in E(\mathbb{Z}^d)$ such that $\left\{ y(u), y(u') \right\} = \left\{ h, h+1 \right\}$. For $\gamma \in \mathrm{SAW}_n$, define the *h-visiting edge-set* $\mathcal{V}_{h,h+1}$ by $\mathcal{V}_{h,h+1}(\gamma) = E_h \cap \left\{ (\gamma_i, \gamma_{i+1}) : 0 \leq i \leq n-1 \right\}$.

Let $\delta > 0$. For $m \geq 1$, write $\mathrm{SAB}_{n,v,\delta}^m \subseteq \mathrm{SAB}_{n,v}$ for the set of $\gamma \in \mathrm{SAB}_{n,v}$ such that there are at least $\delta n$ values of $h \in \mathbb{Z}$ for which $1 \leq |\mathcal{V}_{h,h+1}| \leq m$. Note that

$$(3.2) \qquad \mathrm{SAB}_{n,v,\delta}^1 \subseteq \left\{ \gamma \in \mathrm{SAB}_{n,v} : |\mathsf{R}_\gamma| \geq \delta n \right\},$$

since if a walk $\gamma$ crosses from $E_h$ to $E_{h+1}$ only at $(\gamma_i, \gamma_{i+1})$, then $\gamma_i$ is a renewal point. Thus, to derive Theorem 3.1, we may argue that $\mathrm{SAB}_{n,v,\delta}^1$ is not too small with respect to $\mathrm{SAB}_{n,v}$. The next proposition is the main technical result used in this deduction. It shows that, judged on an exponential scale

and along a suitable subsequence, the sets $\mathrm{SAB}^k_{n,v,\delta}$ and $\mathrm{SAB}^{k-1}_{n,v,\delta'}$ have the same size if $\delta' \ll \delta$.

**Proposition 3.2.** *Let $v > 0$. Let $k \geq 2$. Let $\delta > 0$ and $\{u_n : n \in \mathbb{N}\}$ be a subsequence of $\mathbb{N}$. Then there exist $\delta' > 0$ and a subsequence $\{t_n : n \in \mathbb{N}\}$ of $\{u_n : n \in \mathbb{N}\}$ such that*

$$\limsup_{n \to \infty} \frac{1}{t_n} \log \left( \frac{|\mathrm{SAB}^k_{t_n,v,\delta}|}{|\mathrm{SAB}^{k-1}_{t_n,v,\delta'}|} \right) \leq 0 \,.$$

**Proof of Theorem 3.1.** We claim that

(3.3) $$\mathrm{SAB}_{n,v} \subseteq \mathrm{SAB}^{\lceil 2/v \rceil}_{n,v,v/2} \,.$$

To verify this, note that $\gamma \in \mathrm{SAB}_{n,v}$ implies that $\mathcal{V}_{h,h+1} \neq \emptyset$ for at least $vn$ values of $h \in \mathbb{Z}$. Among these values, $|\mathcal{V}_{h,h+1}|$ can be larger than $2/v$ on at most $vn/2$ occasions. Thus, (3.3) holds.

By (3.3), we may successively apply Proposition 3.2 with $k$ being set equal to $\lceil 2/v \rceil$, $\lceil 2/v \rceil - 1, \ldots, 3$ and $2$ to conclude that, for any given $v > 0$, there exists some $\delta > 0$, some subsequence $\{t_n : n \in \mathbb{N}\}$ of $\{u_n : n \in \mathbb{N}\}$ and some sequence $\{\varepsilon_n : n \in \mathbb{N}\}$ with $\varepsilon_n \searrow 0$ such that $|\mathrm{SAB}^1_{t_n,v,\delta}| \geq e^{-\varepsilon_n t_n} |\mathrm{SAB}_{t_n,v}|$. The theorem follows by invoking (3.2) with $t_n$ replacing $n$. $\qquad\square$

We now introduce a concept which plays a central role in the proof of Proposition 3.2. Figure 4 provides an illustration.

Let $\gamma \in \mathrm{SAB}_n$. A *zigzag* of $\gamma$ is a pair $(i,j)$ of indices belonging to $\{0, \ldots, n\}$ that satisfy $i \leq j$ and for which

- the largest of the values $k$ at which the maximum of $y(\gamma_k)$ over $1 \leq k \leq j$ is achieved equals $i$, and
- the largest of the values $k$ at which the minimum of $y(\gamma_k)$ over $i \leq k \leq n$ is achieved equals $j$.

This definition is similar to the notion of *backtracks*, see [21, Definition 4.3.1]. Let $\mathsf{ZigZag} = \mathsf{ZigZag}(\gamma)$ denote the set of zigzags of $\gamma$. For $(i,j) \in \mathsf{ZigZag}$, we call the image of the subwalk $\gamma_{[i,j]}$ the *central section* of $(i,j)$; by the length of this central section, we mean the length of $\gamma_{[i,j]}$, which is $j - i$. The indices $i$ and $j$ are respectively called the *point of zig* and the *point of zag* of $(i,j)$.

The following basic properties of the zigzags of a bridge are readily verified.

**Lemma 3.3.** *Let $\gamma \in \mathrm{SAB}_n$.*

(1) *The central sections of the zigzags of $\gamma$ are pairwise disjoint.*
(2) *If $(i,i) \in \mathsf{ZigZag}$, then $\gamma_i \in \mathsf{R}_\gamma$.*

We now describe a one-step operation which acts on a bridge and unfolds a given zigzag. To do so, for $v \in \mathbb{R}^d$, we let $\mathcal{R}_v : \mathbb{R}^d \to \mathbb{R}^d$ denote the orthogonal reflection with respect to the hyperplane $\{u \in \mathbb{R}^d : y(u) = y(v)\}$.
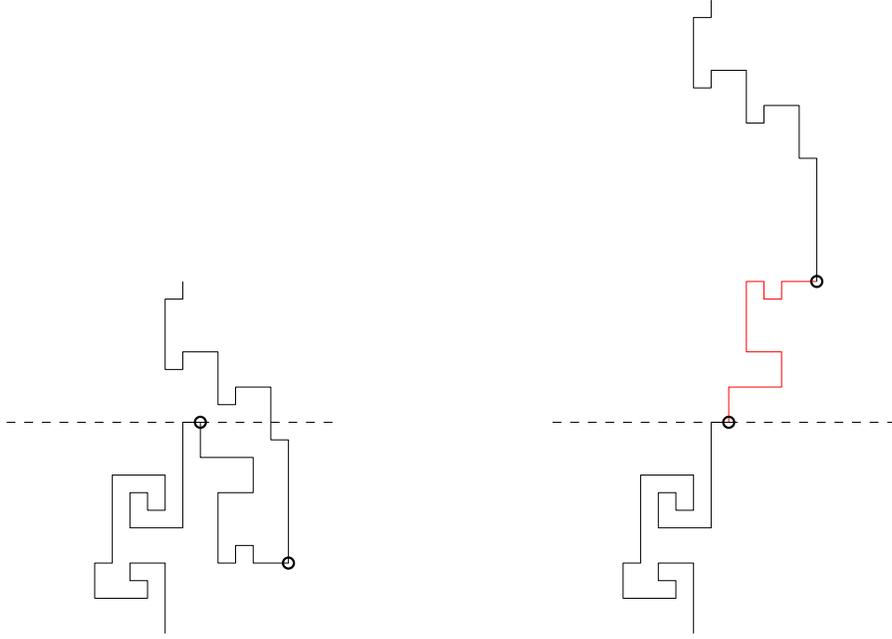
FIGURE 4. In the left-hand sketch, an element of $\mathrm{SAB}_n$ and one of its zigzags is illustrated. The zigzag's central section is shown in red and the points of zig and of zag are marked by circles. The unfolding of the walk indexed by this zigzag is depicted on the right.

