

# SHARP PHASE TRANSITION IN THE RANDOM STIRRING MODEL ON TREES

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ABSTRACT. We establish that the phase transition for infinite cycles in the random stirring model on an infinite regular tree of high degree is sharp. That is, we prove that there exists  $d_0$  such that, for any  $d \geq d_0$ , the set of parameter values at which the random stirring model on the rooted regular tree with offspring degree  $d$  almost surely contains an infinite cycle consists of a semi-infinite interval. The critical point at the left-hand end of this interval is at least  $d^{-1} + \frac{1}{2}d^{-2}$  and at most  $d^{-1} + 2d^{-2}$ .

## 1. INTRODUCTION

Suppose given a graph  $G = (V(G), E(G))$ . To each edge  $e \in E(G)$  is associated an independent Poisson process of rate one on  $[0, \infty)$ . The random stirring model on  $G$  is a stochastic process  $\sigma$  defined on  $[0, \infty)$  and taking values in permutations of  $V(G)$ . The initial condition  $\sigma_0$  is the identity permutation. As time  $t$  increases, on each occasion that a point  $(e, t) \in E(G) \times [0, 1)$  of one of the Poisson processes is encountered,  $\sigma$  is instantaneously modified by composing with the transposition of the two vertices incident to the edge  $e$ .

Let  $d \geq 2$ . Let  $\mathcal{T}$  denote the rooted regular tree of offspring degree  $d$ , and let  $\phi$  denote the root of  $\mathcal{T}$ . Our main theorem shows that, if  $d$  is high, the random stirring model on  $\mathcal{T}$  has a critical value for the transition to infinite cycles. The theorem confirms Conjecture 9 of [2] for such trees.

**Theorem 1.1.** *Suppose that  $d \geq 1640$ . There exists  $T_c(d) \in (0, \infty)$  such that number of vertices in the cycle of  $\phi$  in  $\sigma_t$  is almost surely finite if  $t < T_c(d)$  and is infinite with positive probability if  $t > T_c(d)$ . For such  $d$ ,  $T_c(d) \in [d^{-1} + \frac{1}{2}d^{-2}, d^{-1} + 2d^{-2}]$ .*

*For each  $\varepsilon > 0$ , there exists  $d' \in \mathbb{N}$  such that for  $d \geq d'$ ,  $T_c(d)$  exists and satisfies  $T_c(d) \in [d^{-1} + \frac{1}{2}d^{-2}, d^{-1} + (\frac{7}{6} + \varepsilon)d^{-2}]$ .*

As [2] mentions, on a regular tree, it is simple to see that, for each  $t \in [0, \infty)$ , there being positive probability that the cycle of  $\phi$  under  $\sigma_t$  is infinite is equivalent to the almost sure existence of some infinite cycle under  $\sigma_t$ .

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**1.1. Glossary of notation.** Here we list the notation which is commonly used in the article. A summarizing phrase is provided for each item, as well as the page number at which the concept is introduced.

$\mathcal{T}$	the rooted regular tree with offspring degree $d$	1
$\phi$	the root of $\mathcal{T}$	1
$e^+, e^-$	the parent and child endpoint vertices of an edge $e \in E(\mathcal{T})$	3
the pole at $v$	for $v \in V(\mathcal{T})$ , the set $\{v\} \times [0, 1)$	3
bar	element of $E(\mathcal{T}) \times [0, 1)$	3
$E(b)$	the edge on which the bar $b$ is supported	3
$b^+, b^-$	the parent and child joints of a bar $b$	3
$X_{(v,s)}^{\mathcal{B}}$	cyclic-time random meander from $(v, s) \in V(\mathcal{T}) \times [0, 1)$	3
$X^{\mathcal{B}}$	shorthand for $X_{(\phi,0)}^{\mathcal{B}}$	3
$\mathcal{B}$	bar collection with Poisson- $t$ law on $E(\mathcal{T}) \times [0, 1)$ under $\mathbb{P}_t$	4
$Y^{\mathcal{B}}$	the vertex component of $X^{\mathcal{B}}$	4
$p_\infty$	$\mathbb{P}_t$ -probability that $X^{\mathcal{B}}$ never returns to $(\phi, 0)$	6
$\mathcal{V}_i$	$\{v \in V(\mathcal{T}) : d(\phi, v) = i\}$	6
$\mathcal{E}_i$	$\{e \in E(\mathcal{T}) : d(\phi, e^+) = i\}$	6
$\mathcal{T}_n$	the subtree of $\mathcal{T}$ induced by vertices at distance at most $n$ from $\phi$	6
$P_{\phi,v}$	the path in $\mathcal{T}$ from $\phi$ to $v \in V(\mathcal{T})$	6
$\mathcal{T}_{[v]}, \mathcal{T}^{[v]}$	the descendent tree of $v \in V(\mathcal{T})$ ; $\mathcal{T}$ without $\mathcal{T}_{[v]}$	7
$H_n^{\mathcal{B}}$	the hitting time of $\mathcal{V}_n \times [0, 1)$ by $X^{\mathcal{B}}$	7
$p_n$	$\mathbb{P}_t(H_n^{\mathcal{B}} < \infty)$	7
$\mathcal{A}$	the added bar, with uniform law on $E(\mathcal{T}_n) \times [0, 1)$	7
$X^{\mathcal{B} \cup \mathcal{A}}$	shorthand for $X^{\mathcal{B} \cup \{\mathcal{A}\}}$	7
$P^+$	the on-pivotal event $\{H_n^{\mathcal{B}} = \infty\} \cap \{H_n^{\mathcal{B} \cup \mathcal{A}} < \infty\}$	7
$P^-$	the off-pivotal event $\{H_n^{\mathcal{B}} < \infty\} \cap \{H_n^{\mathcal{B} \cup \mathcal{A}} = \infty\}$	7
$C$	the crossing event: $X^{\mathcal{B}}[0, H_n^{\mathcal{B}}] \cap \{\mathcal{A}^+, \mathcal{A}^-\} \neq \emptyset$	8
$e_{\text{BN}}$	bottleneck edge: last $e \in E(P_{\phi, E(\mathcal{A}^+)})$ supporting unique bar in $\mathcal{B}$	9
$\text{BN}$	the event that $e_{\text{BN}}$ exists	9
$b_{\text{BN}}$	bottleneck bar, the unique bar in $\mathcal{B}$ supported on $e_{\text{BN}}$	9
$\text{NoEsc}$	the non-escape event: $X_{b_{\text{BN}}^+}^{\mathcal{B}}$ visits $(\phi, 0)$ before $\mathcal{V}_n \times [0, 1)$	9
$\text{Found}_t$	the set of bars in $\mathcal{B}$ crossed by $X^{\mathcal{B}}$ during $[0, t]$	10
$\text{UnTouch}_t$	<i>untouched</i> bars: $\{b \in E(\mathcal{T}) \times [0, 1) : \{b^+, b^-\} \cap X^{\mathcal{B}}[0, t] = \emptyset\}$	10
$\tau$	$td^{-1}$	10
$\text{NoBar}$	the event that $\mathcal{B} \cap (\mathcal{E}_0 \times [0, 1)) = \emptyset$	11
$\mathcal{M}_v$	the multi-cluster (a set of edges) associated to $v \in V(\mathcal{T})$	12
$\text{FB}$	far-from-boundary event, $\{d(\phi, e_{\text{BN}}^-) \leq n - 2n_1\}$	15
$\text{CB}$	close-to-boundary event, $\text{BN} \setminus \text{FB}$	15
$\text{High}$	$\{e \in E(\mathcal{T}_n) : d(\phi, e^-) \leq n - 2n_1\}$	15
$\text{ViLoc}(\mathcal{B})$	$\{b \in E(\mathcal{T}_n) \times [0, 1) : \{b^+, b^-\} \cap X^{\mathcal{B}}[0, H_n^{\mathcal{B}}] \neq \emptyset, E(P_{\phi, E(b^+)}) \subseteq \mathcal{M}_\phi\}$	16
$\mathbb{P}_{t, \mathcal{B}}$	the marginal law of $\mathcal{B}$ under $\mathbb{P}_t$ , i.e., Poisson- $t$	17
$\partial_{\text{ext}} G$	the exterior boundary (a set of edges) of $G \subseteq E(\mathcal{T})$	18
$H_{\mathcal{A}}^{\mathcal{B}}$	$\inf \{s > 0 : X^{\mathcal{B}}(s) \in \{\mathcal{A}^+, \mathcal{A}^-\}\}$	28

**1.2. Cyclic-time random meander and walk.** Our analysis of the random stirring model exploits a closely related dependent random walk that Omer Angel in [2] called the cyclic-time random walk. We now introduce some notation and define this walk.

For each edge  $e \in E(\mathcal{T})$ , the incident vertex of  $e$  closer to  $\phi$  will be called the parent vertex and will be denoted by  $e^+$ ; the other, called the child vertex and labelled  $e^-$ .

For convenience, suppose that  $\mathcal{T}$  is embedded in  $\mathbb{R}^2$ , so that each element of  $V(\mathcal{T})$  is identified with a point in  $\mathbb{R}^2$  and each element  $e \in E(\mathcal{T})$  with the line segment  $[v_1, v_2] \subseteq \mathbb{R}^2$  where  $e = (v_1, v_2)$  for  $v_1, v_2 \in V(\mathcal{T})$ . For each  $v \in V(\mathcal{T})$ , let the pole at  $v$ ,  $\{v\} \times [0, 1] \subseteq \mathbb{R}^3$ , denote the unit line segment that rises vertically from  $v$ . Elements of  $E(\mathcal{T}) \times [0, 1]$  will be called bars. A bar  $b = (e, h)$  is said to be supported on the edge  $e$  and to have height  $h$ ; we also record the edge on which  $b$  is supported as  $E(b)$ . Note that the bar  $(e, h)$  is a horizontal line segment which intersects the poles at  $e^+$  and  $e^-$ ; the intersection points  $(e^+, h)$  and  $(e^-, h)$  will be called the parent and child joints of  $(e, h)$ .

The bar set  $E(\mathcal{T}) \times [0, 1]$  carries the product of counting and Lebesgue measure on its components. (As a shorthand, we will refer to this product measure simply as Lebesgue measure.)

Let  $\mathcal{B}_0 \subseteq E(\mathcal{T}) \times [0, 1]$  be a collection of bars. Cyclic-time random meander  $X_{(v,h)}^{\mathcal{B}_0} : [0, \infty) \rightarrow V(\mathcal{T}) \times [0, 1]$ , among  $\mathcal{B}_0$  and with initial condition  $(v, h) \in V(\mathcal{T}) \times [0, 1]$ , is the following process. First,  $X_{(v,h)}^{\mathcal{B}_0}(0) = (v, h)$ ; the process then rises at unit speed on the pole at  $v$  until either it reaches  $(v, 1)$ , when it jumps to  $(v, 0)$ , or until it reaches the joint of a bar in  $\mathcal{B}_0$ , when it jumps to the other joint of this bar. After either of these events,  $X_{(v,h)}^{\mathcal{B}_0}$  continues by iterating the same rule, until it is defined on all of  $[0, \infty)$ . The process is chosen to be right-continuous with left limits. Note that this choice implies that, if  $(v, h)$  is the joint of a bar  $b$  in  $\mathcal{B}_0$ , then  $X_{(v,h)}^{\mathcal{B}_0}$  remains at the pole at  $v$  at small times, rather than crossing  $b$  at time zero. We abbreviate  $X^{\mathcal{B}_0}$  for  $X_{(\phi,0)}^{\mathcal{B}_0}$ . (There are choices of  $\mathcal{B}_0$  for which these rules fail to define  $X_{(v,t)}^{\mathcal{B}_0}$  on all of  $[0, \infty)$ . It is a simple matter to verify that this difficulty does not arise in the case that is relevant to us and which we now discuss.)

Let  $s \in (0, \infty)$ . We will refer to the Poisson law on bar collections of intensity measure  $s$  with respect to Lebesgue measure on  $E(\mathcal{T}) \times [0, 1]$  as the Poisson- $s$  law. Let  $\{\mathcal{B}_s : s \geq 0\}$  be a coupled collection of random bar collections, where  $\mathcal{B}_s$  has the Poisson- $s$  law for each  $s \in [0, \infty)$  and  $\mathcal{B}_s \subseteq \mathcal{B}_{s'}$  whenever  $0 \leq s < s' < \infty$ . Define  $\sigma_t : V(\mathcal{T}) \rightarrow V(\mathcal{T})$  by setting  $\sigma_t(v)$  equal to the vertex component of  $X_{(v,0)}^{\mathcal{B}_t}(1)$ , and note that  $\sigma_t$  is a random

permutation of  $V(\mathcal{T})$ . With this new notation, the random stirring process on  $\mathcal{T}$  is the stochastic process, mapping  $[0, \infty)$  to permutations of  $V(\mathcal{T})$ , given by  $s \rightarrow \sigma_s$ .

We now fix  $t \in (0, \infty)$  and write  $\mathbb{P}_t$  for a probability measure carrying a bar collection  $\mathcal{B} \subseteq E(\mathcal{T}) \times [0, 1)$  having Poisson- $t$  law. Cyclic-time random meander with parameter  $t$  is the random process  $X^{\mathcal{B}}$ .

Cyclic-time random walk (begun at  $\phi$ ) is the vertex-valued process given by projecting  $X^{\mathcal{B}} : [0, \infty) \rightarrow V(\mathcal{T}) \times [0, 1)$  onto  $V(\mathcal{T})$ . We denote it by  $Y^{\mathcal{B}}$ . (In fact, under our definition, cyclic-time random walk moves at a rate which is a factor of  $t$  greater than it does under the definition in [2].) We will more commonly discuss  $X^{\mathcal{B}}$  than  $Y^{\mathcal{B}}$ , and will refer to  $X^{\mathcal{B}}$  in shorthand as a meander. See Figure 1 for an illustration.

**1.3. Different perspectives on the random stirring model.** The random stirring (or random interchange) model was introduced in [8]. Its physical relevance was indicated by Bálint Tóth [12], who used it to give a representation of the spin-1/2 Heisenberg ferromagnet; the lecture notes [5] contain an overview of this topic. Recent mathematical progress on the model includes the resolution of Aldous' conjecture identifying its spectral gap [4], and a formula for the probability that the random permutation consists of a single cycle [1].

The emergence of a giant component under percolation on the complete graph as the percolation parameter increases through values near  $1/n$  has been intensively studied. Oded Schramm [11] showed that this transition is accompanied by the appearance of large-scale cycles in the associated random stirring model: in the composition of  $(1 + \varepsilon)n$  independent uniform transposition on a given  $n$ -set, there exists a giant component of edges transposed at least once, of some density  $\theta(\varepsilon) \in (0, 1)$ ; when the cycle lengths in this random permutation are normalized by  $\theta(\varepsilon)n$  and listed in decreasing order, they converge in law to the Poisson-Dirichlet distribution with parameter one. Nathanaël Berestycki [3] has given a short proof that a cycle exists of size  $\Theta(n)$  when  $(1 + \varepsilon)n$  transpositions are made.

**1.4. Monotonicity near the transition.** For any given graph  $G$  on which the random stirring model is well-defined, let  $\mathcal{T}^G$  denote the set of  $t > 0$  such that the random stirring process on  $G$  at parameter  $t$  has infinite cycles almost surely. Note that  $t \notin \mathcal{T}^G$  unless the bond percolation on  $G$  given by the set of edges that support at least one bar in  $\mathcal{B}$  has an infinite component. As noted in [2], this implies that  $[0, -\log(1 - p_c(G))] \cap \mathcal{T}^G = \emptyset$ , where  $p_c(G)$  denotes the critical value for bond percolation on  $G$ . Noting that  $p_c(\mathcal{T}) = d^{-1}$ , we find that

$$[0, d^{-1} + \frac{1}{2}d^{-2}] \cap \mathcal{T}^{\mathcal{T}} = \emptyset \quad (1.1)$$

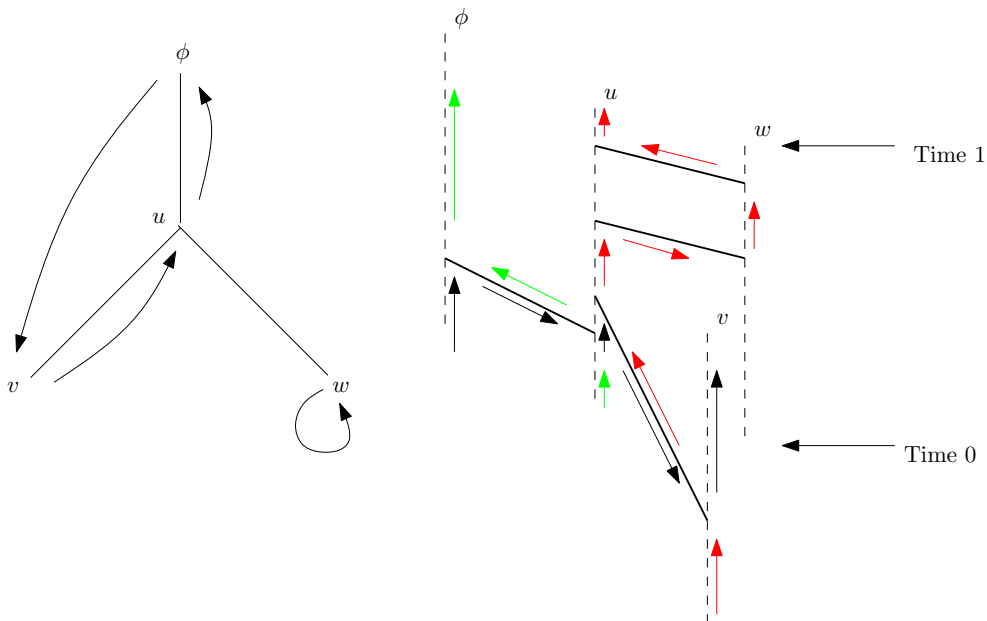


FIGURE 1. For the graph shown on the left, cyclic-time random meander  $X^{\mathcal{B}}$  departing from  $(\phi, 0)$  is illustrated on the right. The right-hand sketch depicts a construction in  $\mathbb{R}^3$  in which the poles associated to vertices are the vertical dashed lines and the bars in  $\mathcal{B}$  are the horizontal black lines. Assume that there are no bars in  $\mathcal{B}$  supported on edges that connect vertices  $v$  and  $w$  of  $\phi$  to their offspring. The trajectory of the meander from  $(\phi, 0)$  is divided into three intervals of unit duration, at the end of which, the meander returns to  $(\phi, 0)$ . These three sub-trajectories are indicated in black, red and green in the right-hand sketch. As the left-hand sketch shows, the cycle of  $\phi$  in the associated permutation thus has three elements.

if  $d \geq 8$ . The papers [2] and [7] provide two different approaches to proving the existence of infinite cycles in the random stirring process. The appendix in [7] draws on these approaches to provide the following quantitative summary. See Figure 2 for an overview of which ranges of  $t$  are handled by the two techniques.