**Definition 3.4.** *For $\gamma \in \mathrm{SAB}_n$ and $(i, j) \in \mathsf{ZigZag}$, let $\mathsf{Unf}_{i,j}(\gamma) \in \mathrm{SAB}_n$ be given by $\mathsf{Unf}_{i,j}(\gamma) = \gamma_{[0,i]} \circ \mathcal{R}_{\gamma_i}\big(\gamma_{[i,j]}\big) \circ \gamma_{[j,n]}$.*

**Lemma 3.5.** *Let $\gamma \in \mathrm{SAB}_n$.*

(1) *Let $(i, j) \in \mathsf{ZigZag}$. Then $\gamma_i$ and $\mathsf{Unf}_{(i,j)}(\gamma)_j$ each belong to $\mathsf{R}_{\mathsf{Unf}_{(i,j)}(\gamma)}$.*

(2) *Let $(i, j) \in \mathsf{ZigZag}$. Then $\mathsf{ZigZag}(\gamma) \setminus \{(i, j)\} \subseteq \mathsf{ZigZag}\big(\mathsf{Unf}_{i,j}(\gamma)\big)$.*

(3) *Let $z_1, z_2 \in \mathsf{ZigZag}$. Then $\mathsf{Unf}_{z_2}\mathsf{Unf}_{z_1}(\gamma) = \mathsf{Unf}_{z_1}\mathsf{Unf}_{z_2}(\gamma)$.*

The proof of Lemma 3.5 is straightforward and omitted. Lemma 3.5(3) shows that the order of application of the maps $\mathsf{Unf}_z$ is immaterial. This permits us to extend the action of $\mathsf{Unf}$ so that several zigzags are undone at once.

**Definition 3.6.** *Let $\gamma \in \mathrm{SAB}_n$ and $Z \subseteq \mathsf{ZigZag}$. Set $\mathsf{Unf}_Z(\gamma) \in \mathrm{SAB}_n$ equal to the bridge obtained by iteratively applying to $\gamma$ the maps $\mathsf{Unf}_z$ for $z \in Z$.*

We finish these preliminaries with a trivial lemma.

**Lemma 3.7.** *Let $\gamma \in \mathrm{SAB}_n$ and $Z \subseteq \mathsf{ZigZag}$. Then $\mathsf{Unf}_Z(\gamma) \in \mathrm{SAB}_n$ and $y(\mathsf{Unf}_Z(\gamma)_n) \geq y(\gamma_n)$.*

We are now in a position to prove Proposition 3.2.

**Proof of Proposition 3.2.** The proof is presented as a study of three cases. The walks in $\mathrm{SAB}^k_{t_n,v,\delta}$ are divided into the three following types, and the respective cases consider the event that walks of the given type form a positive proportion of $\mathrm{SAB}^k_{t_n,v,\delta}$:

(1) **Many renewal points**: walks with a positive proportion of renewal points are common, and there is nothing to prove;

(2) **Few zigzags**: walks with few zigzags are common; here, we unfold *all* zigzags at subexponential cost, thereby reducing the cardinality of $\mathcal{V}_{h,h+1}$ for all levels $h$ for which $|\mathcal{V}_{h,h+1}|$ is not already one;

(3) **Many zigzags and few renewal points**: if such walks as these are common, we unfold a collection of short zigzags of such a walk, the number of unfolded zigzags being both a tiny proportion of all zigzags but at the same time of far greater size than the existing number of renewal points; the former bound yields a large choice for such joint unfoldings, while the latter provides for efficient reconstruction of this choice given the outcome. Thus, the multi-valued map principle shows that image walks, which contain many renewal points because such points are generated wherever unfoldings of zigzags were made, are far more numerous than the preimage walks (which are common by assumption in this case). In spirit, this aspect of the argument has similarities to Kesten's proof [17] of the pattern theorem for self-avoiding walk.

It may be that one or more of the three cases does not occur. What matters is only that at least one of them must.

**Case 1: many renewal points.** Suppose that there exists $\delta' > 0$ and a subsequence $\{t_n : n \in \mathbb{N}\}$ of $\{u_n : n \in \mathbb{N}\}$ such that, for each $n \in \mathbb{N}$,

$$\left| \left\{ \gamma \in \mathrm{SAB}^k_{t_n,v,\delta} : |\mathsf{R}_\gamma| \geq \delta' t_n \right\} \right| \geq \tfrac{1}{4} \left| \mathrm{SAB}^k_{t_n,v,\delta} \right|.$$

If $\gamma \in \mathrm{SAB}_n$ and $\gamma_i \in \mathsf{R}_\gamma$ for some $0 \leq i \leq n-1$, note that $|\mathcal{V}_{h,h+1}| = 1$ for $h = y(\gamma_i)$. Thus if $\gamma \in \mathrm{SAB}^k_{t_n,v,\delta}$ satisfies $|\mathsf{R}_\gamma| \geq \delta' t_n$ then $\gamma \in \mathrm{SAB}^{k-1}_{t_n,v,\delta'}$ because $k \geq 2$.

**Case 2: few zigzags.** Suppose that there exists a sequence $\{\varepsilon_n : n \in \mathbb{N}\}$ with $\varepsilon_n \searrow 0$ for which

$$(3.4) \qquad \left| \left\{ \gamma \in \mathrm{SAB}^k_{u_n,v,\delta} : |\mathsf{ZigZag}(\gamma)| \leq \varepsilon_n u_n \right\} \right| \geq \tfrac{1}{4} \left| \mathrm{SAB}^k_{u_n,v,\delta} \right|.$$

Let $\gamma \in \mathrm{SAB}_n$ and $(i,j) \in \mathsf{ZigZag}(\gamma)$. If $(i,j)$ is a zigzag satisfying $i < j$ and if $h \in \mathbb{Z}$ satisfies $y(\gamma_j) \leq h < y(\gamma_i)$, then $|\mathcal{V}_{h,h+1}(\gamma')| \leq |\mathcal{V}_{h,h+1}(\gamma)| - 2$ where $\gamma' = \mathsf{Unf}_{(i,j)}(\gamma)$.

Suppose $h \in \mathbb{Z}$ is such that $|\mathcal{V}_{h,h+1}(\gamma)| \geq 2$. (In fact, for a bridge, such as $\gamma$ is, the cardinality of this set must be odd.) The successive unfoldings performed by $\mathsf{Unf}_{\mathsf{ZigZag}(\gamma)}$ will leave the set $\mathcal{V}_{h,h+1}$ unchanged except for changing the value of $h$ until the zigzag whose central section includes an edge of $\mathcal{V}_{h,h+1}(\gamma)$ is unfolded, at which time, the set $\mathcal{V}_{h,h+1}$ (with the value of $h$ updated) will be split into three disjoint pieces; each of these sets has size at most $|\mathcal{V}_{h,h+1}| - 2$, since the other two sets are non-empty; subsequently, each of these three sets will remain unchanged except for further changes to the value of $h$ (which now assumes a different value for each of the three sets). Writing $\gamma' = \mathsf{Unf}_{\mathsf{ZigZag}(\gamma)}(\gamma)$ for the outcome of unfolding all of $\gamma$'s zigzags, we thus see that the number of $h \in \mathbb{Z}$ such that $1 \leq |\mathcal{V}_{h,h+1}(\gamma')| \leq k - 1$ is at least the number of $h \in \mathbb{Z}$ such that $1 \leq |\mathcal{V}_{h,h+1}(\gamma)| \leq k$. (In fact, when $k \geq 3$, we may improve $k - 1$ to $k - 2$ here. Note also that, in the case $k = 2$, $1 \leq |\mathcal{V}_{h,h+1}(\gamma)| \leq k$ forces $\mathcal{V}_{h,h+1}(\gamma)$ to be a singleton.) Also applying Lemma 3.7, we find that, for $k \geq 2$,

$$(3.5) \qquad \mathsf{Unf}_{\mathsf{ZigZag}(\gamma)}\big(\mathrm{SAB}^k_{u_n,v,\delta}\big) \subseteq \mathrm{SAB}^{k-1}_{u_n,v,\delta}\,.$$

For any $\phi \in \mathrm{SAB}_{u_n}$, we claim that

$$(3.6) \qquad \Big|\mathsf{Unf}^{-1}_{\mathsf{ZigZag}(\gamma)}(\phi) \cap \big\{\gamma \in \mathrm{SAB}_{u_n} : |\mathsf{ZigZag}(\gamma)| \leq \varepsilon_n u_n\big\}\Big| \leq \binom{u_n}{2\varepsilon_n u_n}\,.$$

Indeed, if $\phi = \mathsf{Unf}_{\mathsf{ZigZag}(\gamma)}(\gamma)$ for given $\phi$ and unknown $\gamma$, then, to determine $\gamma$, it is enough to know the union of the points in $\phi$ that correspond to all of the unfolded points of zig and of zag generated as $\mathsf{Unf}_{\mathsf{ZigZag}(\gamma)}$ is formed. This set of points has cardinality $2\varepsilon_n u_n$, whence (3.6).