**Theorem 1.2.** *If  $d \geq 1286$  then  $[d^{-1} + 2d^{-2}, \infty) \subseteq \mathcal{I}^{\mathcal{T}}$ . For each  $\varepsilon > 0$ , there exists  $d'(\varepsilon)$  such that if  $d \geq d'$  then  $[d^{-1} + (\frac{7}{6} + \varepsilon)d^{-2}, \infty) \subseteq \mathcal{I}^{\mathcal{T}}$ .*

**Proof of Theorem 1.1.** In light of (1.1) and Theorem 1.2, Theorem 1.1 is reduced to the next two results.  $\square$

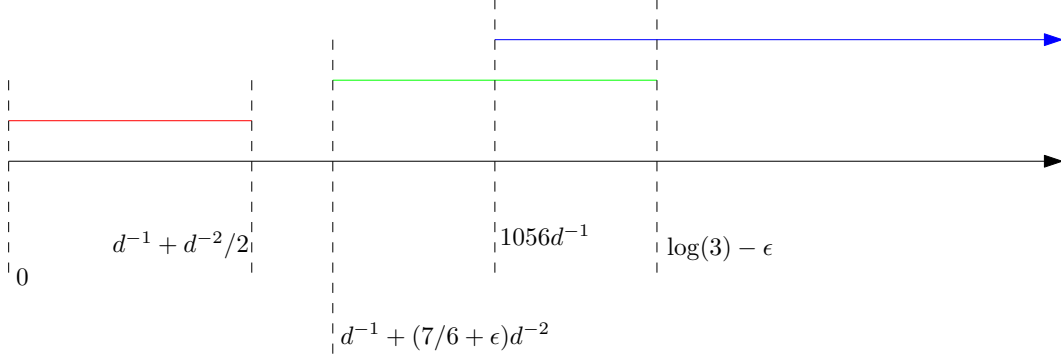


FIGURE 2. The red zone is disjoint from  $\mathcal{T}^T$  by (1.1). Angel’s argument [2] for infinite cycles works well at small  $t$  values, and proves that the green zone is contained in  $\mathcal{T}^T$  if  $d$  is high. The argument in [7] is valid for all high enough  $t$ , and shows that the blue zone is contained in  $\mathcal{T}^T$  if  $d$  is high.

**Proposition 1.3.** *Suppose that  $d \geq 1640$ . Let  $0 < s < s' \leq d^{-1} + 2d^{-2}$ . If  $s \in \mathcal{T}^T$  then  $s' \in \mathcal{T}^T$ .*

In fact, our monotonicity result is valid on a slightly longer interval, as we now record.

**Proposition 1.4.** *There exists  $d_0 \in \mathbb{N}$  such that, for  $d \geq d_0$ , if  $0 < s < s' \leq \frac{1}{7}d^{-1} \log d$ , then  $s \in \mathcal{T}^T$  implies  $s' \in \mathcal{T}^T$ .*

Cyclic-time random meander  $X^B$  is the tool that we will use to prove Propositions 1.3 and 1.4. Let  $p_\infty = p_\infty(t)$  denote the  $\mathbb{P}_t$ -probability that  $(\phi, 0) \notin X^B(0, \infty)$ . Note that in the random stirring model at parameter  $t$ , the cycle of  $\phi$  is infinite with probability  $p_\infty(t)$ .

We will prove Proposition 1.3 by establishing that  $p_\infty : [0, d^{-1} + 2d^{-2}] \rightarrow [0, 1]$  is non-decreasing (for high enough  $d$ ). To do so, we will work with local approximations  $\{p_n : n \in \mathbb{N}\}$  for  $p_\infty$ . To define these, we need some notation for describing the graph  $\mathcal{T}$ . We pause to collect together such general notation.

**Definition 1.5.** *We write  $d(\cdot, \cdot) : V(\mathcal{T}) \times V(\mathcal{T}) \rightarrow \mathbb{N}$  for graphical distance on  $\mathcal{T}$ . For  $i \in \mathbb{N}$ , set  $\mathcal{V}_i = \{v \in V(\mathcal{T}) : d(\phi, v) = i\}$  and  $\mathcal{E}_i = \{e \in E(\mathcal{T}) : d(\phi, e^+) = i\}$ . For  $n \in \mathbb{N}$ , let  $\mathcal{T}_n$  denote the subtree of  $\mathcal{T}$  induced by vertices at distance at most  $n$  from  $\phi$ . For  $v, w \in V(\mathcal{T})$ , let  $P_{v,w}$  denote the unique simple path in  $\mathcal{T}$  connecting  $v$  and  $w$ , and write  $E(P_{v,w})$  for its set of edges. For  $v \in V(\mathcal{T})$ , let  $\mathcal{T}_{[v]}$  denote the subtree of  $\mathcal{T}$  induced by descendants of  $v$  ( $\mathcal{T}_{[v]}$  may be viewed as a rooted tree with root  $v$ ). Let  $\mathcal{T}^{[v]}$  denote “ $\mathcal{T}$  above  $v$ ”, the*

subtree of  $\mathcal{T}$  induced by all elements of  $V(\mathcal{T})$  that are not strict descendants of  $v$ .

We also record notation for sizes of sets:

**Definition 1.6.** For  $\mathcal{B}_0 \subseteq E(\mathcal{T}_n) \times [0, 1)$  a set of bars, write  $|\mathcal{B}_0|$  for the Lebesgue measure of  $\mathcal{B}_0$ . For  $E \subseteq E(\mathcal{T}_n)$  a set of edges, write  $\#E$  for the cardinality of  $E$ .

For  $n \in \mathbb{N}$ , let  $H_n^{\mathcal{B}} \in [0, \infty]$  denote the hitting time  $\inf \{s > 0 : X^{\mathcal{B}}(s) \in \mathcal{V}_n \times [0, 1)\}$ . Let  $p_n = p_n(t)$  denote  $\mathbb{P}_t(H_n^{\mathcal{B}} < \infty)$ . Evidently,  $p_n$  decreases pointwise to  $p_\infty$ .

**1.5. Pivotality and the added bar.** The tool for deriving Propositions 1.3 and 1.4 is now stated. The derivative of  $p_n$  is expressed in terms of the mean effect on  $\mathbb{P}_t(H_n^{\mathcal{B}} < \infty)$  caused by adding to  $\mathcal{B}$  a single “uniformly” placed bar; the formula is analogous to Russo’s formula from percolation theory [6, Theorem 2.25].

**Definition 1.7.** Let  $n \in \mathbb{N}$ . Augment the probability space  $(\Omega, \mathbb{P}_t)$  to include a random bar  $\mathcal{A} = \mathcal{A}_n$  whose law is normalized Lebesgue measure on  $E(\mathcal{T}_n) \times [0, 1)$  and which is independent of  $\mathcal{B}$ . We call  $\mathcal{A}$  the added bar. We abuse notation by writing  $\mathcal{B} \cup \mathcal{A}$  for the bar collection  $\mathcal{B} \cup \{\mathcal{A}\}$ ; thus,  $X^{\mathcal{B} \cup \mathcal{A}}$  denotes cyclic-time random meander among  $\mathcal{B} \cup \{\mathcal{A}\}$ .

The on-pivotal event  $\mathbf{P}^+ = \mathbf{P}_n^+$  is defined to be  $\{H_n^{\mathcal{B}} = \infty, H_n^{\mathcal{B} \cup \mathcal{A}} < \infty\}$ , and the off-pivotal event  $\mathbf{P}^- = \mathbf{P}_n^-$  to be  $\{H_n^{\mathcal{B}} < \infty, H_n^{\mathcal{B} \cup \mathcal{A}} = \infty\}$ .

**Lemma 1.8.** For each  $n \in \mathbb{N}$ ,  $p_n : (0, \infty) \rightarrow [0, 1]$  is differentiable; for  $t > 0$ ,

$$\frac{dp_n(t)}{dt} = \#E(\mathcal{T}_n) \left( \mathbb{P}_t(\mathbf{P}^+) - \mathbb{P}_t(\mathbf{P}^-) \right). \quad (1.2)$$

**Proof.** Let  $\{\mathcal{B}_s : s \geq 0\}$  be a coupled system of random bar collections, where  $\mathcal{B}_s$  has the Poisson- $s$  law on  $E(\mathcal{T}_n) \times [0, 1)$ , and where  $\mathcal{B}_s \subseteq \mathcal{B}_{s'}$  whenever  $0 \leq s \leq s' < \infty$ . Let  $N_{s,s'} \in \mathbb{N}$  denote the cardinality of  $\mathcal{B}_{s'} \setminus \mathcal{B}_s$ . Note that

$$\begin{aligned} & \{H_n^{\mathcal{B}_t} < \infty\} \cup \{H_n^{\mathcal{B}_t} = \infty, H_n^{\mathcal{B}_{t+\varepsilon}} < \infty, N_{t,t+\varepsilon} = 1\} \cup \{N_{t,t+\varepsilon} \geq 2\} \\ &= \{H_n^{\mathcal{B}_{t+\varepsilon}} < \infty\} \cup \{H_n^{\mathcal{B}_t} < \infty, H_n^{\mathcal{B}_{t+\varepsilon}} = \infty, N_{t,t+\varepsilon} = 1\} \cup \{N_{t,t+\varepsilon} \geq 2\}. \end{aligned}$$

The first two sets in the union of the left-hand side are disjoint. Note that, conditionally on  $N_{t,t+\varepsilon} = 1$ , the unique element in  $\mathcal{B}_{t+\varepsilon} \setminus \mathcal{B}_t$  has the distribution of  $\mathcal{A}$ . Thus, taking expectations, we find that

$$\begin{aligned} & p_n(t) + \varepsilon \#E(\mathcal{T}_n) \exp \{ - \varepsilon \#E(\mathcal{T}_n) \} \mathbb{P}_t(\mathbf{P}^+) \\ & \leq p_n(t + \varepsilon) + \varepsilon \#E(\mathcal{T}_n) \exp \{ - \varepsilon \#E(\mathcal{T}_n) \} \mathbb{P}_t(\mathbf{P}^-) + (\varepsilon \#E(\mathcal{T}_n))^2. \end{aligned}$$

which implies that

$$p_n(t) + \varepsilon \#E(\mathcal{T}_n) \left( \mathbb{P}_t(\mathbf{P}^+) - \mathbb{P}_t(\mathbf{P}^-) \right) \leq p_n(t + \varepsilon) + 2\varepsilon^2 (\#E(\mathcal{T}_n))^2. \quad (1.3)$$

Similarly, the first two sets in the union of the right-hand side being disjoint, we find that

$$p_n(t) + \varepsilon \#E(\mathcal{T}_n) \left( \mathbb{P}_t(\mathbf{P}^+) - \mathbb{P}_t(\mathbf{P}^-) \right) + 2\varepsilon^2 (\#E(\mathcal{T}_n))^2 \geq p_n(t + \varepsilon). \quad (1.4)$$

From (1.3) and (1.4) follows the statement of the lemma.  $\square$

In light of Lemma 1.8, it is natural to aim to prove Propositions 1.3 and 1.4 by using the lemma to show that  $\frac{dp_n(t)}{dt} \geq 0$  for high  $n$  and when  $t$  is in the relevant range. In fact, the statement that we will establish is slightly weaker:

**Proposition 1.9.** *Let  $d \geq 1640$  and let  $\varepsilon > 0$ . There exists  $n_0 = n_0(d, \varepsilon) \in \mathbb{N}$  such that if  $n \geq n_0$  and  $0 < t \leq d^{-1} + 2d^{-2}$  satisfy  $p_n(t) \geq \varepsilon$ , then  $\frac{dp_n(t)}{dt} \geq 0$ . There exists  $d_0 \in \mathbb{N}$  such that if  $d \geq d_0$ , the same statement holds when “ $d^{-1} + 2d^{-2}$ ” is replaced by “ $\frac{1}{7}d^{-1} \log d$ ”.*

In the body of the article, we prove Proposition 1.9 for  $d \geq 3 \cdot 10^6$ . In the appendix, several of the lemmas are sharpened so that the cases  $d \geq 1640$  are included.

**Proof of Propositions 1.3 and 1.4.** Proposition 1.9 implies that its statement holds when the expression “ $\frac{dp_n(t)}{dt} \geq 0$ ” is replaced by “ $p_n$  is non-decreasing on  $[t, d^{-1} + 2d^{-2}]$ ”. The functions  $p_n$  decrease pointwise to  $p_\infty$ , so that  $p_\infty$  is also non-decreasing on  $[t, d^{-1} + 2d^{-2}]$  (or on  $[t, \frac{1}{7}d^{-1} \log d]$ ). However, this implies that  $p_\infty$  is non-decreasing on any interval  $[t, d^{-1} + 2d^{-2}]$  such that  $p_\infty(t) > 0$ . This yields the two propositions.  $\square$

The two scenarios depicted in Figure 3 show how the monotonicity  $\mathbb{P}_t(\mathbf{P}^+) \geq \mathbb{P}_t(\mathbf{P}^-)$  is not readily apparent: the appearance of  $\mathcal{A}$  may lengthen the trajectory of the meander from  $(\phi, 0)$ , so that  $\mathbf{P}^+$  occurs, or it may shorten this trajectory and force  $\mathbf{P}^-$ . To suggest our approach in a few words, we will argue that the coagulating mechanism causing  $\mathbf{P}^+$  is stronger than the fragmenting one causing  $\mathbf{P}^-$  when  $t \leq Cd^{-1}$  because, for such  $t$ , the bar collection  $\mathcal{B}$  is dilute: the added bar  $\mathcal{A}$  will probably arrive over an edge where no bar of  $\mathcal{B}$  is present, and then (as we will prove in the straightforward Lemma 4.15) the meander  $X^{\mathcal{B} \cup \mathcal{A}}$  follows either the same route as does  $X^{\mathcal{B}}$  or a longer one.

## 1.6. Two necessary conditions for pivotality of the added bar.

1.6.1. *The meander must encounter the added bar.* Let  $\mathbf{C} = \mathbf{C}_n$  denote the crossing event that  $X^{\mathcal{B}}$  meets a joint of  $\mathcal{A}$  before time  $H_n^{\mathcal{B}}$ . If  $\mathbf{C}$  does not occur, then the trajectories of  $X^{\mathcal{B}}$  and  $X^{\mathcal{B} \cup \mathcal{A}}$  are equal at least on the interval  $[0, H_n]$  (where the value of  $H_n$  is shared by the two processes); this proves the following fact.

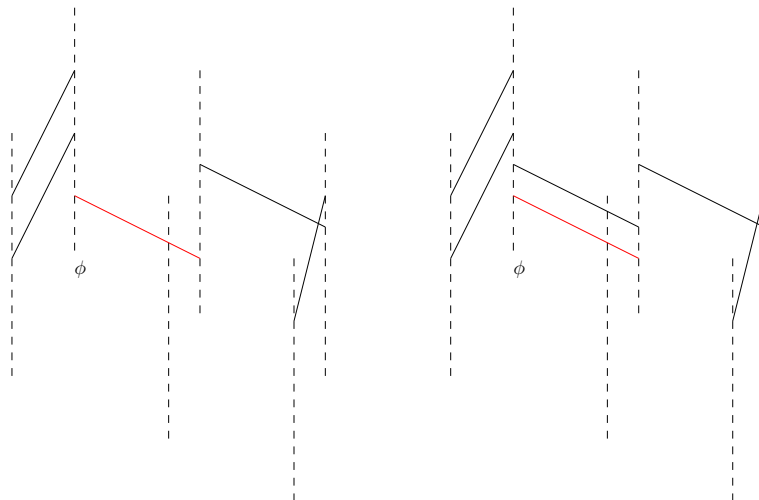


FIGURE 3. In each image, the bars in  $\mathcal{B}$  are black and the added bar  $\mathcal{A}$  is red. In the left-hand case, the meander  $X^{\mathcal{B}}$  from  $(\phi, 0)$  completes a circuit in time one; when  $\mathcal{A}$  appears, the meander  $X^{\mathcal{B} \cup \mathcal{A}}$  makes a longer journey, perhaps never returning to  $(\phi, 0)$ . In the right-hand case, the appearance of  $\mathcal{A}$  has the opposite effect, curtailing the trajectory of the meander from  $(\phi, 0)$ .

**Lemma 1.10.** *We have that*

$$\mathbf{P}^+ \cup \mathbf{P}^- \subseteq \mathbf{C}.$$

1.6.2. *No escape above the bottleneck bar.* In the case that  $\mathbf{C}$  occurs, we now provide a further necessary condition for the occurrence of  $\mathbf{P}^+ \cup \mathbf{P}^-$ . If  $\mathbf{C}$  occurs, note that each element of  $E(P_{\phi, E(\mathcal{A})+})$  supports at least one bar in  $\mathcal{B}$ . The bottleneck event  $\mathbf{BN}$  occurs if one of these elements supports exactly one bar in  $\mathcal{B}$ . If  $\mathbf{BN}$  occurs, define the *bottleneck edge*  $e_{\mathbf{BN}}$  to be the edge on  $P_{\phi, E(\mathcal{A})+}$  supporting exactly one bar in  $\mathcal{B}$  that is furthest from the root. Let  $b_{\mathbf{BN}}$  denote the unique bar on  $e_{\mathbf{BN}}$ . If  $\mathbf{C} \cap \mathbf{BN}$  occurs, then  $X^{\mathcal{B}}$  certainly crosses  $b_{\mathbf{BN}}$ . If also  $X^{\mathcal{B}}$  has a periodic trajectory, then  $X^{\mathcal{B}}$  must later cross back along  $b_{\mathbf{BN}}$  to arrive at  $b_{\mathbf{BN}}^+$ . The *non-escape* event  $\mathbf{NoEsc}$  occurs if the meander  $X_{b_{\mathbf{BN}}^+}^{\mathcal{B}}$  visits  $(\phi, 0)$  before  $\mathcal{V}_n \times [0, 1)$ . We claim that

$$\mathbf{C} \cap \mathbf{BN} \cap \{H_n^{\mathcal{B}} = \infty\} \subseteq \mathbf{NoEsc}. \quad (1.5)$$

Indeed, as we have seen, occurrence of the left-hand event implies that  $X^{\mathcal{B}}$  at some time recrosses  $b_{\mathbf{BN}}$  to arrive at  $b_{\mathbf{BN}}^+$ ; after this time,  $X^{\mathcal{B}}$  follows the route of  $X_{b_{\mathbf{BN}}^+}^{\mathcal{B}}$ , so that  $H_n^{\mathcal{B}} = \infty$  forces  $\mathbf{NoEsc}$ , and we have (1.5). The inclusion (1.5) holds equally if  $H_n^{\mathcal{B} \cup \mathcal{A}}$  replaces  $H_n^{\mathcal{B}}$ ; the same argument works after we note

that  $\mathcal{A} \in E(\mathcal{T}_{[e_{\text{BN}}^-]}) \times [0, 1)$ , and thus  $X_{b_{\text{BN}}^+}^{\mathcal{B} \cup \mathcal{A}}$  and  $X_{b_{\text{BN}}^+}^{\mathcal{B}}$  coincide at least until return to  $b_{\text{BN}}^+$ , by which time the two processes have visited  $(\phi, 0)$  because  $X^{\mathcal{B}}$  (from  $(\phi, 0)$ ) visits  $b_{\text{BN}}^+$ . These inferences form the basis for the following claim.

**Lemma 1.11.** *We have that*

$$(\mathbf{P}^+ \cup \mathbf{P}^-) \cap \text{BN} \subseteq \mathbf{C} \cap \text{NoEsc}.$$

**Proof.** It suffices in light of Lemma 1.10 to argue that  $\mathbf{C} \cap \text{BN} \cap \text{NoEsc}^c \cap (\mathbf{P}^+ \cup \mathbf{P}^-) = \emptyset$ . However, we have argued that  $\mathbf{C} \cap \text{BN} \cap \text{NoEsc}^c$  forces both  $H_n^{\mathcal{B}} < \infty$  and  $H_n^{\mathcal{B} \cup \mathcal{A}} < \infty$ ; this suffices, because then neither  $\mathbf{P}^+$  nor  $\mathbf{P}^-$  may occur.  $\square$

**1.7. Some basic tools.** Here we record two simple observations regarding cyclic-time random meander.

**Lemma 1.12.** *The distribution of the return time to  $(\phi, h)$  of  $X_{(\phi, h)}^{\mathcal{B}} : [0, \infty) \rightarrow V(\mathcal{T}) \rightarrow [0, 1)$  under  $\mathbb{P}_t$  is independent of  $h \in [0, 1)$ .*

**Proof.** The bar collection  $\mathcal{B}$  has the Poisson- $t$  distribution on  $E(\mathcal{T}) \times [0, 1)$  and thus is invariant under the map which increases the height of all bars by  $h$  and reduces modulo 1.  $\square$

**Lemma 1.13.** *Let  $s > 0$ . Consider the law  $\mathbb{P}_t$  given  $X^{\mathcal{B}} : [0, s] \rightarrow V(\mathcal{T}) \times [0, 1)$ . Let  $\text{Found}_s \subseteq E(\mathcal{T}) \times [0, 1)$  denote the set of bars in  $\mathcal{B}$  that  $X^{\mathcal{B}}$  has crossed during  $[0, s]$ . Let the set of time- $s$  untouched bar locations  $\text{UnTouch}_s \subseteq E(\mathcal{T}) \times [0, 1)$  denote the set of bars  $b \in E(\mathcal{T}) \times [0, 1)$  neither of whose joints belongs to  $X[0, s]$ . Then the conditional distribution of  $\mathcal{B}$  is given by  $\text{Found}_s \cup \mathcal{B}_{(s, \infty)}$ , where  $\mathcal{B}_{(s, \infty)}$  is a random bar collection with Poisson law of intensity  $t \mathbb{1}_{\text{UnTouch}_s}$ .*

**Proof.** That  $\text{Found}_s \subseteq \mathcal{B}$  is known given  $X^{\mathcal{B}}$  on  $[0, s]$ ; similarly, if  $X^{\mathcal{B}}[0, s]$  visits the joint of some bar in  $\mathcal{B}$ , that bar belongs to  $\text{Found}_s$ . The time-0 distribution of the remaining bars, those in  $\text{UnTouch}_s$ , is undisturbed by the data  $X^{\mathcal{B}}[0, s]$ .  $\square$

## 2. AN OVERVIEW OF THE PROOF

Throughout the paper, we set  $\tau = td^{-1}$ . We aim to prove that  $\mathbb{P}_t(\mathbf{P}^+) \geq \mathbb{P}_t(\mathbf{P}^-)$  for  $t$  of order  $d^{-1}$ , so that  $\tau$  is of unit order; here, we sketch the argument. By Lemma 1.10, in comparing the probabilities of  $\mathbf{P}^+$  and  $\mathbf{P}^-$ , we may restrict attention to choices of  $\mathcal{B}$  and  $\mathcal{A}$  such that  $\mathbf{C}$  occurs. We will further divide into cases according to whether  $\text{BN}$  occurs, and, if it does, according to the location of the bottleneck edge  $e_{\text{BN}}$ .

For the time being, we consider the case  $\mathbf{C} \cap \mathbf{BN}^c$ . (We will later explain how the case  $\mathbf{C} \cap \mathbf{BN}$  may be treated very similarly.) For this case, it is our aim to establish that

$$\mathbb{P}_t(\mathbf{P}^+ | \mathbf{C} \cap \mathbf{BN}^c) \geq \mathbb{P}_t(\mathbf{P}^- | \mathbf{C} \cap \mathbf{BN}^c) \quad (2.1)$$

whenever  $t$  is say at most a large constant multiple of  $d^{-1}$ .

We are conditioning the joint randomness of  $\mathcal{B}$  and  $\mathcal{A}$  by the occurrence of the event  $\mathbf{C} \cap \mathbf{BN}^c$ . To aid the exposition in the section, we will invoke an assumption.

*Simplifying assumption.* The distribution of  $\mathcal{B}$  under  $\mathbb{P}_t(\cdot | \mathbf{C} \cap \mathbf{BN}^c)$  coincides with the unconditioned Poisson- $t$  law on  $E(\mathcal{T}_n) \times [0, 1)$ .

This assumption is false: a given choice of  $\mathcal{B}$  is weighted under the conditional law according to the size of the set of bar locations at which the appearance of  $\mathcal{A}$  would cause  $\mathbf{C} \cap \mathbf{BN}^c$ . This size-biasing effect will require a little notation to describe precisely, but, as we will later see, the effect is minor enough that the rigorous estimates we will derive are in essence the same as those arising under the simplifying assumption.

We begin by explaining how to obtain a bound of the form

$$\mathbb{P}_t(\mathbf{P}^+ | \mathbf{C} \cap \mathbf{BN}^c) \geq e^{-\tau} p_\infty. \quad (2.2)$$

To argue that (2.2) holds, consider the event  $\mathbf{NoBar}$  that no edge incident to  $\phi$  supports a bar in  $\mathcal{B}$ . Invoking the simplifying assumption, we see that

$$\mathbb{P}_t(\mathbf{NoBar} | \mathbf{C} \cap \mathbf{BN}^c) = e^{-\tau}, \quad (2.3)$$

since the left-hand side equals the  $d^{\text{th}}$  power of  $e^{-t}$ .

The next claim allied with (2.3) and the trivial inequality  $p_{n-1} \geq p_\infty$  yields (2.2).

**Lemma 2.1.**

$$\mathbb{P}_t(\mathbf{P}^+ | \mathbf{C} \cap \mathbf{BN}^c \cap \mathbf{NoBar}) = p_{n-1}.$$

**Proof given the simplifying assumption.** Given  $\mathbf{C} \cap \mathbf{BN}^c \cap \mathbf{NoBar}$ ,  $X^\mathcal{B}$  stays at the pole at  $\phi$  at all times, so that the conditional distribution of  $\mathcal{A}$  is normalized Lebesgue measure on  $\mathcal{E}_0 \times [0, 1)$ . Given  $\mathbf{C} \cap \mathbf{BN}^c \cap \mathbf{NoBar}$  and  $\mathcal{A} = (e, h)$  for given  $(e, h) \in \mathcal{E}_0 \times [0, 1)$ , the meander  $X^{\mathcal{B} \cup \mathcal{A}}$  leaves the pole at  $\phi$  by crossing  $\mathcal{A}$  to arrive at  $(e^-, h)$ . By the simplifying assumption, the distribution of  $\mathcal{B}$  restricted to  $E(\mathcal{T}_{[e^-]}) \times [0, 1)$  is Poisson- $t$ . As such, by Lemma 1.12, there is probability  $p_{n-1}$  that  $X^{\mathcal{B} \cup \mathcal{A}}$  visits  $\mathcal{V}_n \times [0, 1)$  before returning to  $(e^-, h)$ . Under this circumstance,  $H_n^{\mathcal{B} \cup \mathcal{A}} < \infty$  but  $H_n^\mathcal{B} = \infty$ .  $\square$

Later, it will be a simple matter to remove the use of the simplifying assumption in the above proof.

In order to obtain (2.1), we need an upper bound on the off-pivotal event probability  $\mathbb{P}_t(\mathbf{P}^- | \mathbf{C} \cap \mathbf{BN}^c)$  of the form of a small constant multiple of  $p_\infty$ . Some definitions are needed. Say that an element  $e \in E(\mathcal{T})$  is multi-open if  $e$  supports at least two bars in  $\mathcal{B}$ . For  $v \in V(\mathcal{T})$ , the multi-open component of  $v$  is the set of  $w \in V(\mathcal{T})$  such that every edge in  $E(P_{v,w})$  is multi-open. Define the multi-cluster  $\mathcal{M}_v$  to be the edge-set of the subgraph of  $\mathcal{T}$  induced by the multi-open component of  $v$ .

A bound on  $\mathbb{P}(\mathbf{P}^- | \mathbf{C} \cap \mathbf{BN}^c)$  may be expressed in terms of the size of the multi-cluster  $\mathcal{M}_\phi$  of the root:

**Lemma 2.2.** *For  $n \in \mathbb{N}$ ,  $0 \leq m \leq n/2$  and  $t \leq \frac{1}{10}d$ ,*

$$\mathbb{P}_t\left(H_n^{\mathcal{B}} < \infty \mid \mathbf{C} \cap \mathbf{BN}^c, \#\mathcal{M}_\phi = m\right) \leq 3p_{n/2}d(m+1).$$

An outline of the proof of Lemma 2.2 is sketched in the image and text of Figure 4. We will not give a formal proof, because we will later state and apply a more general Lemma 4.10.

We may now provide an upper bound on  $\mathbb{P}_t(\mathbf{P}^- | \mathbf{C} \cap \mathbf{BN}^c)$ . By  $\mathbf{P}^- \subseteq \{H_n^{\mathcal{B}} < \infty\}$ , Lemma 2.2 and the bound  $p_{n/2} \leq \frac{4}{3}p_\infty$  (which is valid if  $p_\infty > 0$  and if  $n$  is high enough), we find that

$$\mathbb{P}_t(\mathbf{P}^- | \mathbf{C} \cap \mathbf{BN}^c, \#\mathcal{M}_\phi = m) \leq 4p_\infty d(m+1) \quad (2.4)$$

for  $0 \leq m \leq n/2$ . Under  $\mathbb{P}_t$ , the vertex set of the multi-cluster  $\mathcal{M}_\phi$  is a Galton-Watson tree in which a given vertex has order  $dt^2 = \tau^2 d^{-1}$  offspring; the simplifying assumption and a forthcoming estimate for subcritical Galton-Watson trees thus yield that some large constant  $C > 0$ ,

$$\mathbb{P}_t(\#\mathcal{M}_\phi = m \mid \mathbf{C} \cap \mathbf{BN}^c) \leq C(C\tau^2 d^{-1})^m.$$

Combining the last two estimates, we find that, for each  $0 \leq m \leq n/2$ ,

$$\mathbb{P}_t(\mathbf{P}^-, \#\mathcal{M}_\phi = m \mid \mathbf{C} \cap \mathbf{BN}^c) \leq 4p_\infty d(m+1)C(C\tau^2 d^{-1})^m.$$

Thus,

$$\begin{aligned} \mathbb{P}_t(\mathbf{P}^- | \mathbf{C} \cap \mathbf{BN}^c) &\simeq \sum_{m=2}^{n/2} \mathbb{P}_t(\mathbf{P}^-, \#\mathcal{M}_\phi = m \mid \mathbf{C} \cap \mathbf{BN}^c) \\ &\leq 4p_\infty d \sum_{m=2}^{n/2} (m+1)C(C\tau^2 d^{-1})^m \leq C\tau^4 p_\infty d^{-1}, \end{aligned} \quad (2.5)$$

which upper bound is a constant multiple of  $p_\infty d^{-1}$  if  $t \leq Cd^{-1}$ ; alongside (2.2), we obtain (2.1) by choosing the offspring degree  $d$  to be high enough. There are however several problems with this analysis:

- the simplifying assumption is false;



- in the bumpy equality in (2.5), cases  $m > n/2$  must be treated; here, the argument sketched by Figure 4 is not useful, because there may be outlying bars close to the boundary of  $\mathcal{T}_n$ , so that the meander from one of the red dots may be likely to reach  $\mathcal{V}_n \times [0, 1]$ ;
- to flatten the same bump, the cases  $m \in \{0, 1\}$  must also be handled; with the present tools, these summands would be shown to contribute at most  $dp_\infty$  or  $p_\infty$  (up to a further constant factor), which is too large to be bounded above by the right-hand side of (2.2).
- the bound (2.4) may be derived only if  $p_\infty = p_\infty(t) > 0$ , so that (2.1) and ultimately  $\frac{dp_n(t)}{dt}(t) \geq 0$  for all high  $n$  may be derived by the argument only for such  $t$ . This is not sufficient to verify that  $p_\infty$  is non-decreasing.

We will resolve each of these difficulties when we give the full proof shortly.

It remains to outline how we compare  $\mathbb{P}_t(\mathbf{P}^+ | \mathbf{C} \cap \mathbf{BN})$  and  $\mathbb{P}_t(\mathbf{P}^- | \mathbf{C} \cap \mathbf{BN})$ . Given  $\mathbf{C} \cap \mathbf{BN}$ , we further condition on the location of the bottleneck bar,  $b_{\mathbf{BN}} = (e, h) \in E(\mathcal{T}_n) \times [0, 1]$ . By Lemma 1.11, each of  $\mathbf{P}^+$  or  $\mathbf{P}^-$  is impossible without **NoEsc**. So, in our conditioning, we may as well include **NoEsc**. The event **NoEsc** is “decided in  $\mathcal{T}$  above  $e_{\mathbf{BN}}^+$ ”: it is measurable with respect to  $\mathcal{B}$  on  $E(\mathcal{T}^{[e_{\mathbf{BN}}^+]}) \times [0, 1]$ . Note then that, given  $\mathbf{C} \cap \mathbf{BN} \cap \mathbf{NoEsc} \cap \{e_{\mathbf{BN}} = (e, h)\}$ , the question of whether  $H_n^{\mathcal{B}} < \infty$  or  $H_n^{\mathcal{B} \cup \mathcal{A}} < \infty$  occurs is entirely determined by what happens over the subtree  $\mathcal{T}_{[e^-]}$ :  $H_n^{\mathcal{B}} < \infty$  if the meander  $X_{(e^-, h)}^{\mathcal{B}}$  reaches  $\mathcal{V}_n \times [0, 1]$  before returning to  $(e^-, h)$ , and likewise for  $H_n^{\mathcal{B} \cup \mathcal{A}} < \infty$  and  $X_{(e^-, h)}^{\mathcal{B} \cup \mathcal{A}}$ . For this reason, the analysis for this case is precisely analogous to that for  $\mathbb{P}_t(\cdot | \mathbf{C} \cap \mathbf{BN}^e)$ , where the meanders  $X_{(e^-, h)}^{\mathcal{B}}$  and  $X_{(e^-, h)}^{\mathcal{B} \cup \mathcal{A}}$  in  $E(\mathcal{T}_{[e^-]}) \times [0, 1]$  form counterparts to  $X^{\mathcal{B}}$  and  $X^{\mathcal{B} \cup \mathcal{A}}$  from  $(\phi, 0)$ . The same proof outline applies to this case as to the earlier one, with the same four difficulties arising. Note that the second listed difficulty – in this case, that  $\mathcal{M}_{e^-}$  may reach close to the boundary of  $\mathcal{T}_n$ , invalidating the “constant multiple of  $p_\infty$ ” comparison made before (2.4) – is particularly troublesome should the edge  $e$  itself be close to the boundary. For this reason, we will split our formal analysis into two further cases, according to whether or not the edge  $e$  in this conditioning is far from, or close to, the boundary of  $\mathcal{T}_n$ .