Applying the multi-valued map principle in light of (3.5) and (3.6), and then using (3.4), we find that

$$\begin{aligned}
|\mathrm{SAB}^{k-1}_{u_n,v,\delta}| &\geq \binom{u_n}{2\varepsilon_n u_n}^{-1}\Big|\big\{\gamma \in \mathrm{SAB}^k_{u_n,v,\delta} : |\mathsf{ZigZag}(\gamma)| \leq \varepsilon_n u_n\big\}\Big| \\
&\geq \frac{1}{4}\binom{u_n}{2\varepsilon_n u_n}^{-1}\big|\mathrm{SAB}^k_{u_n,v,\delta}\big| \\
&\geq \tfrac{1}{4}\exp\big\{ - 2\varepsilon_n u_n \log\big(\tfrac{e}{2\varepsilon_n}\big) + 1\big\}\big|\mathrm{SAB}^k_{u_n,v,\delta}\big|\,,
\end{aligned}$$

where we also used $\binom{m}{\ell} \leq (m/\ell)^\ell e^{\ell-1}$ for $m \geq \ell \geq 1$.

**Case 3: many zigzags and few renewal points.** We now treat the remaining case by supposing that there exist $\delta' > 0$ as well as $\big\{t_n : n \in \mathbb{N}\big\}$ and $\big\{\varepsilon_n : n \in \mathbb{N}\big\}$ with $\varepsilon_n \searrow 0$ such that, for all $n$ sufficiently high,

$$(3.7) \qquad \Big|\big\{\gamma \in \mathrm{SAB}^k_{t_n,v,\delta} : |\mathsf{ZigZag}(\gamma)| \geq 2\delta't_n\,, |\mathsf{R}_\gamma| \leq \varepsilon_n t_n\big\}\Big| \geq \tfrac{1}{2}\big|\mathrm{SAB}^k_{t_n,v,\delta}\big|\,.$$

Set ManyZFewR equal to the set on the left-hand side of (3.7).

The central sections of a walk's zigzags being disjoint by Lemma 3.3(1), an element $\gamma \in \mathrm{SAB}_{t_n}$ may have at most $\delta' t_n$ zigzags whose central section has length at least $\lceil 1/\delta' \rceil$. A zigzag of $\gamma$ is called *short* if its central section has length at most $\lceil 1/\delta' \rceil$. Writing $\mathsf{ShortZZ}_\gamma$ as the set of short zigzags, note then that $\mathrm{ManyZFewR} \subseteq \{ |\mathsf{ShortZZ}_\gamma| \geq \delta' t_n \}$.

Let $\delta'' > 0$ be a parameter whose value will be fixed later (at a value much less than $\delta'$). We now construct a multi-valued unfolding map called $\mathsf{MultiUnf}$ that will be defined on the set of $\gamma \in \mathrm{SAB}_{t_n}$ such that $|\mathsf{ShortZZ}_\gamma| \geq \delta' t_n$. For such $\gamma$, set

$$\mathsf{MultiUnf}(\gamma) = \Big\{ \mathsf{Unf}_Z(\gamma) : Z \subseteq \mathsf{ShortZZ}_\gamma \,,\ |Z| = \lfloor \delta'' t_n \rfloor \Big\}.$$

We now aim to apply the multi-valued map principle. First, if $\phi \in \mathsf{MultiUnf}(\gamma)$ for some $\gamma \in \mathrm{SAB}_{t_n}$ such that $|\mathsf{ShortZZ}_\gamma| \geq \delta'' t_n$, then $|\mathsf{R}_\phi| \geq \delta'' t_n$ because, by Lemma 3.5(1), $\mathsf{R}_\phi$ contains the set of vertices corresponding to the points of zig and of zag generated by the successive unfoldings which comprise the map whose output is $\phi$ (note that the points of zig and zag of a zigzag can be identical). Since $\delta''$ is chosen to be less than $\delta'$, we see that

(3.8)     if $\phi \in \mathsf{MultiUnf}(\gamma)$ for some $\gamma \in \mathrm{ManyZFewR}$, then $|\mathsf{R}_\phi| \geq \delta'' n$.

In other words, $\mathsf{MultiUnf}$ is a multivalued map into $\mathrm{SAB}^1_{t_n, v, \delta''}$.

Second, for each fixed $\gamma \in \mathrm{SAB}_n$, the map $Z \to \mathsf{Unf}_Z(\gamma)$ from the power-set of $\mathsf{ZigZag}(\gamma)$ to $\mathrm{SAB}_n$ is one-to-one. Hence, if $\gamma \in \mathrm{ManyZFewR}$, then

(3.9)     $$|\mathsf{MultiUnf}(\gamma)| = \binom{|\mathsf{ShortZZ}_\gamma|}{\delta'' t_n} \geq \binom{\delta' t_n}{\delta'' t_n}.$$

The only remaining question to address is the size of preimages. Let $\phi \in \mathsf{MultiUnf}(\gamma)$ for some $\gamma \in \mathrm{ManyZFewR}$. We claim that

(3.10)     $$|\mathsf{R}_\phi| \leq \varepsilon_n t_n + 3\lceil 1/\delta' \rceil \delta'' t_n.$$

Indeed, $|\mathsf{R}_\gamma| \leq \varepsilon_n t_n$ by assumption. Furthermore, the number of newly created renewal points is bounded above by $3\lceil 1/\delta' \rceil \delta'' t_n$. Indeed, let $Z \subseteq \mathsf{ZigZag}(\gamma)$ be such that $\phi = \mathsf{Unf}_Z(\gamma)$. Consider a single unfolding operation $\mathsf{Unf}_{(i,j)}$ applied to a walk $\chi$ (so that $(i,j) \in \mathsf{ZigZag}_\chi$). Note that in the half-space given by $y < y(\chi_j)$, $\mathsf{Unf}_{(i,j)}(\chi)$ coincides with $\chi$, while the intersection of $\mathsf{Unf}_{(i,j)}(\chi)$ with the half-space $y > y(\chi_i) + 2\big(y(\chi_i) - y(\chi_j)\big)$ is merely a translate of the intersection with $\chi$ with $y > y(\chi_i)$. As such, renewal points in $\mathsf{Unf}_{(i,j)}(\chi)$ either lie in one of these two half-spaces, in which case, they have counterparts in $\chi$, or they lie in the slab given by the complement of the union of the hyperplanes. This slab has width $3\big(y(\chi_i) - y(\chi_j)\big)$, which is at most three times the length $j - i$ of the central section of $(i,j)$. We see then that the number of renewal points for $\mathsf{Unf}_{(i,j)}(\chi)$ exceeds this number for $\chi$ by at most $3(j - i)$. From $j - i \leq \lceil 1/\delta' \rceil$, we verify that the number of newly

created renewal points satisfies the bound we claimed. From $|Z| = \delta''n$, we obtain (3.10).

Suppose given $\phi = \mathsf{Unf}_Z(\gamma)$ for some unknown $\gamma \in \mathrm{ManyZFewR}$ and for some unknown subset $Z \subseteq \mathsf{ShortZZ}_\gamma$ such that $|Z| = \delta''n$. Recalling Lemma 3.5, the data $\gamma$ may be reconstructed from $\phi$ provided that the $|Z|$ pairs of points corresponding to the points of zig and of zag generated by the successive unfoldings of elements of $Z$ are known. Also note that given a certain point $i$ assumed to be the point of zig of some $(i,j) \in Z$, then $j - i$ is the length of the central section and therefore $i \leq j \leq i + \lceil 1/\delta' \rceil$. For this reason, the number of possibilities for the point of zag given the point of zig is bounded by $\lceil 1/\delta' \rceil$.

Hence, the number of pairs $(\gamma, Z)$ such that $\mathsf{Unf}_Z(\gamma) = \phi$ is at most the product of the number of subsets of $\mathsf{R}_\phi$ of size $\delta''n$ and the quantity $\lceil 1/\delta' \rceil^{\delta''n}$, the first factor counting possible values of the set of points of zig of $Z$ and the second counting such values for the set of points of zag. In light of (3.10), this quantity is at most

$$(3.11) \qquad \binom{(\varepsilon_n + 3\lceil 1/\delta' \rceil \delta'')t_n}{\delta'' t_n} \lceil 1/\delta' \rceil^{\delta'' t_n}.$$

Suppose that $n_1 \geq n_2 \geq m$. Then it is trivial that $\frac{\binom{n_1}{m}}{\binom{n_2}{m}} \geq \left(\frac{n_1 - m}{n_2}\right)^m$. Setting $n_1 = \delta' t_n$, $n_2 = (\varepsilon_n + 3\lceil 1/\delta' \rceil \delta'')t_n$ and $m = \delta'' t_n$, we may bound from below the ratio of the expressions in (3.9) and (3.11) provided that we stipulate that $\delta' \geq \varepsilon_n + 3\lceil 1/\delta' \rceil \delta''$, a condition which may be realized for all $n$ sufficiently high by an appropriate choice of $\delta''$ since $\varepsilon_n \searrow 0$. Note that the other inequality we need, $\varepsilon_n + 3\lceil 1/\delta' \rceil \delta'' \geq \delta''$, is automatically satisfied. The resulting lower bound on the ratio is

$$\frac{\binom{\delta' t_n}{\delta'' t_n}}{\binom{(\varepsilon_n + 3\lceil 1/\delta' \rceil \delta'')t_n}{\delta'' t_n}} \lceil 1/\delta' \rceil^{-\delta'' t_n} \geq \left(\frac{\delta' - \delta''}{(\varepsilon_n + 3\lceil 1/\delta' \rceil \delta'')\lceil 1/\delta' \rceil}\right)^{\delta'' t_n}.$$

By further imposing on the choice of $\delta'' > 0$ the requirement that, for all $n$ sufficiently high,

$$\frac{\delta' - \delta''}{(\varepsilon_n + 3\lceil 1/\delta' \rceil \delta'')\lceil 1/\delta' \rceil} \geq 2\,,$$

we can apply the multi-valued map principle when $n$ is large enough to deduce that

$$|\mathrm{SAB}_{t_n,v,\delta''}^{k-1}| \geq |\mathrm{SAB}_{t_n,v,\delta''}^1| = \left|\left\{\gamma \in \mathrm{SAB}_{t_n,v} : |\mathsf{R}_\gamma| \geq \delta'' t_n\right\}\right|$$

$$\geq \quad 2^{\delta'' t_n}\left|\left\{\gamma \in \mathrm{SAB}_{t_n,v,\delta}^k : |\mathsf{ZigZag}(\gamma)| \geq 2\delta' t_n \text{ and } |\mathsf{R}_\gamma| \leq \varepsilon_n t_n\right\}\right|,$$

whose right-hand side is at least $2^{\delta'' t_n - 1}\left|\mathrm{SAB}_{t_n,v,\delta}^k\right|$ in view of (3.7). This provides the sought inequality in the third case. $\qquad \square$

Recall that $\mathbb{P}_{\text{iSAB}}$ denotes the law on iSAB given by $\mathbb{P}_{\text{iSAB}}(\gamma) = \mu_c^{-|\gamma|}$ and $\mathbb{P}_{\text{iSAB}}^{\otimes \mathbb{N}}$ the law on semi-infinite walks $\gamma : \mathbb{N} \to \mathbb{Z}^d$ formed by concatenating infinitely many independent samples of $\mathbb{P}_{\text{iSAB}}$. We extend the notion of renewal points to infinite bridges in an obvious way. The set of renewal points is still denoted by $\mathsf{R}_\gamma$.

**Lemma 3.8.** *For each $n \in \mathbb{N}$, we have that*

$$|\text{SAB}_n|\, \mu_c^{-n} = \mathbb{P}_{\text{iSAB}}^{\otimes \mathbb{N}}\big(n \in \mathsf{R}_\gamma\big)\,.$$

*The conditional distribution of $\gamma_{[0,n]}$ under $\mathbb{P}_{\text{iSAB}}^{\otimes \mathbb{N}}\big(\,\cdot\,\big|n \in \mathsf{R}_\gamma\big)$ equals $\mathsf{P}_{\text{SAB}n}$.*

**Proof.** Note that

$$|\text{SAB}_n|\, \mu_c^{-n} = \sum_{\gamma \in \text{SAB}:|\gamma|=n} \mu_c^{-|\gamma|} = \sum_{\ell \geq 0}\ \sum_{\gamma^{[1]},\ldots,\gamma^{[\ell]} \in \text{iSAB}:\sum_{i=1}^{\ell}|\gamma^{[i]}|=n}\ \prod_{i=1}^{\ell} \mu_c^{-|\gamma^{[i]}|}$$

$$= \mathbb{P}_{\text{iSAB}}^{\otimes \mathbb{N}}\Big(\exists\, \ell \in \mathbb{N} : \sum_{i=1}^{\ell} |\gamma^{[i]}| = n\Big) = \mathbb{P}_{\text{iSAB}}^{\otimes \mathbb{N}}\big(\gamma_n \in \mathsf{R}_\gamma\big)\,.$$

The latter statement of the lemma follows directly. $\qquad\square$

**Proof of Corollary 2.4.** Assume that $\mathbb{E}_{\text{iSAB}}\big(|\gamma|\big) = \infty$. Fix $\delta > 0$. We may select $k \in \mathbb{N}$ such that

$$\mathbb{E}_{\text{iSAB}}\big(\min\{k,|\gamma|\}\big) \geq 2/\delta\,.$$

With $\gamma$ denoting a sample of $\mathbb{P}_{\text{iSAB}}^{\otimes \mathbb{N}}$, we write $\gamma^{[j]}$ for the $j^{\text{th}}$ irreducible bridge entering in the decomposition of $\gamma$ (translated in such a way that it starts from 0). Define

$$\mathsf{X}_j = \mathsf{X}_j(\gamma) := \sum_{i=1}^{j} \min\{k, |\gamma^{[i]}|\}\,.$$

Note that, irrespectively of the choice of $k \in \mathbb{N}$,

(3.12) $\qquad \mathbb{P}_{\text{iSAB}}^{\otimes \mathbb{N}}\big(\big|\{i \in [0,n] : \gamma_i \in \mathsf{R}_\gamma\}\big| \geq \delta n\big) \leq \mathbb{P}_{\text{iSAB}}^{\otimes \mathbb{N}}(\mathsf{X}_{\delta n} \leq n)\,.$

By construction, $\mathsf{X}_j$ is a sum of $\delta n$ independent and identically distributed bounded random variables whose common mean is at least $2/\delta$. By the exponential Markov inequality, there exists $\varepsilon > 0$ such that, for each $n \in \mathbb{N}$,

(3.13) $\qquad\qquad\qquad \mathbb{P}_{\text{iSAB}}^{\otimes \mathbb{N}}(\mathsf{X}_{\delta n} \leq n) \leq e^{-\varepsilon n}\,.$

However, Lemma 3.8 gives

$$\mathsf{P}_{\text{SAB}n}\big(|\mathsf{R}_\gamma| \geq \delta n\big) = \mathbb{P}_{\text{iSAB}}^{\otimes \mathbb{N}}\Big(\big|\{i \in [0,n] : \gamma_i \in \mathsf{R}_\gamma\}\big| \geq \delta n\,\Big|\,\gamma_n \in \mathsf{R}_\gamma\Big)\,.$$

Proposition 2.1 implies that $|\text{SAB}_n| \geq e^{-C\sqrt{n}}|\text{SAW}_n|$ and the classical submultiplicativity of the number of walks together with Fekete's lemma leads

to $|\mathrm{SAW}_n| \geq \mu_c^n$. These two facts in unison with Lemma 3.8 yield $\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}(\gamma_n \in \mathsf{R}_\gamma) \geq e^{-C\sqrt{n}}$. Therefore, (3.12) and (3.13) imply that

$$\mathsf{P}_{\mathsf{SAB}_n}\big(|\mathsf{R}_\gamma| \geq \delta n\big) \leq e^{-\varepsilon n + C\sqrt{n}}$$

for any positive $n$. The quantity $\delta > 0$ being arbitrary, this is in contradiction with Theorem 2.3. $\qquad\square$

## 4. Incompatibility of the renewal theory for bridges

In this section, we prove Theorem 2.5 by making rigorous the argument outlined after the theorem's statement. In the presentation, indices are rounded to the nearest integer. For the sake of simplicity, we omit the rounding operation in the notation.

We extend the semi-infinite walk measure $\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}$ which we introduced after the statement of Theorem 2.5 to doubly infinite walks. Let $\Omega$ be the set of bi-infinite walks $\gamma : \mathbb{Z} \to \mathbb{Z}^d$ such that $\gamma_0 = 0$. Let $\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{Z}}$ denote the law on $\Omega$ formed from $\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}$ by iteratively inserting infinitely many independent samples of $\mathbb{P}_{\mathrm{iSAB}}$ so that the terminal point of the newly inserted renewal block is the starting point of the walk that is presently formed. Let $\mathcal{F}$ be the $\sigma$-algebra generated by events depending on a finite number of vertices of the walk.