### 3. THE MAIN ESTIMATES

We now describe formally how we will split the analysis into cases, state the estimates that we will prove for each case, and then provide the proof of Proposition 1.9 using these estimates.

First we specify a cutoff distance  $n_1 \in \mathbb{N}$  to delineate the cases that  $e_{\mathbf{BN}}$  is far from, or close to, the boundary of  $\mathcal{T}_n$ . In light of the fourth difficulty listed in the preceding section, we work with  $p_n$  as a substitute for  $p_\infty$ .

**Lemma 3.1.** *Let  $\varepsilon > 0$ . There exists  $n_1 = n_1(d, \varepsilon)$  such that if  $n \geq n_1$  and  $t \in [0, d^{-1} + 2d^{-2}]$  are such that  $p_n(t) \geq \varepsilon$ , then  $p_m(t) \leq (1 + \frac{1}{25})p_n(t)$  if  $n \geq m \geq n_1$ . The same statement holds when “ $d^{-1} + 2d^{-2}$ ” is replaced by “ $\frac{1}{7}d^{-1} \log d$ ”.*

Lemma 3.1 is concerned with an upper bound on  $\mathbb{P}_t(H_m^{\mathcal{B}} < \infty, H_n^{\mathcal{B}} = \infty)$  when  $n \gg m \gg 1$ . We will anyway study such questions in treating the “close-to-boundary” case in Section 6, where the proof of Lemma 3.1 appears.

Fixing  $\varepsilon > 0$  as the parameter in Proposition 1.9, the cutoff  $n_1$  is specified by means of Lemma 3.1. We write  $\text{BN} = \text{FB} \cup \text{CB}$  where the *far-from-boundary* event  $\text{FB}$  is defined to be  $\{d(\phi, e_{\text{BN}}^-) \leq n - 2n_1\}$  and the *close-to-boundary* event  $\text{CB}$  is  $\text{BN} \setminus \text{FB}$ .

Lemmas 1.10 and 1.11 imply that the right-hand side of the following equality is a partition into three disjoint sets:

$$\begin{aligned} \text{P}^+ \cup \text{P}^- &= (\text{P}^+ \cup \text{P}^-) \cap \text{C} \cap \text{BN}^c & (3.1) \\ &\cup (\text{P}^+ \cup \text{P}^-) \cap \text{C} \cap \text{NoEsc} \cap \text{FB} \\ &\cup (\text{P}^+ \cup \text{P}^-) \cap \text{C} \cap \text{NoEsc} \cap \text{CB}. \end{aligned}$$

The events  $\text{P}^+$  and  $\text{P}^-$  being disjoint, we have that

$$\mathbb{P}_t(\text{P}^+) - \mathbb{P}_t(\text{P}^-) = A_1 + A_2 + A_3, \quad (3.2)$$

where

$$A_1 = \mathbb{P}_t(\text{P}^+ \cap \text{C} \cap \text{BN}^c) - \mathbb{P}_t(\text{P}^- \cap \text{C} \cap \text{BN}^c), \quad (3.3)$$

$$A_2 = \mathbb{P}_t(\text{P}^+ \cap \text{C} \cap \text{NoEsc} \cap \text{FB}) - \mathbb{P}_t(\text{P}^- \cap \text{C} \cap \text{NoEsc} \cap \text{FB})$$

and

$$A_3 = \mathbb{P}_t(\text{P}^+ \cap \text{C} \cap \text{NoEsc} \cap \text{CB}) - \mathbb{P}_t(\text{P}^- \cap \text{C} \cap \text{NoEsc} \cap \text{CB}). \quad (3.4)$$

Let  $\text{High} = \{e \in E(\mathcal{T}_n) : d(\phi, e^-) \leq n - 2n_1\}$ . Note then that  $\text{FB} = \text{BN} \cap \{e_{\text{BN}} \in \text{High}\}$ . Thus,

$$A_2 = \sum_{e \in \text{High}} \left( \mathbb{P}_t(\text{P}^+ \cap \text{C} \cap \text{NoEsc} \cap \{e_{\text{BN}} = e\}) - \mathbb{P}_t(\text{P}^- \cap \text{C} \cap \text{NoEsc} \cap \{e_{\text{BN}} = e\}) \right). \quad (3.5)$$

**3.1. Estimates for the three cases.** We now state the estimates that we will derive concerning the quantities  $A_1$ ,  $A_2$  and  $A_3$ : throughout, assume that  $(t, n) \in (0, \infty) \times \mathbb{N}$  satisfies  $p_n(t) \geq \varepsilon$ . First, for  $d \geq 3 \cdot 10^6$ ,  $\tau \leq 1 + 2d^{-1}$  and  $n \geq 2n_1$ ,

$$\mathbb{P}_t(\text{P}^+ \cap \text{C} \cap \text{BN}^c) - \mathbb{P}_t(\text{P}^- \cap \text{C} \cap \text{BN}^c) \geq \frac{1}{30} e^{-2\tau} p_n d (\#E(\mathcal{T}_n))^{-1}; \quad (3.6)$$

for the same choices of  $d$  and  $\tau$ , and for all  $e \in \text{High}$ ,

$$\mathbb{P}_t(\mathbf{P}^+ \cap \mathbf{C} \cap \text{NoEsc} \cap \{e_{\text{BN}} = e\}) \geq \mathbb{P}_t(\mathbf{P}^- \cap \mathbf{C} \cap \text{NoEsc} \cap \{e_{\text{BN}} = e\}); \quad (3.7)$$

if  $d \geq \max\{2^6\tau^6(6\tau + 1), 5\}$ , then  $n \geq 4n_1$  implies that

$$\mathbb{P}_t(\mathbf{P}^- \cap \mathbf{C} \cap \text{NoEsc} \cap \text{CB}) \leq 12\varepsilon^{-5}e^{2\tau}d^{2n_1+2}n(\#E(\mathcal{T}_n))^{-1}c_{\varepsilon,\tau,d}^n, \quad (3.8)$$

where  $c_{\varepsilon,\tau,d} \in (0, 1)$  is, for given  $\varepsilon \in (0, 1)$ , bounded away from one uniformly in  $\tau$  on any given compact interval in  $(0, \infty)$ . Moreover, there exists  $d_0 \in \mathbb{N}$  such that (3.6) and (3.7) hold for all  $d \geq d_0$  and for  $\tau \leq \frac{1}{7} \log d$ .

**3.2. Assembling the estimates.** We now use the three preceding estimates to prove Proposition 1.9. In summary, the far-from-boundary case is shown to be at worst neutral in (3.7); the positive contribution of the close-to-boundary case is discarded, while its negative contribution (3.8) is offset against the gain term (3.6) in the case  $\text{BN}^c$ .

**Proof of Proposition 1.9 for  $d \geq 3 \cdot 10^6$ .** Let  $c_d \in (0, 1)$  equal  $\sup c_{\varepsilon,\tau,d}$ , the supremum taken over  $1 \leq \tau \leq 1 + 2d^{-1}$  or over  $1 \leq \tau \leq \frac{1}{7} \log d$  according to whether the first or the second of the proposition's assertions is being considered. Let  $n_0 \geq 4n_1$  be chosen so that, for  $n \geq n_0$ ,

$$n \geq \frac{\log(30 \cdot 12\varepsilon^{-6}\kappa d^{2n_1+1}n)}{\log(c_d^{-1})}.$$

where  $\kappa$  is the supremum of  $e^{4\tau}$  over the same interval of  $\tau$ ; note that  $\kappa$  is respectively at most  $e^8$  and  $d^{2/3}$ . In light of (3.3), (3.4) and (3.5), the proposition follows by applying (3.6), (3.7) and (3.8) to (3.2).  $\square$

It remains to prove the estimates (3.6),(3.7) and (3.8). Sections 4, 5 and 6 treat each of these estimates in turn.

#### 4. CROSSING WITHOUT BOTTLENECK: DERIVING (3.6)

The estimates (3.6) and (3.7) are derived by very similar means. The derivation of (3.7) requires a little further notation and may be seen as an extension of that of (3.6).

##### 4.1. The reweighting of measure due to crossing without bottleneck.

For this reason, we begin by deriving (3.6), a task requiring an inquiry into the joint distribution of  $\mathcal{B}$  and  $\mathcal{A}$  under the conditional law  $\mathbb{P}_t(\cdot | \mathbf{C} \cap \text{BN}^c)$ . We begin with a technical lemma which we deferred from the outline Section 2: we find an explicit description of the conditional law in terms of an unconditioned Poisson- $t$  bar collection.

**Definition 4.1.** Let  $\mathcal{B}_0 \subseteq E(\mathcal{T}_n) \times [0, 1)$ . Let the set  $\text{ViLoc}(\mathcal{B}_0) \subseteq E(\mathcal{T}_n) \times [0, 1)$  of viable bar locations be such that  $b \in \text{ViLoc}(\mathcal{B}_0)$  if and only if both of the following conditions apply:

- the meander  $X^{\mathcal{B}_0}$  visits at least one joint of  $b$  before time  $H_n^{\mathcal{B}_0}$ ;
- every edge in the path  $P_{\phi, E(b)^+}$  supports at least two bars in  $\mathcal{B}_0$ .

The key property of ViLoc is the following. Let  $\mathcal{B}_0 \subseteq E(\mathcal{T}_n) \times [0, 1)$ . Conditionally on  $\mathbb{P}_t$  given  $\mathcal{B} = \mathcal{B}_0$ ,

$$\mathcal{A} \in \text{ViLoc}(\mathcal{B}_0) \text{ if and only if } \mathbf{C} \cap \mathbf{BN}^c. \quad (4.1)$$

**Lemma 4.2.** *Let  $\mathbb{P}_{t, \mathcal{B}}$  denote the marginal distribution of  $\mathcal{B}$  under  $\mathbb{P}_t$  (so that this law does not track  $\mathcal{A}$ ). Let  $\mathbb{P}_{t, \mathcal{B}}^{\mathbf{C} \cap \mathbf{BN}^c}$  denote the marginal distribution of  $\mathcal{B}$  under  $\mathbb{P}_t(\cdot | \mathbf{C} \cap \mathbf{BN}^c)$ . Then, for any given  $\mathcal{B}' \subseteq E(\mathcal{T}_n) \times [0, 1)$ ,*

$$\frac{d\mathbb{P}_{t, \mathcal{B}}^{\mathbf{C} \cap \mathbf{BN}^c}}{d\mathbb{P}_{t, \mathcal{B}}}(\mathcal{B}') = Z^{-1} |\text{ViLoc}(\mathcal{B}')|, \quad (4.2)$$

where  $Z = \int |\text{ViLoc}(\mathcal{B}')| d\mathbb{P}_{t, \mathcal{B}}(\mathcal{B}')$ .

Given a sample  $\mathcal{B}'$  of the marginal distribution  $\mathbb{P}_{t, \mathcal{B}}^{\mathbf{C} \cap \mathbf{BN}^c}$  of the law  $\mathbb{P}_t(\cdot | \mathbf{C} \cap \mathbf{BN}^c)$ , the conditional distribution of  $\mathcal{A}$  is given by normalized Lebesgue measure on  $\text{ViLoc}(\mathcal{B}')$ .

To illustrate, consider two examples. Take  $d = 2$  in either case. In example 1, let  $\mathcal{B}_1$  be any  $\mathcal{B}$  realizing NoBar. In example 2, let  $\mathcal{B}_2$  be a  $\mathcal{B}$  as shown in Figure 4; we assume now that no bars in  $\mathcal{B}_2$  are supported on edges with parent vertex in  $\{v_{11}, v_{12}, v_{21}, v_{22}\}$ . Given  $\mathbf{C} \cap \mathbf{BN}^c$ , any given  $\mathcal{B}_1$  or  $\mathcal{B}_2$  is weighted according to the Lebesgue measure  $|\text{ViLoc}|$  of the set of viable positions to place  $\mathcal{A}$ . In example 1,  $X^{\mathcal{B}}$  performs a circuit of period one on the pole at  $\phi$ , so that  $\text{ViLoc} = \{(\phi, v_1), (\phi, v_2)\} \times [0, 1)$  and  $|\text{ViLoc}| = 2$ . In example 2, the meander  $X^{\mathcal{B}}$  successively visits poles at vertices  $\phi, v_2, v_{21}, v_2, v_{22}, v_2, \phi, v_1, v_{12}, v_1, \phi, v_2, \phi, v_1, \phi$ , then completing a circuit. The trace of the circuit is given by  $\{\phi, v_1, v_{12}, v_2, v_{21}, v_{22}\} \times [0, 1)$ . Thus,  $\text{ViLoc} = \{(\phi, v_1), (\phi, v_2), (v_1, v_{11}), (v_1, v_{12}), (v_2, v_{21}), (v_2, v_{22})\} \times [0, 1)$  and  $|\text{ViLoc}| = 6$ .

**Proof of Lemma 4.2.** Let  $A$  denote a  $\mathcal{B}$ -measurable event. By (4.1), we see that

$$\mathbb{P}_t(A \cap \mathbf{C} \cap \mathbf{BN}^c) = \int_A \mathbb{P}_t(\mathcal{A} \in \text{ViLoc}(\mathcal{B}) | \mathcal{B} = \mathcal{B}') d\mathbb{P}_{t, \mathcal{B}}(\mathcal{B}'). \quad (4.3)$$

By the independence of  $\mathcal{B}$  and  $\mathcal{A}$ ,

$$\mathbb{P}_t(\mathcal{A} \in \text{ViLoc}(\mathcal{B}) | \mathcal{B} = \mathcal{B}') = \frac{|\text{ViLoc}(\mathcal{B}')|}{\#E(\mathcal{T}_n)} \quad (4.4)$$

which implies by the definition of a Radon-Nikodym derivative that

$$\frac{d\mathbb{P}_{t, \mathcal{B}}^{\mathbf{C} \cap \mathbf{BN}^c}}{d\mathbb{P}_{t, \mathcal{B}}}(\mathcal{B}') = \frac{|\text{ViLoc}(\mathcal{B}')|}{\#E(\mathcal{T}_n) \mathbb{P}_t(\mathbf{C} \cap \mathbf{BN}^c)}.$$

Thus, (4.2) holds with  $Z = \#E(\mathcal{T}_n) \mathbb{P}_t(\mathbf{C} \cap \mathbf{BN}^c)$ . Applying (4.3) for  $A = \Omega$  and then (4.4) yields the form for  $Z$  stated in the lemma.  $\square$

In fact, the law  $\mathbb{P}_{t,\mathcal{B}}^{\mathbf{C} \cap \mathbf{BN}^c}$  is formed by weighting  $\mathbb{P}_{t,\mathcal{B}}$  by a factor which is at most the size of the multi-cluster  $\mathcal{M}_\phi$ . The next lemma clarifies why this is so.

**Lemma 4.3.** *Given  $G \subseteq E(\mathcal{T}_n)$ , let  $\partial_{\text{ext}}G$  denote the set of edges in  $E(\mathcal{T}_n) \setminus G$  that are incident to the endpoint of some element of  $G$ . Then*

$$\text{ViLoc}(\mathcal{B}) \subseteq \left( \mathcal{M}_\phi \cup \partial_{\text{ext}}\mathcal{M}_\phi \right) \times [0, 1).$$

**Proof.** A consequence of the second part of the definition of ViLoc.  $\square$

**Corollary 4.4.** *For each  $k \in \mathbb{N}$ ,*

$$|\text{ViLoc}(\mathcal{B})| > dk \implies \#\mathcal{M}_\phi \geq k.$$

**Proof.** For any set  $E \subseteq E(\mathcal{T}_n)$ ,  $\#\partial_{\text{ext}}E \leq d + (d-1)\#E$ .  $\square$

We may now complete a step omitted earlier.

**Proof of Lemma 2.1.** To complete the earlier argument, we now argue that given  $\mathbf{C} \cap \mathbf{BN}^c \cap \text{NoBar}$  and  $\mathcal{A} = (e, h)$  for some  $(e, h) \in \mathcal{E}_0 \times [0, 1)$ , the distribution of  $\mathcal{B}$  restricted to  $E(\mathcal{T}_{[e-]}) \times [0, 1)$  is Poisson- $t$ . Any  $\mathcal{B}$  satisfying NoBar has  $\text{ViLoc}(\mathcal{B}) = \mathcal{E}_0 \times [0, 1)$  and thus  $|\text{ViLoc}(\mathcal{B})| = d$ . Lemma 4.2 implies that  $\mathcal{B}$  restricted to  $E(\mathcal{T}_{[e-]}) \times [0, 1)$  under  $\mathbb{P}_{t,\mathcal{B}}^{\mathbf{C} \cap \mathbf{BN}^c}$  is Poisson- $t$  distributed; independence of  $\mathcal{A}$  and  $\mathcal{B}$  under  $\mathbb{P}_t$  completes the proof.  $\square$

For later use, we record upper and lower bounds on the normalization  $Z$  in Lemma 4.2.

**Lemma 4.5.** *Let  $t \in (0, \infty)$ . Then*

$$Z \geq de^{-\tau}.$$

**Proof.** Due to  $\text{NoBar} \subseteq \{\mathcal{E}_0 \times [0, 1) = \text{ViLoc}(\mathcal{B})\}$  and  $\mathbb{P}_t(\text{NoBar}) = e^{-\tau}$ .  $\square$

**Lemma 4.6.** *Let  $d \geq 15\tau^2$ . Then*

$$Z \leq \left(1 + \frac{1}{5}\right)d.$$

**Proof.** Recall from Lemma 4.2 that  $Z = \int |\text{ViLoc}(\mathcal{B}')| d\mathbb{P}_{t,\mathcal{B}}(\mathcal{B}')$ . For  $k \in \mathbb{N}$ , set  $\Lambda_k = \{dk < |\text{ViLoc}(\mathcal{B}')| \leq d(k+1)\}$ . Note then that

$$Z = \sum_{k \geq 0} \int_{\Lambda_k} |\text{ViLoc}(\mathcal{B}')| d\mathbb{P}_{t,\mathcal{B}}(\mathcal{B}') \leq \sum_{k \geq 0} d(k+1) \mathbb{P}_t(\#\mathcal{M}_\phi \geq k),$$

the inequality invoking Corollary 4.4.