We begin by describing a few properties of the measure $\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{Z}}$. Let $\mathbf{r}$ be the bi-infinite sequence of integers defined by $\mathbf{r}_0 = 0$ and $\mathbf{r}_{k+1} = \inf\{j > \mathbf{r}_k : \gamma_j \in \mathsf{R}_\gamma\}$ for each $k \in \mathbb{Z}$. That is, for $k \neq 0$, $\mathbf{r}_k$ is the index of the $|k|^{\mathrm{th}}$ renewal point, counted northwards or southwards away from the origin according to the sign of $k$. Let $\tau : \Omega \to \Omega$ be the *shift* defined by the formula $\tau(\gamma)_i = \gamma_{i+\mathbf{r}_1} - \gamma_{\mathbf{r}_1}$ for every $i \in \mathbb{Z}$. Let $\sigma$ denote reflection in the hyperplane $\{u \in \mathbb{R}^d : y(u) = 0\}$.

**Lemma 4.1.** *The measure $\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{Z}}$ has the following properties.*

(1) *It is invariant under the shift $\tau$.*
(2) *The shift $\tau$ is ergodic for $(\Omega, \mathcal{F}, \mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{Z}})$.*
(3) *Under $\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{Z}}$, the random sequences $(\sigma \gamma_n)_{n \leq 0}$ and $(\gamma_n - \gamma_1)_{n \geq 1}$ are independent and identically distributed.*

**Proof.** Part (1) is fairly straightforward. Indeed, the law of $\gamma_{[\mathbf{r}_{-n}, \mathbf{r}_n]}$ determines, in the high-$n$ limit, the law of $\gamma$ since we work with the $\sigma$-algebra $\mathcal{F}$. Now, the laws of $\tau(\gamma_{[\mathbf{r}_{-n-1}, \mathbf{r}_{n-1}]})$ and $\gamma_{[\mathbf{r}_{-n}, \mathbf{r}_n]}$ are the same by construction (the common law is simply the law of $2n$ consecutively concatenated independent irreducible bridges). The claim follows by letting $n \to \infty$.

Part (2) is classical. Let $A$ be a shift-invariant event. We aim to prove that $\mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{Z}}(A) \in \{0, 1\}$. In order to do so, let $A_n$ be an event depending only on vertices $\gamma_{-n}, \ldots, \gamma_n$ such that $\mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{Z}}(A_n \Delta A) \leq \varepsilon$. By extension, $A_n$ depends only on vertices $\gamma_{\mathbf{r}-n}, \ldots, \gamma_{\mathbf{r}_n}$. Invariance under $\tau$ implies that

$$\mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{Z}}(A) = \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{Z}}(A \cap A) = \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{Z}}[A \cap \tau^{-2n}(A)].$$

Moreover,

$$\left| \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{Z}}\big(A \cap \tau^{-2n}(A)\big) - \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{Z}}\big(A_n \cap \tau^{-2n}(A_n)\big) \right|$$
$$\leq \ \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{Z}}\big(A \Delta A_n\big) + \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{Z}}\big(\tau^{-2n}(A) \Delta \tau^{-2n}(A_n)\big) \leq 2\varepsilon \,,$$

the latter inequality again invoking invariance. Using the independence between the walk before and after $\mathbf{r}_n$, we find that

$$\left| \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{Z}}(A) - \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{Z}}(A_n)^2 \right| \leq 2\varepsilon$$

which implies that

$$\left| \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{Z}}(A) - \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{Z}}(A)^2 \right| \leq 4\varepsilon \,.$$

The value of $\varepsilon > 0$ being arbitrary, we conclude that $\mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{Z}}(A) = \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{Z}}(A)^2$ and thus that $\mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{Z}}(A) \in \{0, 1\}$.

Part (3) follows from an explicit construction of the law $\mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{Z}}$ which emphasises its near symmetry under $\sigma$. Noting that any bridge begins by making a move to the north, let $\mathsf{P}_{\text{iSAB}}^*$ denote the law on SAW formed by removing the first edge of the walk and translating the resulting subwalk one unit southwards. Note that the support of $\mathsf{P}_{\text{iSAB}}^*$ includes the trivial walk of length zero. Note also that $\mathsf{P}_{\text{iSAB}}^*$ assigns probability $\mu_c^{-|\gamma|-1}$ to all $\gamma$ (of positive length) having the property that $\gamma$ visits only $y$-coordinates comprised between the $y$-coordinates of its starting and ending points, and no such $y$-coordinate is visited exactly once; the residue $\mu_c^{-1}$ of the probability is assigned to the trivial walk. From this description of $\mathsf{P}_{\text{iSAB}}^*$, it is clear that this law is invariant under application of $\sigma$ (and the necessary translation).

As such, $\mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{Z}}$ may be constructed as follows. First a north-going edge is placed incident to the origin. An element of $\mathsf{P}_{\text{iSAB}}^*$ is then concatenated to its northerly endpoint. The construction is then iterated. To build the walk in the southerly direction, independent elements of $\mathsf{P}_{\text{iSAB}}^*$ and north-south edges are alternately joined so that the terminal point of each is the initial point of the presently constructed walk. The invariance of $\mathsf{P}_{\text{iSAB}}^*$ under $\sigma$ shows that the two random variables in part(3) each have the law of the concatenation of the sequences formed alternately by independent elements of $\mathsf{P}_{\text{iSAB}}^*$ and north-south edges. $\qquad\square$

Recall the notion of diamond point from Section 2.3. As we explained in the sketch in that section, our goal is now to prove that under the assumption that $\mathbb{E}_{\mathrm{iSAB}}(|\gamma|) < \infty$, a positive density of renewal points are in fact diamond points. Let $\mathsf{D}_\gamma$ be the set of diamond points of $\gamma$. Extend the definition of $(\mathbf{r}_k)_{k \geq 0}$ from $\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{Z}}$ to $\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}$.

**Proposition 4.2.** *If $\mathbb{E}_{\mathrm{iSAB}}(|\gamma|) < \infty$, there exists $\delta > 0$ such that*

$$\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}} \left( \liminf_{n \to \infty} \frac{|\mathsf{D}_\gamma \cap [0, \mathbf{r}_n]|}{n} \geq \delta \right) = 1 \,.$$

That the length of the renewal block has finite mean implies that the bridge under $\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}$ is tall and narrow, with height growing linearly and width sublinearly. So the probability under $\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{Z}}$ that the positively (or negatively) indexed walk stays within the cone

$$\left\{ u \in \mathbb{Z}^d : \ |y(u)| > |x(u)| \right\}$$

is strictly positive. This separation of future and past occurs at a positive density of renewal points; when it does, the renewal point is actually a diamond point. A little care is needed as we make this argument rigorous.

**Proof of Proposition 4.2.** Assume $\mathbb{E}_{\mathrm{iSAB}}(|\gamma|) < \infty$. Let us first prove that $\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{Z}}(\gamma_0 \in \mathsf{D}_\gamma) > 0$.

The law of an irreducible bridge is invariant under the reflection with respect to the hyperplane $\{u \in \mathbb{Z}^d : x(u) = 0\}$. Therefore, the law of large numbers applied to $(x, y)(\gamma_{\mathbf{r}_n})$ implies that, $\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}$-almost surely,

$$(4.1) \qquad \left( \frac{x(\gamma_{\mathbf{r}_n})}{n}, \frac{y(\gamma_{\mathbf{r}_n})}{n} \right) \longrightarrow (0, \mu) \quad \text{as } n \to \infty \,,$$

where $\mu$ denotes some strictly positive constant. An irreducible bridge having finite expected size, we deduce that each of $\inf_{k \geq 0}(x + y)(\gamma_k)$ and $\inf_{k \geq 0}(y - x)(\gamma_k)$ is finite $\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}$-almost surely. For $\ell \in \mathbb{N}$, write $\rho_\ell$ for $\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}$-probability that both of these random variables is at least $-\ell$, and choose $K \in \mathbb{N}$ so that $\rho_K > 0$.