By Lemma 4.7,

$$Z \leq d + \left(1 + \frac{1}{10}\right)e^{-1}d \sum_{k \geq 1} (k+1)(e\tau^2 d^{-1})^k = d + \left(1 + \frac{1}{10}\right)e^{-1}d \left( (1 - e\tau^2 d^{-1})^{-2} - 1 \right),$$

so that  $e\tau^2 d^{-1} \leq 1 - \left(\frac{11}{11+2e}\right)^{1/2}$  gives the result.  $\square$

**Lemma 4.7.** *Assume that  $d \geq 11\tau^2$ . Then, for each  $\ell \in \mathbb{N}^+$ ,*

$$\mathbb{P}_t(\#\mathcal{M}_\phi \geq \ell) \leq \left(1 + \frac{1}{10}\right)e^{-1}(e\tau^2 d^{-1})^\ell.$$

**Proof.** Consider the following procedure for determining  $\mathcal{M}_\phi$ , which is similar to the coding of trees by Lukasiewicz paths presented in [9, Section 1.1]. At time zero, the  $d$  elements of  $\mathcal{E}_0$  are called candidates; at each time step from time one, one candidate is examined. On being examined, a candidate changes status, either being found to belong to  $\mathcal{M}_\phi$ , or not. In the first case, the edges incident to the candidate's child vertex join the candidate list; in the second, no new candidates join. The process stops when there are no candidates left. The set of edges that are candidates at some time is  $\mathcal{M}_\phi \cup \partial_{\text{ext}}\mathcal{M}_\phi$ .

For  $r \in [0, 1]$ , let  $Z = Z_r : \mathbb{N} \rightarrow \mathbb{Z}$ ,  $Z(0) = d$ , denote the Markov chain on  $\mathbb{Z}$  with two transitions, namely a  $d - 1$  displacement with probability  $r$  and a  $-1$  displacement with probability  $1 - r$ . Let  $\sigma_r = \inf \{\ell \geq 0 : Z(\ell) = 0\}$ .

Any given candidate belongs to  $\mathcal{M}_\phi$  with probability  $1 - (1+t)e^{-t}$ . Hence, for each  $s \in \mathbb{N}$ ,  $Z_{1-(1+t)e^{-t}}(s)$  is the number of candidates after the examination at time  $s$ . We see that  $\#\mathcal{M}_\phi + \#\partial_{\text{ext}}\mathcal{M}_\phi$  under  $\mathbb{P}_t$  has the distribution of  $\sigma_{1-(1+t)e^{-t}}$ . Noting that  $\#\mathcal{M}_\phi + \#\partial_{\text{ext}}\mathcal{M}_\phi = d(\#\mathcal{M}_\phi + 1)$  and also that  $\sigma_r$  is stochastically increasing in  $r$ , we find that

$$\mathbb{P}_t(\#\mathcal{M}_\phi \geq \ell) \leq \mathbb{P}_t(\#\mathcal{M}_\phi + \#\partial_{\text{ext}}\mathcal{M}_\phi \geq d\ell + 1) \leq \mathbb{P}(\sigma_{t^2} \geq d\ell + 1).$$

Note now that  $\sigma_{t^2} \geq d\ell + 1$  entails that at least  $\ell$  among the first  $d\ell$  transitions made by  $Z_{t^2}$  are up moves. The probability of this is at most  $\sum_{k=\ell}^{d\ell} \binom{d\ell}{k} (t^2)^k$ . The first summand here is at most  $\frac{(d\ell)^\ell}{\ell!} t^{2\ell}$  which since  $\ell \geq 1$  is bounded above by  $d^\ell e^{\ell-1} t^{2\ell} = e^{-1} d^{-\ell} e^\ell \tau^{2\ell}$ , while the ratio of successive summands is always at most  $d^{-1}\tau^2$ . Hence, the sum is at most  $(1 - d^{-1}\tau^2)^{-1} e^{-1} d^{-\ell} (e\tau^2)^\ell$ . From the assumption that  $d \geq 11\tau^2$  follows the statement of the lemma.  $\square$

**4.2. Estimating the gain and loss terms.** We will derive a gain term: if  $d \geq 10e\tau^2$ , then

$$\mathbb{P}_t(\mathbf{P}^+ | \mathbf{C} \cap \mathbf{BN}^c) \geq \left(1 - \frac{1}{6}\right)e^{-\tau} p_n, \quad (4.5)$$

this being the rigorous analogue of (2.2). We will prove a loss term: for  $d \geq 3 \cdot 10^6$  and  $d^{-1} \leq t \leq d^{-1} + 2d^{-2}$ , (and for  $d \geq d_0$  for some  $d_0$  and  $\tau \leq \frac{1}{7} \log d$ ),

$$\mathbb{P}_t(\mathbf{P}^- | \mathbf{C} \cap \mathbf{BN}^c) \leq \left(1 - \frac{1}{5}\right)e^{-\tau} p_n. \quad (4.6)$$

We will also argue that, for all  $t \in (0, \infty)$ ,

$$\mathbb{P}_t(\mathbf{C} \cap \mathbf{BN}^c) \geq e^{-\tau} d (\#E(\mathcal{T}_n))^{-1}. \quad (4.7)$$

From (4.5) and (4.6) follows

$$\mathbb{P}_t(\mathbf{P}^+ \cap \mathbf{C} \cap \mathbf{BN}^c) - \mathbb{P}_t(\mathbf{P}^- \cap \mathbf{C} \cap \mathbf{BN}^c) \geq \frac{1}{25} \mathbb{P}_t(\mathbf{P}^+ \cap \mathbf{C} \cap \mathbf{BN}^c).$$

Hence, (3.6) follows from (4.5), (4.6) and (4.7).

**4.3. Deriving the gain term.** We now prove (4.5).

Recall the event **NoBar** from after (2.2).

**Lemma 4.8.** *If  $d \geq 15\tau^2$ , then*

$$\mathbb{P}_t(\mathbf{NoBar} | \mathbf{C} \cap \mathbf{BN}^c) \geq (1 - \frac{1}{6})e^{-\tau},$$

**Proof.** If **NoBar** occurs then  $\text{ViLoc}(\mathcal{B}) = \mathcal{E}_0 \times [0, 1)$  and thus  $|\text{ViLoc}(\mathcal{B})| = d$ . By Lemmas 4.2 and 4.6, we find that

$$\mathbb{P}_t(\mathbf{NoBar} | \mathbf{C} \cap \mathbf{BN}^c) = dZ^{-1} \mathbb{P}_t(\mathbf{NoBar}) \geq (1 - \frac{1}{6})e^{-\tau};$$

here we used  $\mathbb{P}_t(\mathbf{NoBar}) = e^{-td} = e^{-\tau}$ .  $\square$

Lemma 2.1 and the trivial  $p_{n-1} \geq p_n$  imply that  $\mathbb{P}_t(\mathbf{P}^+ | \mathbf{NoBar} \cap \mathbf{C} \cap \mathbf{BN}^c) \geq p_n$ . Applying Lemma 4.8 yields (4.5).

We also prove (4.7). Note that  $\mathbf{NoBar} \cap \{\mathcal{A} \in \mathcal{E}_\phi \times [0, 1)\} \subseteq \mathbf{C} \cap \mathbf{BN}^c$ . Under  $\mathbb{P}_t$ , the events **NoBar** and  $\{\mathcal{A} \in \mathcal{E}_\phi \times [0, 1)\}$  are independent, with  $\mathbb{P}_t(\mathbf{NoBar}) = e^{-\tau}$  and  $\mathbb{P}_t(\mathcal{A} \in \mathcal{E}_\phi \times [0, 1)) = \frac{|\mathcal{E}_\phi|}{|E(\mathcal{T}_n)|} = d|E(\mathcal{T}_n)|^{-1}$ , as required to verify (4.7).

**4.4. Deriving the loss term from several estimates.** To establish (4.6), we partition according to the size of  $\mathcal{M}_\phi$ :

$$\mathbb{P}_t(\mathbf{P}^- | \mathbf{C} \cap \mathbf{BN}^c) = \sum_{m=0}^{\infty} \mathbb{P}_t(\mathbf{P}^-, \#\mathcal{M}_\phi = m | \mathbf{C} \cap \mathbf{BN}^c).$$

We provide upper bounds on the right-hand side by distinguishing four cases. The cutoff  $n_1 \in \mathbb{N}$  specified at the start of Section 3 may be increased if necessary to ensure that

$$6e^{\tau-1}(2e\tau^2d^{-1})^{n_1} \leq \frac{1}{50}e^{1-\tau}\varepsilon. \quad (4.8)$$

The four cases are:  $m = 0$ ,  $m = 1$ ,  $2 \leq m \leq n_1 - 1$  and  $m \geq n_1$ . The four bounds are now stated, alongside the assumptions on parameters required; in addition to these, assume that  $p_n(t) \geq \varepsilon$  and that  $n \geq 2n_1$ .

- for  $d \geq 40000$ ,  $1 \leq \tau \leq 1 + 2d^{-1}$ , or for  $d$  sufficiently high and  $1 \leq \tau \leq \frac{1}{7} \log d$ ,

$$\mathbb{P}_t(\mathbf{P}^-, \#\mathcal{M}_\phi = 0 | \mathbf{C} \cap \mathbf{BN}^c) \leq (1 + \frac{2}{5})p_n e^{3\tau} \left( (\tau(1+\tau)+6)d^{-1/2} + \frac{3}{4}\tau d^{-1/2} \log d \right); \quad (4.9)$$

- for  $d \geq 11\tau^2$ ,

$$\mathbb{P}_t\left(\mathbf{P}^-, \#\mathcal{M}_\phi = 1 \mid \mathbf{C} \cap \mathbf{BN}^c\right) \leq 4\left(1 + \frac{1}{5}\right)p_n\tau^3 e^{2\tau} d^{-1}; \quad (4.10)$$

- for  $m \geq 0$  (and in particular for  $2 \leq m \leq n_1 - 1$ ),  $d \geq 10e\tau^2$ ,  $\tau \geq 1$ ,

$$\mathbb{P}_t\left(\mathbf{P}^-, \#\mathcal{M}_\phi = m \mid \mathbf{C} \cap \mathbf{BN}^c\right) \leq 6\left(1 + \frac{1}{25}\right)dp_n e^{2\tau-1}(m+1)(2e\tau^2 d^{-1})^m; \quad (4.11)$$

- and, for  $d \geq 10e\tau^2$  and  $\tau \geq 1$ ,

$$\mathbb{P}_t\left(\mathbf{P}^-, \#\mathcal{M}_\phi \geq n_1 \mid \mathbf{C} \cap \mathbf{BN}^c\right) \leq \frac{1}{50}e^{-\tau}p_n. \quad (4.12)$$

If, in addition to the conditions required for the four preceding bounds, we have that  $d \geq \max\{40e\tau^2, 2000\}$  and  $1 \leq \tau \leq 1 + 2d^{-1}$ , then (4.9), (4.10), (4.11) and (4.12) show that  $\mathbb{P}_t(\mathbf{P}^- \mid \mathbf{C} \cap \mathbf{BN}^c)$  is at most the product of  $p_n$  and

$$\begin{aligned} & \left(1 + \frac{2}{5}\right)e^{3\left(1 + \frac{1}{1000}\right)} \left( \left(1 + \frac{1}{1000}\right)\left(2 + \frac{1}{1000}\right) + 6 + \frac{3}{4}\left(1 + \frac{1}{1000}\right) \log d \right) d^{-1/2} \\ & + 4\left(1 + \frac{1}{5}\right)\left(1 + \frac{1}{1000}\right)^3 e^{2\left(1 + \frac{1}{1000}\right)} d^{-1} \\ & + 6 \cdot 16\left(1 + \frac{1}{25}\right)\left(1 + \frac{1}{9}\right)\left(1 + \frac{1}{1000}\right)^4 e^{2\left(1 + \frac{1}{1000}\right)+1} d^{-1} + \frac{1}{50} \\ & \leq 226d^{-1/2} + 18.2d^{-1/2} \log d + (42 + 2242)d^{-1} + \frac{1}{50}. \end{aligned} \quad (4.13)$$

If  $t \leq d^{-1} + 2d^{-2}$  and  $d \geq 2000$  then  $1 - \tau \geq -\frac{1}{1000}$ , so that, in this case, if (4.13) is at most  $\left(1 - \frac{1}{5}\right)e^{-1 - \frac{1}{1000}} \geq 0.367$  then (4.6) holds. Given that  $t \rightarrow t^{-1/2} \log t$  is decreasing on  $[e^{e^2}, \infty) \supseteq [1619, \infty)$  and that (4.13) at  $d = 3 \cdot 10^6$  is at most 0.308, we confirm (4.6) for the first of the two stated parameter choices.

If  $\tau \leq \frac{1}{7} \log d$ , then considering an analogous upper bound to (4.13) shows that (4.6) holds for all sufficiently high  $d$ . This completes the derivation of (4.6).

**4.5. The loss term with  $2 \leq \#\mathcal{M}_\phi < n_1$ .** Following the presentation of the outline Section 2, we begin by treating the case  $\#\mathcal{M}_\phi \geq 2$  and then turn to the cases  $\#\mathcal{M}_\phi = 1$  and  $\#\mathcal{M}_\phi = 0$ . Here, then, we prove (4.11).

**Lemma 4.9.** *Assume that  $d \geq 10e\tau^2$  and that  $\tau \geq 1$ . For each  $m \in \mathbb{N}^+$ ,*

$$\mathbb{P}_t\left(\#\mathcal{M}_\phi \geq m \mid \mathbf{C} \cap \mathbf{BN}^c\right) \leq 6e^{\tau-1}(2e\tau^2 d^{-1})^m.$$

**Proof.** By Lemma 4.2,

$$\mathbb{P}_t\left(|\text{ViLoc}(\mathcal{B})| \geq d(m+1) \mid \mathbf{C} \cap \mathbf{BN}^c\right) = \int_{|\text{ViLoc}(\mathcal{B})| \geq d(m+1)} |\text{ViLoc}(\mathcal{B})| d\mathbb{P}_t.$$

Following the notation and argument in the proof of Lemma 4.6,

$$\begin{aligned}
& \int_{|\text{ViLoc}(\mathcal{B})| \geq d(m+1)} |\text{ViLoc}(\mathcal{B})| \, d\mathbb{P}_t \\
&= \sum_{k \geq m+1} \int_{\Lambda_k} |\text{ViLoc}(\mathcal{B})| \, d\mathbb{P}_t \leq \sum_{k \geq m+1} d(k+1) \mathbb{P}_t(\#\mathcal{M}_\phi \geq k) \\
&\leq \left(1 + \frac{1}{10}\right) e^{-1} d \sum_{k \geq m+1} (k+1) (e\tau^2 d^{-1})^k \\
&= \left(1 + \frac{1}{10}\right) e^{-1} d (e\tau^2 d^{-1})^{m+1} \sum_{k \geq 0} (m+1+k) (e\tau^2 d^{-1})^k \\
&= \left(1 + \frac{1}{10}\right) e^{-1} d (e\tau^2 d^{-1})^{m+1} \left( (m+1)(1 - e\tau^2 d^{-1})^{-1} + e\tau^2 d^{-1} (1 - e\tau^2 d^{-1})^{-2} \right) \\
&\leq \left(1 + \frac{1}{10}\right) \tau^2 (2e\tau^2 d^{-1})^m \left(1 + \frac{1}{9} + \frac{1}{8}\right) \leq \left(1 + \frac{2}{5}\right) \tau^2 (2e\tau^2 d^{-1})^m,
\end{aligned}$$

where we used  $e\tau^2 d^{-1} \leq \frac{1}{10}$ . From this bound and Lemmas 4.2, 4.7 and 4.5, we find that, if  $m \geq 1$ ,

$$\begin{aligned}
& \mathbb{P}_t(\#\mathcal{M}_\phi \geq m \mid \mathbf{C} \cap \mathbf{BN}^c) \\
&\leq \mathbb{P}_t(|\text{ViLoc}(\mathcal{B})| \geq d(m+1) \mid \mathbf{C} \cap \mathbf{BN}^c) + Z^{-1} d(m+1) \mathbb{P}_t(\#\mathcal{M}_\phi \geq m) \\
&\leq \left(1 + \frac{2}{5}\right) \tau^2 (2e\tau^2 d^{-1})^m + e^{\tau^{-1}} \left(1 + \frac{1}{10}\right) (2e\tau^2 d^{-1})^m.
\end{aligned}$$

Using  $\tau^2 \leq 2e^{\tau^{-1}}$  for  $\tau \geq 1$  completes the proof.  $\square$

The next lemma strengthens Lemma 2.2. Although the proof has already been sketched in Section 2, the reweighting of measure caused by conditioning on  $\mathbf{C} \cap \mathbf{BN}^c$  must be treated; this makes the proof among the more technical ones in the paper.

**Lemma 4.10.** *Assume that  $n \geq 2n_1$ . Consider  $\mathbb{P}_t$  given  $\mathbf{C} \cap \mathbf{BN}^c$  and further given  $\mathcal{B} \cap ((\mathcal{M}_\phi \cup \partial_{\text{ext}} \mathcal{M}_\phi) \times [0, 1])$  for a choice such that  $\max_{v \in V(\mathcal{M}_\phi)} d(\phi, v) \leq n_1 - 1$ . Note that each edge in  $\partial_{\text{ext}} \mathcal{M}_\phi$  supports at most one bar in  $\mathcal{B}$ ; let  $\ell$  denote the number of elements in  $\partial_{\text{ext}} \mathcal{M}_\phi$  that do support a bar in  $\mathcal{B}$ . Then the conditional probability of  $H_n^{\mathcal{B}} < \infty$  is at most  $(1 + \frac{1}{25}) e^\tau p_n \ell$ .*

**Proof.** Denote by  $\mathcal{B}^{\mathcal{M}}$  the set of bars in  $\mathcal{B}$  supported on elements of  $\mathcal{M}_\phi$ , and let  $\mathcal{B}^{\text{ext}}$  denote the set of bars in  $\mathcal{B}$  supported on elements of  $\partial_{\text{ext}} \mathcal{M}_\phi$  (so that  $\ell$  is the cardinality of  $\mathcal{B}^{\text{ext}}$ ).

Consider the meander  $X^{\mathcal{B}^{\mathcal{M}}}$  from  $(\phi, 0)$ , where note that only bars in  $\mathcal{B}^{\mathcal{M}}$  are used to define the meander's trajectory. The process  $X^{\mathcal{B}^{\mathcal{M}}}$  has a periodic trajectory during which it encounters a certain subset of the parent joints of the elements of  $\mathcal{B}^{\text{ext}}$ . Let  $\mathcal{B}^* \subseteq \mathcal{B}^{\text{ext}}$  denote the set of bars whose parent joints are so encountered; for  $b \in \mathcal{B}^*$ , set  $u_b = d(\phi, E(b)^-)$ .

The trajectory of the meander  $X^{\mathcal{B}}$  coincides with that of  $X^{\mathcal{B}^{\mathcal{M}}}$  until the two processes reach the parent joint of a bar in  $\mathcal{B}^*$  (which bar we call  $b_1$ ); at

this time,  $X^{\mathcal{B}}$  crosses  $b_1$  and begins a sojourn over the edges of the descendent tree  $T_{[E(b_1)^-]}$ . If this sojourn is of finite duration, it finishes by the recrossing of  $b_1$  by  $X^{\mathcal{B}}$ ; from  $b_1^+$ ,  $X^{\mathcal{B}}$  then pursues the same course as does  $X^{\mathcal{B}^{\mathcal{M}}}$  until the two processes encounter the parent joint of a second bar  $b_2$  in  $\mathcal{B}^*$ . In this way, the meanders are coupled, following a common trajectory, and with  $X^{\mathcal{B}}$  pursuing sojourns in the descendent trees of the vertices at the child joints of the successively encountered elements of  $\mathcal{B}^*$ . Each such sojourn must not include a visit by  $X^{\mathcal{B}}$  to  $\mathcal{V}_n \times [0, 1)$  if  $H_n^{\mathcal{B}} = \infty$  is to occur.

For each  $b \in \mathcal{B}^{\text{ext}}$ , let  $\text{Esc}_b$  denote the event that the meander  $X_{b^-}^{\mathcal{B}}$  visits  $\mathcal{V}_n \times [0, 1)$  before returning to  $b^-$ . The coupling in the preceding paragraph shows that  $\{H_n^{\mathcal{B}} < \infty\} \subseteq \cup_{b \in \mathcal{B}^{\text{ext}}} \text{Esc}_b$ . Hence, the conditional probability in the lemma's statement is at most  $\sum_{b \in \mathcal{B}^{\text{ext}}} q_b$ , where  $q_b$  is the probability of  $\text{Esc}_b$  under the conditional law in the statement.

It suffices then to argue that

$$q_b \leq \left(1 + \frac{1}{25}\right) e^\tau p_n \text{ for each } b \in \mathcal{B}^{\text{ext}}. \quad (4.14)$$

We begin by arguing that a similar inequality holds under the unconditioned law: that is, for  $v \in V(\mathcal{T})$ , we write  $\mathbb{P}_{t, \mathcal{B}}^v$  for the Poisson- $t$  distribution on  $E(\mathcal{T}_{[v]}) \times [0, 1)$ , and claim that

$$\mathbb{P}_{t, \mathcal{B}}^{E(b)^-}(\text{Esc}_b) \leq \left(1 + \frac{1}{25}\right) p_n \text{ for each } b \in \mathcal{B}^{\text{ext}}. \quad (4.15)$$

By Lemma 1.12,  $\mathbb{P}_{t, \mathcal{B}}^{E(b)^-}(\text{Esc}_b) = p_{n-u_b}$ . Note that for  $b \in \mathcal{B}^{\text{ext}}$ ,  $u_b = d(\phi, E(b)^+) + 1$ ; however,  $E(b)^+ \in V(\mathcal{M}_\phi)$  so that our hypotheses imply that  $u_b \leq n_1$  and thus that  $n - u_b \geq n_1$ . By the definition of  $n_1$  in Lemma 3.1,  $p_{n-u_b} \leq \left(1 + \frac{1}{25}\right) p_n$ . Thus (4.15) indeed holds.

Were  $\mathcal{B}$  on  $E(\mathcal{T}_{[b^-]})$  to have the Poisson- $t$  distribution under the conditional law, then  $q_b$  would equal  $\mathbb{P}_{t, \mathcal{B}}^{E(b)^-}(\text{Esc}_b)$  and by (4.15) we would be (more than) done in seeking (4.14). However, a size-biasing effect arising from our conditioning on  $\mathbf{C} \cap \mathbf{BN}^c$  means that this is not the case.

We will prove (4.14) by establishing a stronger statement. We will condition on more information, and make use of a variant of Lemma 4.2.

**Lemma 4.11.** *Let  $v \in V(\mathcal{T}_n)$ . Suppose given an arbitrary bar collection  $\mathcal{B}^v \subseteq E(\mathcal{T}^{[v]}) \times [0, 1)$ . Let  $\mathbb{P}_{t, \mathcal{B}}^{v, \mathbf{C} \cap \mathbf{BN}^c}$  denote the marginal distribution of  $\mathcal{B}$  on  $E(\mathcal{T}_{[v]}) \times [0, 1)$  under  $\mathbb{P}_t$  given that  $\mathcal{B} \cap (E(\mathcal{T}^{[v]}) \times [0, 1)) = \mathcal{B}^v$  and given  $\mathbf{C} \cap \mathbf{BN}^c$ . Then, for each  $\mathcal{B}' \subseteq E(\mathcal{T}_{[v]}) \times [0, 1)$ ,*

$$\frac{d\mathbb{P}_{t, \mathcal{B}}^{v, \mathbf{C} \cap \mathbf{BN}^c}}{d\mathbb{P}_{t, \mathcal{B}}^v}(\mathcal{B}') = Z_0^{-1} |(\text{ViLoc})(\mathcal{B}^v, \mathcal{B}')|,$$

where  $(\mathcal{B}^v, \mathcal{B}')$  is the bar collection in  $E(\mathcal{T}_n) \times [0, 1)$  whose intersections with  $E(\mathcal{T}^{[v]}) \times [0, 1)$  and  $E(\mathcal{T}_{[v]}) \times [0, 1)$  are  $\mathcal{B}^v$  and  $\mathcal{B}'$ . Here,

$$Z_0 = \int |(\text{ViLoc})(\mathcal{B}^v, \mathcal{B}')| d\mathbb{P}_{t, \mathcal{B}}^v(\mathcal{B}'). \quad (4.16)$$

**Proof.** Let  $A$  be an event measurable with respect to  $\mathcal{B} \cap (E(\mathcal{T}_{[v]}) \times [0, 1))$ . Note that

$$\begin{aligned} & \mathbb{P}_{t, \mathcal{B}}^v \left( A \cap \mathbf{C} \cap \mathbf{BN}^c \mid \mathcal{B} \cap (E(\mathcal{T}^{[v]}) \times [0, 1)) = \mathcal{B}^v \right) \\ &= \int_A \mathbb{P}_t \left( \mathcal{A} \in \text{ViLoc}(\mathcal{B}^v, \mathcal{B}') \mid \mathcal{B} = (\mathcal{B}^v, \mathcal{B}') \right) d\mathbb{P}_{t, \mathcal{B}}^v(\mathcal{B}'), \end{aligned} \quad (4.17)$$

because, given  $\mathcal{B} = (\mathcal{B}^v, \mathcal{B}')$ , the condition  $\mathcal{A} \in \text{ViLoc}(\mathcal{B}^v, \mathcal{B}')$  is equivalent to  $\mathbf{C} \cap \mathbf{BN}^c$ . By the independence of  $\mathcal{B}$  and  $\mathcal{A}$ ,

$$\mathbb{P}_t \left( \mathcal{A} \in \text{ViLoc}(\mathcal{B}^v, \mathcal{B}') \mid \mathcal{B} = (\mathcal{B}^v, \mathcal{B}') \right) = \frac{|\text{ViLoc}(\mathcal{B}^v, \mathcal{B}')|}{\#E(\mathcal{T}_n)}, \quad (4.18)$$

which implies by the definition of a Radon-Nikodym derivative that

$$\frac{d\mathbb{P}_{t, \mathcal{B}}^{v, \mathbf{C} \cap \mathbf{BN}^c}}{d\mathbb{P}_{t, \mathcal{B}}^v}(\mathcal{B}') = \frac{|\text{ViLoc}(\mathcal{B}^v, \mathcal{B}')|}{\#E(\mathcal{T}_n) \mathbb{P}_t \left( \mathbf{C} \cap \mathbf{BN}^c \mid \mathcal{B} \cap (E(\mathcal{T}_{[v]}) \times [0, 1)) = \mathcal{B}^v \right)}.$$

Thus, (4.16) holds with  $Z_0 = \#E(\mathcal{T}_n) \mathbb{P}_t(\mathbf{C} \cap \mathbf{BN}^c \mid \mathcal{B} \cap (E(\mathcal{T}_{[v]}) \times [0, 1)) = \mathcal{B}^v)$ . Applying (4.17) for  $A = \Omega$  and then (4.18) yields the form for  $Z_0$  stated in the lemma.  $\square$

Condition  $\mathbb{P}_t$  on the data specified in Lemma 4.10's statement; then select  $b_0 \in \mathcal{B}^{\text{ext}}$  measurably with respect to this data and further condition on  $\mathcal{B} \cap (E(\mathcal{T}^{[b_0^-]}) \times [0, 1))$ . We will argue that the conditional probability of  $\text{Esc}_{b_0}$  is at most a  $e^\tau$ -multiple of its unconditioned probability. Set  $e_0 = E(b_0)$  and use the shorthand  $v = e_0^-$ . In the notation of Lemma 4.11, our conditioning is comprised of an instance of  $\mathcal{B}^v$  and the occurrence of  $\mathbf{C} \cap \mathbf{BN}^c$ , so that the marginal distribution of  $\mathcal{B}$  on  $E(\mathcal{T}_{[v]}) \times [0, 1)$  is  $\mathbb{P}_{t, \mathcal{B}}^{v, \mathbf{C} \cap \mathbf{BN}^c}$ ; hence, the conditional probability of  $\text{Esc}_{b_0}$  is  $\mathbb{P}_{t, \mathcal{B}}^{v, \mathbf{C} \cap \mathbf{BN}^c}(\text{Esc}_{b_0})$ , and our claim is

$$\mathbb{P}_{t, \mathcal{B}}^{v, \mathbf{C} \cap \mathbf{BN}^c}(\text{Esc}_{b_0}) \leq e^\tau \mathbb{P}_{t, \mathcal{B}}^v(\text{Esc}_{b_0}). \quad (4.19)$$

Using Lemma 4.11, we now find such an upper bound for  $\mathbb{P}_{t, \mathcal{B}}^{v, \mathbf{C} \cap \mathbf{BN}^c}(\text{Esc}_{b_0})$  by analysing  $\text{ViLoc}$  as a function of  $\mathcal{B}'$  for the given  $\mathcal{B}^v$ . Our deduction will be based on the information stated in the next paragraph.

Write  $\text{NoBar}_v(\mathcal{B}')$  for the event that no bar in  $\mathcal{B}'$  lies in  $\{e \in E(\mathcal{T}) : e^+ = v\} \times [0, 1)$ . Set  $\chi \in [0, H_n^{\mathcal{B}^v}]$  to equal  $\inf\{s > 0 : X^{\mathcal{B}^v}(s) = b_0^+\} \wedge H_n^{\mathcal{B}^v}$ . We

claim that

$$\text{if } \chi = H_n^{\mathcal{B}^v} \text{ then } \text{ViLoc}(\mathcal{B}^v, \mathcal{B}') = \text{ViLoc}(\mathcal{B}^v) \text{ for all } \mathcal{B}' \subseteq E(\mathcal{T}_{[v]}) \times [0, 1), \quad (4.20)$$

while if  $\chi < H_n^{\mathcal{B}^v}$  then

$$\text{ViLoc}(\mathcal{B}^v, \mathcal{B}') \begin{cases} \subseteq \text{ViLoc}(\mathcal{B}^v) \cup (\{e_0\} \times [0, 1)) & \text{for all } \mathcal{B}' \subseteq E(\mathcal{T}_{[v]}) \times [0, 1), \\ = \text{ViLoc}(\mathcal{B}^v) \cup (\{e_0\} \times [0, 1)) & \text{if } \text{NoBar}_v(\mathcal{B}') \text{ occurs.} \end{cases} \quad (4.21)$$

To derive (4.20) and (4.21) we will employ a little further notation. For  $t \in [0, \infty]$ , let  $\text{Admis}_t$  denote the set of bar locations  $b \in E(\mathcal{T}_n) \times [0, 1)$  at least one of whose joints belongs to  $X^{\mathcal{B}^v}[0, t)$  and for which each edge on the path  $P_{\phi, E(b)^+}$  supports at least two bars in  $\mathcal{B}^v$ . Note that  $\text{Admis}_{H_n^{\mathcal{B}^v}} = \text{ViLoc}(\mathcal{B}^v)$ .

Note firstly that the meanders  $X^{\mathcal{B}^v}$  and  $X^{(\mathcal{B}^v, \mathcal{B}')}$  coincide until time  $\chi$ . Should  $\chi = H_n^{\mathcal{B}^v}$ , then  $\text{ViLoc}(\mathcal{B}^v, \mathcal{B}') = \text{Admis}_{H_n^{\mathcal{B}^v}}$  for all choices of  $\mathcal{B}' \subseteq E(\mathcal{T}_{[v]}) \times [0, 1)$ ; thus (4.20) is verified. In the case that  $\chi < H_n^{\mathcal{B}^v}$ , we claim that

$$\text{ViLoc}(\mathcal{B}^v, \mathcal{B}') \begin{cases} \subseteq \text{Admis}_\chi \cup (\{e_0\} \times [0, 1)) & \text{if } \text{Esc}_{b_0}(\mathcal{B}') \text{ occurs,} \\ = \text{Admis}_{H_n^{\mathcal{B}^v}} \cup (\{e_0\} \times [0, 1)) & \text{if } \text{NoBar}_v(\mathcal{B}') \text{ occurs,} \\ \subseteq \text{Admis}_{H_n^{\mathcal{B}^v}} \cup (\{e_0\} \times [0, 1)) & \text{if } \text{Esc}_{b_0}(\mathcal{B}')^c \text{ occurs.} \end{cases} \quad (4.22)$$

Note that (4.22) indeed implies (4.21), since  $\text{Admis}_\chi \subseteq \text{Admis}_{H_n^{\mathcal{B}^v}}$  and  $\text{Admis}_{H_n^{\mathcal{B}^v}} = \text{ViLoc}(\mathcal{B}^v)$ .

To verify (4.22), note that, if  $\chi < H_n^{\mathcal{B}^v}$ , then each of  $X^{\mathcal{B}^v}$  and  $X^{(\mathcal{B}^v, \mathcal{B}')}$  arrives at  $b_0^+$  at time  $\chi$ , at which moment  $X^{(\mathcal{B}^v, \mathcal{B}')}$  crosses  $b_0$  to arrive at  $b_0^-$ . Should  $\text{Esc}_{b_0}$  occur,  $X^{(\mathcal{B}^v, \mathcal{B}')}$  remains in  $E(\mathcal{T}_{[v]}) \times [0, 1)$  until time  $H_n^{(\mathcal{B}^v, \mathcal{B}')}$ ; if for any  $s \in [\chi, H_n^{(\mathcal{B}^v, \mathcal{B}')}]$ ,  $X^{(\mathcal{B}^v, \mathcal{B}')(s)}$  is to be a joint of some element of  $\text{ViLoc}(\mathcal{B}^v, \mathcal{B}')$ , it is necessary that the path from  $\phi$  to the parent of the vertex component of  $X^{(\mathcal{B}^v, \mathcal{B}')(s)}$  support at least two bars in  $(\mathcal{B}^v, \mathcal{B}')$ . However, unless  $X^{(\mathcal{B}^v, \mathcal{B}')(s)} \in \{v\} \times [0, 1)$ , this path contains the edge  $e_0$ , which supports just one bar in  $(\mathcal{B}^v, \mathcal{B}')$ ; for a similar reason, when  $X^{(\mathcal{B}^v, \mathcal{B}')(s)} \in \{v\} \times [0, 1)$ , a bar of which  $X^{(\mathcal{B}^v, \mathcal{B}')(s)}$  is a joint may belong to  $\text{ViLoc}(\mathcal{B}^v, \mathcal{B}')$  only if it is supported on  $e_0$ . This argument yields that  $\text{ViLoc}(\mathcal{B}^v, \mathcal{B}')$  equals the union of  $\text{Admis}_\chi$  and some subset of  $e_0 \times [0, 1)$  in the case that  $\{\chi < H_n^{\mathcal{B}^v}\} \cap \text{Esc}_{b_0}$  occurs; we have obtained the first claim of (4.22).