We now claim that

$$(4.2) \qquad \rho_0 \geq \mu_c^{-2K} \rho_K \,.$$

To verify (4.2), consider an experiment under which the law $\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}$ is constructed by concatenating $2K$ independent samples of $\mathbb{P}_{\mathrm{iSAB}}$ to the terminal point of which is concatenated an independent sample of $\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}$. If each of the $2K$ samples happens to be a walk of length one (which entails that each is a north-south edge) and if the independent copy of $\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}$ realizes

$$\min \left( \inf_{k > 0}(x + y)(\gamma_k) \,, \ \inf_{k > 0}(y - x)(\gamma_k) \right) > -K \,,$$

then the constructed sample realizes

$$\min \left( \inf_{k>0}(x+y)(\gamma_k) \ , \ \inf_{k>0}(y-x)(\gamma_k) \right) > 0 \,.$$

By the two assertions of Lemma 3.8, the probability that the $i^{\text{th}}$ sample of $\mathbb{P}_{\text{iSAB}}$ is a north-south edge is $\mu_c^{-1}$. Thus, the experiment behaves as described with probability $\mu_c^{-2K}\rho_K$, and we obtain (4.2).

From (4.2) and Lemma 4.1(3), we deduce that

$$\delta := \mathbb{P}_{\text{iSAB}}^{\otimes \mathbb{Z}}(\gamma_0 \in \mathsf{D}_\gamma) > 0 \,.$$

The shift $\tau$ being ergodic by Lemma 4.1(2), we obtain

(4.3)
$$\mathbb{P}_{\text{iSAB}}^{\otimes \mathbb{Z}} \left( \lim_{n\to\infty} \frac{|\mathsf{D}_\gamma \cap [0, \mathbf{r}_n]|}{n} = \delta \right) = 1 \,.$$

From this, the statement of the proposition follows immediately. The equality "$= \delta$" becomes an inequality "$\geq \delta$" because of the possibility that the negatively indexed semi-infinite bridge under $\mathbb{P}_{\text{iSAB}}^{\otimes \mathbb{Z}}$ is responsible for destroying diamond points; in fact, it would be simple to argue that (4.3) holds for the law $\mathbb{P}_{\text{iSAB}}^{\otimes \mathbb{N}}$, but we have no need of this assertion. $\square$

We now modify long bridges using the stickbreak operation in order to obtain Theorem 2.5.

**Proof of Theorem 2.5.** The proof entails some interplay between infinite and finite bridges: in denoting generic infinite or finite bridges, we write $\phi$ and $\gamma$.

For a finite bridge $\gamma$, let

$$\mathbf{width}(\gamma) = \max \left\{ x(\gamma_i) - x(\gamma_j) : 0 \leq i, j \leq |\gamma| \right\} \,.$$

The notion of diamond point may naturally be defined also for finite bridges. We enumerate $\gamma$'s successive diamond points as $\left\{ \mathbf{d}_i(\gamma) : 1 \leq i \leq |\mathsf{D}_\gamma| \right\}$.

We aim to prove that $\mathbb{E}_{\text{iSAB}}(|\gamma|) = \infty$. We proceed by contradiction. Set $\mathbb{E}_{\text{iSAB}}(|\gamma|) < \nu < \infty$ and $0 < \varepsilon < \delta/20$, where $\delta$ is given by Proposition 4.2.

Let $\Omega^+$ denote the set of semi-infinite bridges, so that $\phi \in \Omega^+$ whenever $\phi : \mathbb{N} \to \mathbb{Z}^d$ is a self-avoiding walk for which $y(\phi_i) > 0$ for $i > 0$. To any $(\phi, n) \in \Omega^+ \times \mathbb{N}$ for which $\mathbf{r}_n(\phi)$ exists, associate the truncated bridge $\phi^{(n)} := \phi_{[0, \mathbf{r}_n(\phi)]}$. Note then that $\mathbf{r}_n(\phi)$ coincides with $\mathbf{r}_n(\phi^{(n)})$; we write $\mathbf{r}_n$ for the common value in what follows. Let $\overline{\text{SAB}}_n = \overline{\text{SAB}}_n(\varepsilon)$ denote the set of $\phi \in \Omega^+$ such that

(1) $\mathbf{r}_n \leq \nu n$,
(2) $\mathbf{width}(\phi^{(n)}) \leq \varepsilon n$,
(3) $\mathbf{d}_{\delta n/2}(\phi^{(n)}) \leq \mathbf{r}_n$,

and note for later use that the data $\phi^{(n)}$ determines whether these properties hold.

These three properties will be important in implementing the stickbreak operation which was depicted in Figure 3 as we outlined our approach. We now prove that

$$(4.4) \qquad \mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}\big(\overline{\mathrm{SAB}}_n\big) \longrightarrow 1 \text{ as } n \to \infty \,.$$

From $\mathbb{E}_{\mathrm{iSAB}}(|\gamma|) < \nu < \infty$ and $\mathbb{E}_{\mathrm{iSAB}}[x(\mathbf{t}_\gamma)] = 0$, where $\mathbf{t}_\gamma$ is the endpoint of the bridge $\gamma$, we find that

$$\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}\Big(\mathbf{r}_n \leq \nu n \text{ and } \mathbf{width}(\phi^{(n)}) \leq \varepsilon n\Big) \longrightarrow 1$$

thanks to the law of large numbers (which holds since the renewal blocks are independent and identically distributed under $\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}$). Note that, whenever $\phi^{(n)}$ has at least $i$ diamond points, we have that $\mathbf{d}_i(\phi^{(n)}) \leq \mathbf{d}_i(\phi)$, because the truncation operation $\phi \to \phi^{(n)}$ can only create (and not annihilate) diamond points. Thus, Proposition 4.2 yields that

$$\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}\big(\mathbf{d}_{\delta n/2}(\phi^{(n)}) \leq \mathbf{r}_n\big) \longrightarrow 1 \,.$$

This concludes the proof of (4.4).

The convergence (4.4) implies that we will reach a contradiction if we prove that, for each $n \in \mathbb{N}$,

$$(4.5) \qquad \mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}\Big(\mathbf{width}(\phi^{(n)}) > \varepsilon n\Big) \geq \left(\frac{\delta}{20\nu}\right)^2 \mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}\big(\overline{\mathrm{SAB}}_n\big) \,.$$

For $\phi \in \Omega^+$ and $\gamma$ a finite bridge, set $\gamma \lhd \phi$ if $\phi_{[0,|\gamma|]} = \gamma$ and if $\phi_{|\gamma|}$ is a renewal point of $\phi$. Lemma 3.8 implies that

$$(4.6) \qquad \mu_c^{-|\gamma|} = \mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}(\phi \in \Omega^+ : \gamma \lhd \phi) \,.$$

Let $\overline{\mathrm{SAB}}_n^+ = \big\{\phi^{(n)} : \phi \in \overline{\mathrm{SAB}}_n\big\}$; that is, $\overline{\mathrm{SAB}}_n^+$ is the set of finite bridges $\gamma$ having $n$ renewal points and for which $\gamma \lhd \phi$ for some infinite bridge $\phi$ belonging to $\overline{\mathrm{SAB}}_n$. The definition of $\overline{\mathrm{SAB}}_n$ has been chosen so that the membership of any $\phi \in \Omega^+$ in this set is determined by the data $\phi^{(n)}$; thus, we may sum (4.6) to learn that

$$(4.7) \qquad \mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}\big(\overline{\mathrm{SAB}}_n\big) = \sum_{\gamma \in \overline{\mathrm{SAB}}_n^+} \mu_c^{-|\gamma|} \,.$$

We now focus our attention on $\overline{\mathrm{SAB}}_n^+$. Consider $\gamma \in \overline{\mathrm{SAB}}_n^+$. In what follows, we adopt the shorthand $\mathbf{r}_n = \mathbf{r}_n(\gamma)$ and $\mathbf{d}_i = \mathbf{d}_i(\gamma)$. For each $i \in \big[\frac{\delta}{10}n, \frac{2\delta}{10}n\big]$ and $j \in \big[\frac{3\delta}{10}n, \frac{4\delta}{10}n\big]$, let the $(i,j)$-stickbreak $\mathsf{StBr}_{i,j}(\gamma)$ of $\gamma$ be given by

$$\mathsf{StBr}_{i,j}(\gamma) = \gamma_{[0,\mathbf{d}_i]} \circ \rho_{\pi/2,\mathbf{d}_i}\big(\gamma_{[\mathbf{d}_i,\mathbf{d}_j]}\big) \circ \gamma_{[\mathbf{d}_j,\mathbf{r}_n]} \,.$$

Here, $\rho_{\pi/2,\mathbf{d}_i} : \mathbb{R}^d \to \mathbb{R}^d$ is the orthogonal symmetry of $\mathbb{R}^d$ given by a clockwise rotation of angle $\pi/2$ about $\big(x(\gamma_{\mathbf{d}_i}), y(\gamma_{\mathbf{d}_i})\big)$ in each hyperplane $\{z \in \mathbb{R}^d : z_3 = x_3, \ldots, z_d = x_d\}$. It is $\mathsf{StBr}_{i,j}(\gamma)$ which was illustrated in Figure 3; the figure shows how this walk is self-avoiding, a property which is a consequence of the definition of a diamond point.

We now check that $\mathsf{StBr}_{i,j}(\gamma)$ is a bridge for $i \in \big[\frac{\delta}{10}n, \frac{2\delta}{10}n\big]$ and $j \in \big[\frac{3\delta}{10}n, \frac{4\delta}{10}n\big]$. To do so, it suffices to confirm that for $0 \le k \le \mathbf{r}_n$,

$$(4.8) \qquad 0 \le y\big(\mathsf{StBr}_{i,j}(\gamma)_k\big) \le y\big(\mathsf{StBr}_{i,j}(\gamma)_{\mathbf{r}_n}\big).$$

We prove the second inequality only, the first one being proved in exactly the same way. For $\mathbf{d}_j \le k \le \mathbf{r}_n$,

$$y\big(\mathsf{StBr}_{i,j}(\gamma)_{\mathbf{r}_n}\big) \ge y\big(\mathsf{StBr}_{i,j}(\gamma)_k\big)$$

by construction. For $\mathbf{d}_i \le k \le \mathbf{d}_j$, we have

$$
\begin{aligned}
& y\big(\mathsf{StBr}_{i,j}(\gamma)_{\mathbf{r}_n}\big) \; - \; y\big(\mathsf{StBr}_{i,j}(\gamma)_k\big) \\
= \; & \Big(y\big(\mathsf{StBr}_{i,j}(\gamma)_{\mathbf{r}_n}\big) \; - \; y\big(\mathsf{StBr}_{i,j}(\gamma)_{\mathbf{d}_j}\big)\Big) \\
& + \Big(y\big(\mathsf{StBr}_{i,j}(\gamma)_{\mathbf{d}_j}\big) \; - \; y\big(\mathsf{StBr}_{i,j}(\gamma)_k\big)\Big) \\
\ge \; & \tfrac{\delta}{10}n - \mathbf{width}(\gamma_{[\mathbf{d}_i,\mathbf{d}_j]}) \; \ge \; \big(\tfrac{\delta}{10} - \varepsilon\big)n \; > \; \varepsilon n \,.
\end{aligned}
$$

Regarding the first inequality, we used property (3) of $\overline{\mathrm{SAB}}_n$ and $j \le \frac{4\delta}{10}n$ to find that there are at least $\frac{\delta}{10}n$ diamond points of $\gamma$ whose index lies between $\mathbf{d}_j$ and $\mathbf{r}_n$, and thus bounded the first term; in the second inequality, the second term was bounded by using property (2). For $0 \le k \le \mathbf{d}_i$, we get

$$
\begin{aligned}
& y\big(\mathsf{StBr}_{i,j}(\gamma)_{\mathbf{r}_n}\big) \; - \; y\big(\mathsf{StBr}_{i,j}(\gamma)_k\big) \\
= \; & \Big(y\big(\mathsf{StBr}_{i,j}(\gamma)_{\mathbf{r}_n}\big) \; - \; y\big(\mathsf{StBr}_{i,j}(\gamma)_{\mathbf{d}_i}\big)\Big) \\
& + \Big(y\big(\mathsf{StBr}_{i,j}(\gamma)_{\mathbf{d}_i}\big) \; - \; y\big(\mathsf{StBr}_{i,j}(\gamma)_k\big)\Big) \; > \; \varepsilon n \,,
\end{aligned}
$$

where the inequality follows from the last display with $k = d_i$, as well as the fact that $y\big(\mathsf{StBr}_{i,j}(\gamma)_{\mathbf{d}_i}\big) \ge y\big(\mathsf{StBr}_{i,j}(\gamma)_k\big)$.

Let

$$\Phi \; = \; \big[\tfrac{\delta}{10}n, \tfrac{2\delta}{10}n\big] \; \times \; \big[\tfrac{3\delta}{10}n, \tfrac{4\delta}{10}n\big] \; \times \; \overline{\mathrm{SAB}}_n^+ \,.$$

We introduce the quantity

$$S := \sum_{(i,j,\gamma)\in\Phi} \mu_c^{-|\mathsf{StBr}_{i,j}(\gamma)|} \,.$$

One can express $S$ in terms of $\mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{N}}(\overline{\mathrm{SAB}}_n)$. Indeed, $|\mathsf{StBr}_{i,j}(\gamma)| = |\gamma|$, and therefore

$$(4.9) \qquad S = \sum_{(i,j,\gamma)\in\Phi} \mu_c^{-|\gamma|} = \left(\frac{\delta n}{10}\right)^2 \sum_{\gamma\in\overline{\mathrm{SAB}}_n^+} \mu_c^{-|\gamma|} = \left(\frac{\delta n}{10}\right)^2 \mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{N}}(\overline{\mathrm{SAB}}_n).$$

In the last equality, we used (4.7). However, $S$ can be expressed in another way, which we now present.

First note that the width of $\mathsf{StBr}_{i,j}(\gamma)$ is larger than $\frac{\delta}{10}n > \varepsilon n$ (since the height of the rotated piece is larger than its number of diamond points). Furthermore, every renewal point of $\mathsf{StBr}_{i,j}(\gamma)$ corresponds to a renewal point of $\gamma$, except for renewal points possibly created whose index lies between $\mathbf{d}_i$ and $\mathbf{d}_j$. There are at most $\mathbf{width}(\gamma_{[\mathbf{d}_i,\mathbf{d}_j]}) \leq \varepsilon n$ of them. But at least $\frac{\delta}{10}n$ renewal points (in fact even diamond points) were destroyed in the rotation of the central part, so that $\mathsf{StBr}_{i,j}(\gamma)$ contains at most $n+\varepsilon n - \frac{\delta}{10}n \leq n$ renewal points. All together, we find that if $\mathsf{StBr}_{i,j}(\gamma) \triangleleft \phi$ holds for some $\phi \in \Omega^+$, then $r_n(\phi) \geq |\mathsf{StBr}_{i,j}(\gamma)|$ and thus $\mathbf{width}(\phi_{[0,\mathbf{r}_n(\phi)]}) > \varepsilon n$. We deduce that, for any $(i,j,\gamma) \in \Phi$,

$$\mu_c^{-|\mathsf{StBr}_{i,j}(\gamma)|}$$
$$= \mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{N}}\big(\phi \in \Omega^+ : \mathsf{StBr}_{i,j}(\gamma) \triangleleft \phi\big)$$
$$= \mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{N}}\big(\phi \in \Omega^+ : \mathsf{StBr}_{i,j}(\gamma) \triangleleft \phi \text{ and } \mathbf{width}(\phi_{[0,\mathbf{r}_n(\phi)]}) > \varepsilon n\big).$$

Therefore,

$$S = \sum_{(i,j,\gamma)\in\Phi} \mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{N}}\big(\phi \in \Omega^+ : \mathsf{StBr}_{i,j}(\gamma) \triangleleft \phi \text{ and } \mathbf{width}(\phi_{[0,\mathbf{r}_n(\phi)]}) > \varepsilon n\big)$$
$$= \mathbb{E}_{\mathrm{iSAB}}^{\otimes\mathbb{N}}\Big(\big|\{(i,j,\gamma) \in \Phi : \mathsf{StBr}_{i,j}(\gamma) \triangleleft \phi\}\big| \cdot \mathbf{1}_{\{\mathbf{width}(\phi_{[0,\mathbf{r}_n(\phi)]})>\varepsilon n\}}\Big)$$
$$\leq (2\nu n)^2 \,\mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{N}}\big(\mathbf{width}(\phi_{[0,\mathbf{r}_n(\phi)]}) > \varepsilon n\big).$$
(4.10)

The inequality here made use of the fact that, for any given $\phi \in \Omega^+$, the number of elements $(i,j,\gamma)$ of $\Phi$ that satisfy $\mathsf{StBr}_{i,j}(\gamma) \triangleleft \phi$ is at most $(2\nu n)^2$. To see this bound, note that the solution of this reconstruction problem is uniquely determined once we know the indices in $\phi$ of the westerly and easterly endpoints of the first and second one-edge walks $\tau^{e_1}$ used in the stickbreak operation. Each of these indices is at most $\mathbf{d}_{\delta n/2}(\gamma) \leq \nu n \leq 2\nu n$ by virtue of $\gamma \in \overline{\mathrm{SAB}}_n^+$; hence the desired bound.