If  $\{\chi < H_n^{\mathcal{B}^v}\} \cap \text{Esc}_{b_0}^c$  occurs, then, after time  $\chi$ ,  $X^{(\mathcal{B}^v, \mathcal{B}')}$  remains in  $E(\mathcal{T}_{[v]}) \times [0, 1)$  until returning to  $b_0^-$  and recrossing  $b_0$ , after which it then pursues the same trajectory as  $X^{\mathcal{B}^v}$  does from  $b_0^+$ . As we saw in the preceding paragraph, this sojourn in  $E(\mathcal{T}_{[v]}) \times [0, 1)$  may contribute to  $\text{ViLoc}(\mathcal{B}^v, \mathcal{B}')$

only elements in  $e_0 \times [0, 1)$ . Whether or not  $H_n^{\mathcal{B}^v}$  is finite, we have obtained the third claim of (4.22).

Note that  $\text{NoBar}_v \subseteq \text{Esc}_{b_0}^c$ , so that the last paragraph applies equally in this case. In fact, in this case,  $X^{(\mathcal{B}^v, \mathcal{B}^')}$  spends the duration  $[\chi, \chi + 1]$  traversing entirely the pole at  $v$  and thus meeting a joint of every bar supported on  $e_0 \times [0, 1)$ . This confirms the equality asserted by (4.22) in the second case and concludes the proof of (4.22) and thus of (4.20) and (4.21).

We now use (4.20) and (4.21) to argue that (4.19) holds.

If the  $\mathcal{B}^v$ -measurable event  $\chi = H_n^{\mathcal{B}^v}$  occurs, then (4.20) implies that the Radon-Nikodym derivative in Lemma 4.11 is independent of  $\mathcal{B}^'$ , so that  $\mathbb{P}_{t, \mathcal{B}}^{v, \mathcal{C} \cap \text{BN}^c}(\text{Esc}_{b_0}) = \mathbb{P}_{t, \mathcal{B}}^v(\text{Esc}_{b_0})$ , yielding (4.19) in this case.

To treat the case that  $\chi < H_n^{\mathcal{B}^v}$ , note that the normalization  $Z_0$  in Lemma 4.11 is at least  $\int_{\text{NoBar}_v} |\text{ViLoc}(\mathcal{B}^v, \mathcal{B}^')| d\mathbb{P}_{t, \mathcal{B}}^v(\mathcal{B}^')$ ; by the second claim of (4.21) and  $\mathbb{P}_{t, \mathcal{B}}^v(\text{NoBar}_v) = e^{-\tau}$ , this integral equals  $e^{-\tau} |\text{ViLoc}(\mathcal{B}^v) \cup (\{e_0\} \times [0, 1))|$ . By Lemma 4.11,  $\mathbb{P}_{t, \mathcal{B}}^{v, \mathcal{C} \cap \text{BN}^c}(\text{Esc}_{b_0}) = Z_0^{-1} \int_{\text{Esc}_{b_0}} |(\text{ViLoc})(\mathcal{B}^v, \mathcal{B}^')| d\mathbb{P}_{t, \mathcal{B}}^v(\mathcal{B}^')$ ; the first claim of (4.21) and our lower bound on  $Z_0$  show that this expression is bounded above by  $e^{\tau} \mathbb{P}_{t, \mathcal{B}}^v(\text{Esc}_{b_0})$ . This completes the derivation of (4.19).

From (4.15) and (4.19) follows  $\mathbb{P}_{t, \mathcal{B}}^{v, \mathcal{C} \cap \text{BN}^c}(\text{Esc}_{b_0}) \leq (1 + \frac{1}{25}) e^{\tau} p_n$ . The quantity  $q_{b_0}$  being an average of  $\mathbb{P}_{t, \mathcal{B}}^{v, \mathcal{C} \cap \text{BN}^c}(\text{Esc}_{b_0})$  over choices of  $\mathcal{B}^{b_0}$  taking the value  $\mathcal{B} \cup \mathcal{B}^{\text{ext}}$  on  $(\mathcal{M}_\phi \cup \partial_{\text{ext}} \mathcal{M}_\phi) \times [0, 1)$ , we obtain (4.14). This completes the proof of Lemma 4.10.  $\square$

**Corollary 4.12.** *Let  $0 \leq m \leq n_1 - 1$ . Then*

$$\mathbb{P}_t \left( H_n^{\mathcal{B}} < \infty \mid \mathcal{C} \cap \text{BN}^c, \#\mathcal{M}_\phi = m \right) \leq \left( 1 + \frac{1}{25} \right) e^{\tau} p_n d(m+1).$$

**Proof.** Further condition  $\mathbb{P}_t$  on the value of  $\mathcal{B}$  restricted to the product of  $\mathcal{M}_\phi \cup \partial_{\text{ext}} \mathcal{M}_\phi$  and  $[0, 1)$ . In the notation of Lemma 4.10,  $\ell \leq \#\partial_{\text{ext}} \mathcal{M}_\phi$ , which is at most  $d + (d-1)\#\mathcal{M}_\phi$ . By Lemma 4.10, the probability given this further conditioning that  $H_n^{\mathcal{B}} < \infty$  is at most  $(1 + \frac{1}{25}) e^{\tau} p_n \ell$ . Averaging over the further conditioning gives the corollary.  $\square$

Noting that  $\mathcal{P}^- \subseteq \{H_n^{\mathcal{B}} < \infty\}$ , we now obtain the estimate (4.11) from Lemma 4.9 and Corollary 4.12.

**4.6. The loss term with  $\#\mathcal{M}_\phi = 1$ .** The inequality (4.11) for  $m = 1$  yields an upper bound that exceeds  $12\tau^2 e^{2\tau} p_n$ ; we need at least a smaller constant multiple of  $p_n$  so as not to deplete the gain term (4.5). Here we prove the bound (4.10) by replacing the use of Corollary 4.12 with the more informative Lemma 4.10.

In considering in this subsection the case that  $\#\mathcal{M}_\phi = 1$ , we will denote by  $g$  the unique member of  $\mathcal{M}_\phi$  (note that  $g \in \mathcal{E}_0$ ). Let  $S_1$  (and  $S_2$ ) denote the set of edges incident to  $g^+$  (and  $g^-$ ) supporting exactly one bar in  $\mathcal{B}$ . Set

$S = S_1 \cup S_2$ . We will estimate the typical size of  $S$  when  $\#\mathcal{M}_\phi = 1$  and then apply Lemma 4.10 with  $\ell = \#S$ .

**Lemma 4.13.** *Assume that  $n \geq 2n_1$ . For  $k \in \{0, \dots, 2d-1\}$ ,*

$$\mathbb{P}_t\left(H_n^{\mathcal{B}} < \infty \mid \#\mathcal{M}_\phi = 1, \#S = k, \mathbf{C} \cap \mathbf{BN}^c\right) \leq \left(1 + \frac{1}{25}\right)e^\tau k p_n.$$

**Proof.** This is implied by Lemma 4.10.  $\square$

**Lemma 4.14.** *Let  $d \geq 11\tau^2$ . For  $k \in \{1, \dots, 2d-1\}$ ,*

$$\mathbb{P}_t\left(\#\mathcal{M}_\phi = 1, \#S = k \mid \mathbf{C} \cap \mathbf{BN}^c\right) \leq 2\left(1 + \frac{1}{10}\right)\tau^2 e^\tau d^{-1} \tilde{q}_k,$$

where  $\{\tilde{q}_k : 0 \leq k \leq 2d-1\}$  denotes the law  $\text{Bin}(2d-1, \frac{t}{1+t})$ .

**Proof.** If  $\mathcal{B}$  is such that  $\mathcal{M}_\phi$  has a unique element  $g \in E(\mathcal{T}_n)$ , note that  $\text{ViLoc}(\mathcal{B})$  is a subset of bar locations whose parent joint lies in  $\{g^+, g^-\} \times [0, 1)$ ; hence,  $|\text{ViLoc}(\mathcal{B})| \leq 2d$ . By Lemmas 4.2 and 4.5, the conditional probability in Lemma 4.14 is thus at most a multiple  $2dZ^{-1} \leq 2e^\tau$  of

$$\mathbb{P}_t\left(\#\mathcal{M}_\phi = 1, \#S = k\right).$$

Given  $\mathcal{M}_\phi = \{g\}$ , the  $2d-1$  edges other than  $g$  incident to either  $g^+$  or  $g^-$  independently have probability  $\frac{t}{1+t}$  to support precisely one bar. Thus, under  $\mathbb{P}_t$  given  $\mathcal{M}_\phi = \{g\}$ ,  $\#S$  has the  $\text{Bin}(2d-1, \frac{t}{1+t})$  law. Hence, Lemma 4.14 follows from  $\mathbb{P}_t(\#\mathcal{M}_\phi = 1) \leq \left(1 + \frac{1}{10}\right)\tau^2 d^{-1}$ , which is known by Lemma 4.7 when  $d \geq 11\tau^2$ .  $\square$

Lemmas 4.13 and 4.14 imply that, for  $1 \leq k \leq 2d-1$ ,

$$\mathbb{P}_t\left(H_n^{\mathcal{B}} < \infty, \#\mathcal{M}_\phi = 1, \#S = k \mid \mathbf{C} \cap \mathbf{BN}^c\right) \leq 2\left(1 + \frac{1}{5}\right)k p_n \tau^2 e^{2\tau} d^{-1} \tilde{q}_k.$$

Note then that

$$\begin{aligned} & \mathbb{P}_t\left(H_n^{\mathcal{B}} < \infty, \#\mathcal{M}_\phi = 1 \mid \mathbf{C} \cap \mathbf{BN}^c\right) \\ & \leq \sum_{k=0}^{2d-1} \mathbb{P}_t\left(H_n^{\mathcal{B}} < \infty, \#\mathcal{M}_\phi = 1, \#S = k \mid \mathbf{C} \cap \mathbf{BN}^c\right) \\ & \leq 2\left(1 + \frac{1}{5}\right)p_n \tau^2 e^{2\tau} d^{-1} \sum_{k=0}^{2d-1} k \tilde{q}_k \\ & = 2\left(1 + \frac{1}{5}\right)p_n \tau^2 e^{2\tau} d^{-1} (2d-1) \frac{t}{1+t} \leq 4\left(1 + \frac{1}{5}\right)p_n \tau^3 e^{2\tau} d^{-1}. \end{aligned}$$

The inclusion  $\mathbf{P}^- \subseteq \{H_n^{\mathcal{B}} < \infty\}$  yields (4.10).

**4.7. The loss term with  $\#\mathcal{M}_\phi = 0$ .** For  $m = 0$ , (4.11) yields an upper bound that exceeds  $6e^{2\tau-1}dp_n$ , which is certainly insufficient for high  $d$ . The improvement afforded by the approach of the preceding subsection is also not enough when  $m = 0$ . Indeed, it is easy to see that the  $\mathbb{P}_t(\cdot | \mathbf{C} \cap \mathbf{BN}^c \cap \{\mathcal{M}_\phi = \emptyset\})$ -probability that precisely one element of  $\mathcal{E}_0$  supports a bar in  $\mathcal{B}$  is bounded away from zero (uniformly in  $d$ ); use of Lemma 4.10 can thus show an upper bound on  $\mathbb{P}_t(\mathbf{P}^-, \mathcal{M}_\phi = \emptyset | \mathbf{C} \cap \mathbf{BN}^c)$  of the form of some  $d$ -independent multiple of  $p_n$  for  $m = 0$ .

To obtain the further improvement (4.9), we now state a new necessary condition for off-pivotality:

**Lemma 4.15.** *If  $E(\mathcal{A})$  supports no bar in  $\mathcal{B}$  then  $\mathbf{P}^-$  does not occur.*

For use in the proof, write  $H_{\mathcal{A}} = \inf \{s > 0 : X(s) \in \{\mathcal{A}^+, \mathcal{A}^-\}\}$ , and note that  $X^{\mathcal{B}}$  and  $X^{\mathcal{B} \cup \mathcal{A}}$  coincide until  $H_{\mathcal{A}}$ , whose value the two processes share.

**Proof of Lemma 4.15.** By Lemma 1.10, we may assume that  $\mathbf{C}$  occurs. If  $E(\mathcal{A})$  supports no bar in  $\mathcal{B}$ , then the trajectory  $X^{\mathcal{B} \cup \mathcal{A}}$  is formed from that of  $X^{\mathcal{B}}$  as follows. Note that the two processes reach the parent joint of  $\mathcal{A}$  at time  $H_{\mathcal{A}}$ : it is impossible that  $X^{\mathcal{B}}$  be at the pole of  $\mathcal{A}^-$  at any time, because this would entail  $X^{\mathcal{B}}$  crossing  $E(\mathcal{A})$ , an edge which supports no bar in  $\mathcal{B}$ . Thus, at time  $H_{\mathcal{A}}$  it is from the parent to the child joint that  $X^{\mathcal{B} \cup \mathcal{A}}$  crosses  $\mathcal{A}$ . This meander then spends a duration in  $E(\mathcal{T}_{[E(\mathcal{A})^-]}) \times [0, 1)$ . It may visit  $\mathcal{V}_n \times [0, 1)$  during this sojourn, so that  $H_n^{\mathcal{B} \cup \mathcal{A}} < \infty$  occurs, thus excluding  $\mathbf{P}^-$ . If this does not happen,  $X^{\mathcal{B} \cup \mathcal{A}}$  recrosses  $\mathcal{A}$  to reach  $\mathcal{A}^+$  again and then continues to follow the trajectory of  $X^{\mathcal{B}}$  from this point. The two processes have no further opportunity to diverge (except by a return to  $\mathcal{A}^+$  in a later circuit), and this makes  $\mathbf{P}^-$  impossible.  $\square$

In now considering the case that  $\mathcal{M}_\phi$  is empty, we change the notation of the preceding subsection and write  $S$  (which is defined under  $\mathbb{P}_t$ ) for the set of edges incident to  $\phi$  that support exactly one bar in  $\mathcal{B}$ .

To enhance the argument of Subsection 4.6 with the use of Lemma 4.15, it is natural to seek to argue that the conditional probability that  $E(\mathcal{A})$  supports a bar in  $\mathcal{B}$  is small, given  $\mathbf{C} \cap \mathbf{BN}^c$  and that  $\#S$  is not too large. Figure 5 illustrates a difficulty for some approaches to proving such a claim.

In light of this difficulty, we now define an event which  $\mathcal{B}$  typically satisfies whose occurrence will ensure that the conditional probability that  $E(\mathcal{A})$  supports a bar in  $\mathcal{B}$  is indeed small. The *no-quick-getaway* event  $\mathbf{NQG}$  is defined to be  $\mathcal{B} \cap (\mathcal{E}_0 \times [0, d^{-1/2})) = \emptyset$ . First, we check that  $\mathbf{NQG}$  is typical for the measure in question.

**Lemma 4.16.** *If  $d \geq 15\tau^2$  then*

$$\mathbb{P}_t(\mathbf{NQG}^c | \mathbf{C} \cap \mathbf{BN}^c, \mathcal{M}_\phi = \emptyset) \leq (1 + \frac{1}{5})\tau e^{2\tau} d^{-1/2}.$$

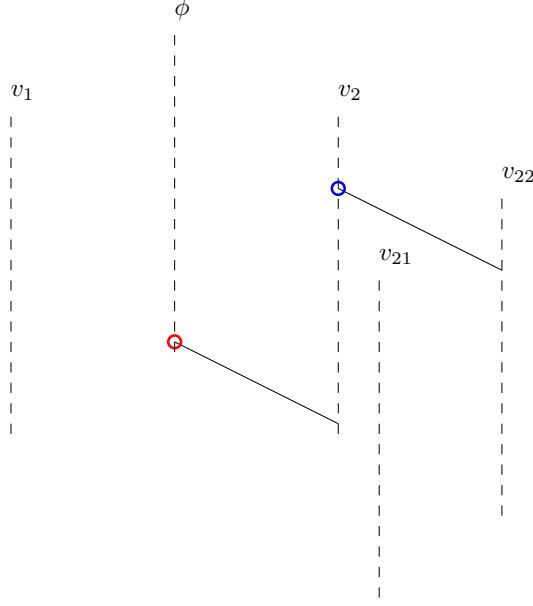


FIGURE 5. Condition  $\mathbb{P}_t$  on a given bar collection  $\mathcal{B}$  and on  $\mathbf{C} \cap \mathbf{BN}^c$ . Suppose that  $\mathcal{B}$  is as depicted: the meander from  $(\phi, 0)$  very quickly (at time  $h_1 > 0$  and at the red dot) crosses a bar supported on  $(\phi, v_1)$ , and remains at the pole of  $v_2$  until a later time  $h_2$  (at the blue dot), at which it crosses a bar supported on  $(v_2, v_{22})$ , not to return later (due to bars over edges not depicted). By Lemma 4.2, the conditional distribution of  $\mathcal{A}$  is uniform over the set  $\text{ViLoc}(\mathcal{B})$ . This set is the union of  $[\phi, v_1] \times [0, h_1]$  and  $[\phi, v_2] \times [0, h_2]$ . Thus,  $\mathcal{A}$  has probability  $\frac{h_2}{h_1+h_2} \gg 1/2$  to appear over the edge  $(\phi, v_2)$ . A similar example with a tree of high offspring degree shows that there are choices of  $\mathcal{B}$  on such a tree with only one bar in  $\mathcal{E}_0 \times [0, 1)$  but where, under  $\mathbb{P}_t$  given  $\mathcal{B}$  and  $\mathbf{C} \cap \mathbf{BN}^c$ ,  $\mathcal{A}$  is highly likely to appear over the element of  $\mathcal{E}_0$  that supports a bar in  $\mathcal{B}$ .

**Proof.** The probability in question equals

$$\frac{\mathbb{P}_t(\text{NQG}^c, \mathcal{M}_\phi = \emptyset | \mathbf{C} \cap \mathbf{BN}^c)}{\mathbb{P}_t(\mathcal{M}_\phi = \emptyset | \mathbf{C} \cap \mathbf{BN}^c)}. \quad (4.23)$$

Note that

$$\mathbb{P}_t(\mathcal{M}_\phi = \emptyset | \mathbf{C} \cap \mathbf{BN}^c) \geq (1 - \frac{1}{6})e^{-\tau}. \quad (4.24)$$

Indeed,  $\text{NoBar} \subseteq \{\mathcal{M}_\phi = \emptyset\}$  so that Lemma 4.8 implies (4.24).