The two inferences (4.9) and (4.10) about $S$ together yield (4.5). This concludes the proof. $\qquad\square$

**Remark 4.3.** *Conjecturally, the law of the length of an irreducible bridge belongs to the domain of attraction of a stable law. This prediction arises from the law having infinite mean, and the scale invariance of the conjectural scaling limit. We refer to [1] for a discussion of the distribution of the scaled renewal points in dimension two.*

## 5. Deriving Corollaries 1.2 and 1.3

Note first that Corollary 1.2 is a trivial consequence of Theorem 1.1.

**Sketch of proof of Corollary 1.3.** We explain how to reduce the proof to a standard application of the Ornstein-Zernike theory developed for subcritical self-avoiding walk by [16]. Fix $K > 0$. Instead of working on $\mathcal{O}_\delta$ with $\delta \searrow 0$, we expand the domain $\mathcal{O}$ by a factor of $n$ and work on $\mathbb{Z}^d$. Without loss of generality, assume that $||a - b|| = 1$. We write $\widehat{an}$ and $\widehat{bn}$ for sites of $n\mathcal{O} \cap \mathbb{Z}^d$ closest to $na$ and $nb$. By Proposition 2.1,

$$(5.1) \qquad \mathbb{P}_{(n\mathcal{O}, \widehat{an}, \widehat{bn}, \mu_c^{-1})}(|\gamma| \leq Kn) \leq \frac{\sum_{n \leq k \leq Kn} e^{C\sqrt{k}} \mathsf{P}_{\mathsf{SAB}k}\big(||\gamma_k|| \geq k/K\big)}{\sum_{\gamma \subseteq n\mathcal{O}:\, \gamma_0 = \widehat{an},\, \gamma_{|\gamma|} = \widehat{bn}} \mu_c^{-|\gamma|}}.$$

Theorem 1.1 implies that the numerator of (5.1) decays exponentially fast. Therefore, it suffices to prove that the denominator does not decay exponentially fast. Fixing $\delta > 0$, we wish to show that

$$(5.2) \qquad \sum_{\gamma \subseteq n\mathcal{O}:\, \gamma_0 = \widehat{an},\, \gamma_{|\gamma|} = \widehat{bn}} \mu_c^{-|\gamma|} \geq \exp\big\{-\delta n\big\}$$

for $n$ large enough. We mention in passing that (5.2) has been established in [20, Theorem 2.1] for the case that $\mathcal{O}$ is a hypercube and $a$ and $b$ its opposite corners. In the spirit of the proof of [16, Theorem 1.1], it may be argued that, for $z < \mu_c^{-1}$,

$$\sum_{\gamma \subseteq n\mathcal{O}:\, \gamma_0 = \widehat{an},\, \gamma_{|\gamma|} = \widehat{bn}} z^{|\gamma|} \geq \exp\Big\{-\mathbf{length}_z(\Omega, a, b)n(1 + o_{n\to\infty}(1))\Big\},$$

where $\mathbf{length}_z(\Omega, a, b)$ is the length of the geodesic from $a$ to $b$ in $\Omega$ for the norm associated to the correlation length of self-avoiding walk of parameter $z$. (See (1.3) of [16] for the definition of this norm, which is roughly inversely proportional to the correlation length.) That this correlation length diverges as $z \nearrow \mu_c^{-1}$ follows from the divergence of $\sum_{\gamma \in \mathrm{SAW}} z^{|\gamma|}$ in this limit (a fact which we noted in the proof of Lemma 2.2). Thus, the parameter $z$ can be chosen close enough to $\mu_c^{-1}$ that $\mathbf{length}_z(\Omega, a, b) < \delta$. We thus obtain (5.2). $\square$

## References

[1] Tom Alberts and Hugo Duminil-Copin. Bridge decomposition of restriction measures. *J. Stat. Phys.*, 140(3):467–493, 2010.

[2] R. Bauerschmidt, H. Duminil-Copin, J. Goodman, and G. Slade. Lectures on self-avoiding walks. In D. Ellwood, C. Newman, V. Sidoravicius, and W. Werner, editors, *Lecture notes, in Probability and Statistical Physics in Two and More Dimensions.* CMI/AMS – Clay Mathematics Institute Proceedings, 2011.

[3] N. Beaton, M. Bousquet-Mélou, J. De Gier, H. Duminil-Copin, and A. Guttmann. The critical fugacity for surface adsorption of self-avoiding walks on the honeycomb lattice is $1 + \sqrt{2}$. arxiv:1109.0358, 2013.

[4] Christian Borgs, Jennifer Chayes, Christopher King, and Neal Madras. Anisotropic self-avoiding walks. *J. Math. Phys.*, 41(3):1321–1337, 2000. Probabilistic techniques in equilibrium and nonequilibrium statistical physics.

[5] D. C. Brydges, J.Z. Imbrie, and G. Slade. Functional integral representations for self-avoiding walk. *Probab. Surv.*, 6:34–61, 2009.

[6] David Brydges and Gordon Slade. Renormalisation group analysis of weakly self-avoiding walk in dimensions four and higher. In *Proceedings of the International Congress of Mathematicians. Volume IV*, pages 2232–2257, New Delhi, 2010. Hindustan Book Agency.

[7] David Brydges and Thomas Spencer. Self-avoiding walk in 5 or more dimensions. *Comm. Math. Phys.*, 97(1-2):125–148, 1985.

[8] H. Duminil-Copin, G. Kozma, and A. Yadin. Supercritical self-avoiding walks are space-filling. to appear in AIHP, arXiv:1110.3074, 2011.

[9] H. Duminil-Copin and S. Smirnov. The connective constant of the honeycomb lattice equals $\sqrt{2 + \sqrt{2}}$. *Ann. of Math.*, 175(3):1653–1665, 2012.

[10] Bertrand Duplantier. Fractals in two dimensions and conformal invariance. *Phys. D*, 38(1-3):71–87, 1989. Fractals in physics (Vence, 1989).

[11] Bertrand Duplantier. Renormalization and conformal invariance for polymers. In *Fundamental problems in statistical mechanics VII (Altenberg, 1989)*, pages 171–223. North-Holland, Amsterdam, 1990.

[12] P. Flory. *Principles of Polymer Chemistry.* Cornell University Press, 1953.

[13] J. M. Hammersley and D. J. A. Welsh. Further results on the rate of convergence to the connective constant of the hypercubical lattice. *Quart. J. Math. Oxford Ser. (2)*, 13:108–110, 1962.

[14] T. Hara and G. Slade. Critical behaviour of self-avoiding walk in five or more dimensions. *Bull. Amer. Math. Soc. (N.S.)*, 25(2):417–423, 1991.

[15] T. Hara and G. Slade. Self-avoiding walk in five or more dimensions. I. The critical behaviour. *Comm. Math. Phys.*, 147(1):101–136, 1992.

[16] D. Ioffe. Ornstein-Zernike behaviour and analyticity of shapes for self-avoiding walks on $\mathbf{Z}^d$. *Markov Process. Related Fields*, 4(3):323–350, 1998.

[17] Harry Kesten. On the number of self-avoiding walks. *J. Mathematical Phys.*, 4:960–969, 1963.

[18] Harry Kesten. On the number of self-avoiding walks. II. *J. Mathematical Phys.*, 5:1128–1137, 1964.

[19] G. F. Lawler, O. Schramm, and W. Werner. On the scaling limit of planar self-avoiding walk. In *Fractal geometry and applications: a jubilee of Benoît Mandelbrot, Part 2*, volume 72 of *Proc. Sympos. Pure Math.*, pages 339–364. Amer. Math. Soc., Providence, RI, 2004.

[20] N. Madras. Critical behaviour of self-avoiding walks that cross a square. *J. Phys. A*, 28(6):1535–1547, 1995.

[21] N. Madras and G. Slade. *The self-avoiding walk*. Probability and its Applications. Birkhäuser Boston Inc., Boston, MA, 1993.

[22] Neal Madras and C. Chris Wu. Self-avoiding walks on hyperbolic graphs. *Combin. Probab. Comput.*, 14(4):523–548, 2005.

[23] B. Nienhuis. Exact critical point and critical exponents of $O(n)$ models in two dimensions. *Phys. Rev. Lett.*, 49:1062–1065, 1982.

[24] B. Nienhuis. Coulomb gas description of 2D critical behaviour. *J. Statist. Phys.*, 34:731–761, 1984.

[25] W.J.C. Orr. Statistical treatment of polymer solutions at infinite dilution. *Transactions of the Faraday Society*, 43:12–27, 1947.

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