Note that  $\mathcal{M}_\phi = \emptyset$  implies that  $\text{ViLoc}(\mathcal{B}) \subseteq \mathcal{E}_\phi \times [0, 1)$  and thus

$$\{\mathcal{M}_\phi = \emptyset\} \subseteq \{|\text{ViLoc}(\mathcal{B})| \leq d\}. \quad (4.25)$$

Hence, Lemmas 4.2 and 4.5 imply that

$$\begin{aligned} & \mathbb{P}_t(\text{NQG}^c, \mathcal{M}_\phi = \emptyset | \mathbf{C} \cap \text{BN}^c) \leq dZ^{-1} \mathbb{P}_t(\text{NQG}^c, \mathcal{M}_\phi = \emptyset) \\ & \leq dZ^{-1} \mathbb{P}_t(\text{NQG}^c) \leq dZ^{-1} (1 - \exp\{-d^{1/2}t\}) \leq e^\tau d^{1/2}t = e^\tau \tau d^{-1/2}. \end{aligned}$$

Hence, (4.23) is at most  $(1 + \frac{1}{5})e^{2\tau} \tau d^{-1/2}$ .  $\square$

We now divide into cases with a view to enhancing the technique of Subsection 4.6 by the use of Lemma 4.15. Let  $r \in (0, \infty)$  be a parameter whose value will be set shortly. Note that

$$\left\{ \mathbf{P}^-, \#S \geq 1, \mathcal{M}_\phi = \emptyset, \mathbf{C} \cap \text{BN}^c \right\} = \mathbf{A}_1 \cup \mathbf{A}_2 \cup \mathbf{A}_3, \quad (4.26)$$

where

$$\begin{aligned} \mathbf{A}_1 &= \left\{ \mathbf{P}^-, 1 \leq \#S \leq r \log d, \text{NQG}, \mathcal{M}_\phi = \emptyset, \mathbf{C} \cap \text{BN}^c \right\}, \\ \mathbf{A}_2 &= \left\{ \mathbf{P}^-, 1 \leq \#S \leq r \log d, \text{NQG}^c, \mathcal{M}_\phi = \emptyset, \mathbf{C} \cap \text{BN}^c \right\}, \end{aligned}$$

and

$$\mathbf{A}_3 = \left\{ \mathbf{P}^-, \#S > r \log d, \mathcal{M}_\phi = \emptyset, \mathbf{C} \cap \text{BN}^c \right\},$$

4.7.1. *Bounding  $\mathbb{P}_t(\mathbf{A}_1)$ .* Note that  $\mathbb{P}_t(\mathbf{A}_1)$  equals

$$\begin{aligned} & \sum_{j=1}^{\lfloor r \log d \rfloor} \mathbb{P}_t \left( \mathbf{P}^-, \#S = j, \text{NQG}, \mathcal{M}_\phi = \emptyset, \mathbf{C} \cap \text{BN}^c \right) \\ & \leq \sum_{j=1}^{\lfloor r \log d \rfloor} \mathbb{P}_t \left( H_n^\mathcal{B} < \infty, \mathcal{B} \cap (E(\mathcal{A}) \times [0, 1)) \neq \emptyset, \#S = j, \text{NQG}, \mathcal{M}_\phi = \emptyset, \mathbf{C} \cap \text{BN}^c \right), \end{aligned}$$

where the inequality is due to Lemma 4.15 and  $\mathbf{P}^- \subseteq \{H_n^\mathcal{B} < \infty\}$ .

We now confirm that conditioning on NQG indeed forces the probability that  $E(\mathcal{A})$  supports a bar in  $\mathcal{B}$  to be small.

**Lemma 4.17.** *For  $1 \leq j \leq d$ ,*

$$\mathbb{P}_t \left( \mathcal{B} \cap (E(\mathcal{A}) \times [0, 1)) \neq \emptyset \mid H_n^\mathcal{B} < \infty, \#S = j, \text{NQG}, \mathcal{M}_\phi = \emptyset, \mathbf{C} \cap \text{BN}^c \right) \leq jd^{-1/2}.$$

**Proof.** Recall from Lemma 4.2 that, given  $\text{ViLoc}(\mathcal{B})$ , the location of  $\mathcal{A}$  has the conditional distribution of normalized Lebesgue measure on  $\text{ViLoc}(\mathcal{B})$ . Thus, given any  $\mathcal{B}$  realizing the conditioning in the lemma, the conditional probability that  $E(\mathcal{A})$  supports at least one bar in  $\mathcal{B}$  is

$$\frac{|\{b \in \text{ViLoc}(\mathcal{B}) : \mathcal{B} \cap (E(b) \times [0, 1)) \neq \emptyset\}|}{|\text{ViLoc}(\mathcal{B})|} \quad (4.28)$$

Note that  $\mathcal{M}_\phi = \emptyset$  implies that  $\text{ViLoc}(\mathcal{B}) \subseteq \mathcal{E}_0 \times [0, 1)$ , so that the set in the numerator in (4.28) is a subset of  $S \times [0, 1)$ ;  $\#S = j$  thus implies that the numerator is at most  $j$ . On the other hand, whenever NQG occurs,  $\mathcal{B}$  is

such that  $\mathcal{E}_\phi \times [0, d^{-1/2}] \subseteq \text{ViLoc}(\mathcal{B})$ , so that the denominator in (4.28) is at least  $d^{1/2}$ .  $\square$

**Lemma 4.18.** *Assume that  $d \geq 12^2(1 + \frac{1}{5})^2 \tau^2 e^{6\tau}$ . For  $1 \leq j \leq d$ ,*

$$\mathbb{P}_t(\#S = j \mid \text{NQG}, \mathcal{M}_\phi = \emptyset, \mathbf{C} \cap \text{BN}^c) \leq (1 + \frac{1}{3})e^{2\tau} \bar{q}_j,$$

where  $\bar{q}_j$  is the probability that  $\text{Bin}(d, \frac{t}{1+t})$  assumes the value  $j$ .

**Proof.** Note that

$$\begin{aligned} & \mathbb{P}_t(\#S = j \mid \text{NQG}, \mathcal{M}_\phi = \emptyset, \mathbf{C} \cap \text{BN}^c) \\ & \leq \frac{\mathbb{P}_t(\#S = j, \mathcal{M}_\phi = \emptyset \mid \mathbf{C} \cap \text{BN}^c)}{\mathbb{P}_t(\text{NQG}, \mathcal{M}_\phi = \emptyset \mid \mathbf{C} \cap \text{BN}^c)}. \end{aligned}$$

Recalling (4.25), Lemmas 4.2 and 4.5 imply that  $\mathbb{P}_t(\#S = j, \mathcal{M}_\phi = \emptyset \mid \mathbf{C} \cap \text{BN}^c) \leq dZ^{-1} \mathbb{P}_t(\#S = j, \mathcal{M}_\phi = \emptyset) \leq e^\tau \mathbb{P}_t(\#S = j \mid \mathcal{M}_\phi = \emptyset) = e^\tau \bar{q}_j$ .

Note also that  $\mathbb{P}_t(\mathcal{M}_\phi = \emptyset \mid \mathbf{C} \cap \text{BN}^c) \geq \mathbb{P}_t(\text{NoBar} \mid \mathbf{C} \cap \text{BN}^c) \geq (1 - \frac{1}{6})e^{-\tau}$  by Lemma 4.8. Applying Lemma 4.16 and  $(1 + \frac{1}{5})\tau e^{3\tau} d^{-1/2} \leq \frac{1}{12}$ , we find that  $\mathbb{P}_t(\text{NQG}, \mathcal{M}_\phi = \emptyset \mid \mathbf{C} \cap \text{BN}^c) \geq (1 - \frac{1}{4})e^{-\tau}$ .  $\square$

**Lemma 4.19.** *For  $1 \leq j \leq d$ ,*

$$\mathbb{P}_t(H_n^{\mathcal{B}} < \infty \mid \#S = j, \text{NQG}^c, \mathcal{M}_\phi = \emptyset, \mathbf{C} \cap \text{BN}^c) \leq (1 + \frac{1}{25})e^\tau p_n j.$$

**Proof.** Given that  $\mathcal{M}_\phi = \emptyset$ , the events  $\text{NQG}^c$  and  $\#S = j$  are measurable with respect to  $\mathcal{B} \cap ((\mathcal{M}_\phi \cup \partial_{\text{ext}} \mathcal{M}_\phi) \times [0, 1))$ . Thus, Lemma 4.19 is implied by Lemma 4.10.  $\square$

We remark that, were the event that  $E(\mathcal{A})$  supports a bar in  $\mathcal{B}$  to be introduced into the conditioning in Lemma 4.19, the use of Lemma 4.10 would not be possible, because this event introduces further weighting effects. It is for this reason that we have chosen the ordering in conditioning in the three preceding lemmas.

Applying  $\mathbf{P}^- \subseteq \{H_n^{\mathcal{B}} < \infty\}$  and Lemmas 4.17, 4.18 and 4.19 to (4.27), we find that, if  $d \geq 12^2(1 + \frac{1}{5})^2 \tau^2 e^{6\tau}$ ,

$$\begin{aligned} \mathbb{P}_t(\mathbf{A}_1) & \leq \sum_{j=1}^{\lfloor \tau \log d \rfloor} \mathbb{P}_t(\mathcal{M}_\phi = \emptyset, \mathbf{C} \cap \text{BN}^c, \text{NQG}) \left( (1 + \frac{1}{3})e^{2\tau} \bar{q}_j \right) \left( (1 + \frac{1}{25})e^\tau p_n j \right) (j d^{-1/2}) \\ & \leq (1 + \frac{2}{5})e^{3\tau} p_n \tau (1 + \tau) d^{-1/2} \mathbb{P}_t(\mathcal{M}_\phi = \emptyset, \mathbf{C} \cap \text{BN}^c, \text{NQG}), \end{aligned}$$

where we used  $\sum_{j=1}^d j^2 \bar{q}_j = d \frac{t}{1+t} + d(d-1) \left( \frac{t}{1+t} \right)^2 \leq \tau + \tau^2$ .

4.7.2. *Bounding  $\mathbb{P}_t(\mathbf{A}_2)$ .* By Lemmas 4.16 and 4.19, if  $d \geq 15\tau^2$ ,

$$\begin{aligned} \mathbb{P}_t(\mathbf{A}_2) &\leq \mathbb{P}_t(\mathcal{M}_\phi = \emptyset, \mathbf{C} \cap \mathbf{BN}^c) \mathbb{P}_t(\mathbf{NQG}^c | \mathcal{M}_\phi = \emptyset, \mathbf{C} \cap \mathbf{BN}^c) \\ &\quad \mathbb{P}_t(H_n^{\mathbf{B}} < \infty | 1 \leq \#S \leq r \log d, \mathbf{NQG}^c, \mathcal{M}_\phi = \emptyset, \mathbf{C} \cap \mathbf{BN}^c) \\ &= \mathbb{P}_t(\mathcal{M}_\phi = \emptyset, \mathbf{C} \cap \mathbf{BN}^c) \left( (1 + \frac{1}{5}) \tau e^{2\tau} d^{-1/2} \right) \left( (1 + \frac{1}{25}) e^\tau p_n r \log d \right) \\ &= \left( 1 + \frac{1}{4} \right) \tau e^{3\tau} p_n r d^{-1/2} \log(d) \mathbb{P}_t(\mathcal{M}_\phi = \emptyset, \mathbf{C} \cap \mathbf{BN}^c). \end{aligned}$$

4.7.3. *Bounding  $\mathbb{P}_t(\mathbf{A}_3)$ .*

**Lemma 4.20.** *If  $r > 1/2$ ,  $d \geq 24$  and  $t \leq d^{-1} + 2d^{-2}$ , then*

$$\mathbb{P}_t(\#S > r \log d | \mathcal{M}_\phi = \emptyset, \mathbf{C} \cap \mathbf{BN}^c) \leq \left( 1 + \frac{1}{5} \right) e^{2\tau} d^{-r} (\log \log d + \log r - \log 5).$$

*If  $d \geq 15\tau^2$ ,  $t \leq \frac{1}{7}d^{-1} \log d$  and  $r = \frac{e^2}{3(e-1)}$ , then the bound holds with right-hand side  $(1 + \frac{1}{5}) e^{2\tau} d^{-r \log 2}$ .*

**Proof.** Note that

$$\begin{aligned} &\mathbb{P}_t(\#S > r \log d | \mathcal{M}_\phi = \emptyset, \mathbf{C} \cap \mathbf{BN}^c) \\ &= \frac{\mathbb{P}_t(\#S > r \log d, \mathcal{M}_\phi = \emptyset | \mathbf{C} \cap \mathbf{BN}^c)}{\mathbb{P}_t(\mathcal{M}_\phi = \emptyset | \mathbf{C} \cap \mathbf{BN}^c)}. \end{aligned} \quad (4.29)$$

We use (4.25) and Lemmas 4.2, 4.5 and 4.21 to find that

$$\mathbb{P}_t(\#S > r \log d, \mathcal{M}_\phi = \emptyset | \mathbf{C} \cap \mathbf{BN}^c) \leq e^\tau \mathbb{P}_t(\#S > r \log d).$$

An upper bound on the numerator of (4.29) is now obtained from Lemma 4.21, while the denominator is bounded below by (4.24).  $\square$

**Lemma 4.21.** *If  $r > 1/2$ ,  $d \geq 24$  and  $t \leq d^{-1} + 2d^{-2}$ , then*

$$\mathbb{P}_t(\#S \geq r \log d) \leq d^{-r} (\log \log d + \log r - \log 5).$$

*If  $t \leq \frac{1}{7}d^{-1} \log d$  and  $r = \frac{e^2}{3(e-1)}$ , then  $\mathbb{P}_t(\#S \geq r \log d) \leq d^{-r \log 2}$ .*

**Proof.** Let  $\lambda = \lambda(t) > 0$  satisfy  $te^{-t} = 1 - e^{-\lambda}$ . Note that

$$te^{-t} \leq \lambda \leq \frac{e}{e-1} te^{-t}. \quad (4.30)$$

Let  $Z$  be a Poisson random variable of mean  $d\lambda$ . A Bernoulli( $te^{-t}$ ) random variable being stochastically dominated by the Poisson distribution of mean  $\lambda$ ,  $\#S$  under  $\mathbb{P}_t$  is stochastically dominated by  $Z$ . Chernoff's bound applied to the Poisson distribution [10] states that

$$\mathbb{P}(Z \geq x) \leq e^{-d\lambda} (d\lambda e x^{-1})^x,$$

provided that  $x > d\lambda$ . In regard to the lemma's first assertion, note that this lower bound on  $x$  is satisfied by the choice  $x = r \log d$ , due to (4.30) and  $d \geq e^{2e/(e-1)}$ . Thus,

$$\begin{aligned} \mathbb{P}(Z \geq r \log d) &\leq \exp \left\{ dte^{-t} \right\} \left( \frac{e^2 dte^{-t}}{(e-1)r \log d} \right)^{r \log d} \\ &\leq \left( \frac{e^{2\tau}}{(e-1)r \log d} \right)^{r \log d} \leq d^{-r(\log \log d + \log r - \log 5)}. \end{aligned} \quad (4.31)$$

since  $\tau \leq 1 + 2d^{-1} \leq 5(e-1)e^{-2}$  if  $d \geq 13$ . This proves the first assertion; the second follows similarly.  $\square$

**Lemma 4.22.**

$$\mathbb{P}_t \left( H_n^{\mathcal{B}} < \infty \mid \#S > r \log d, \mathcal{M}_\phi = \emptyset, \mathbf{C} \cap \mathbf{BN}^c \right) \leq \left( 1 + \frac{1}{25} \right) e^\tau p_n d.$$

**Proof.** The events in the conditioning are measurable with respect to  $\mathcal{B} \cap ((\mathcal{M}_\phi \cup \partial_{\text{ext}} \mathcal{M}_\phi) \times [0, 1))$ . Hence, Lemma 4.10 implies that the probability in question is at most  $(1 + \frac{1}{25})e^\tau p_n \ell'$ , where  $\ell'$  is an upper bound on the number of elements in  $\partial_{\text{ext}} \mathcal{M}_\phi$  that support a bar in  $\mathcal{B}$  over choices of  $\mathcal{B}$  compatible with the conditioning. We may take  $\ell' = d$ , because  $\mathcal{M}_\phi = \emptyset$  implies that  $\partial_{\text{ext}} \mathcal{M}_\phi = \mathcal{E}_\phi$ .  $\square$

By Lemmas 4.20, 4.22 and  $\mathbf{P}^- \subseteq \{H_n^{\mathcal{B}} < \infty\}$ , we find then that, if  $r > 1/2$ ,  $d \geq 24$  and  $t \leq d^{-1} + 2d^{-2}$ ,

$$\mathbb{P}_t(\mathbf{A}_3) \leq \left( 1 + \frac{1}{4} \right) e^{3\tau} p_n d^{-r(\log \log d + \log r - \log 5)} \mathbb{P}_t(\mathcal{M}_\phi = \emptyset, \mathbf{C} \cap \mathbf{BN}^c);$$

and that the similar result with obvious changes holds if  $t \leq \frac{1}{7}d^{-1} \log d$ .

4.7.4. *The resulting bound.* If  $\mathcal{M}_\phi = \emptyset$  then  $S \neq \emptyset$  is necessary for  $\mathbf{P}^-$ . Assume that

$$d \geq 12^2 \left( 1 + \frac{1}{5} \right)^2 \tau^2 e^{6\tau}. \quad (4.32)$$

Assume further either that (1):  $r > 1/2$ ,  $d \geq 24$  and  $t \leq d^{-1} + 2d^{-2}$ , or that (2):  $t \leq \frac{1}{7}d^{-1} \log d$  and  $r = \frac{e^2}{3(e-1)}$ . In case (1), we find that

$$\begin{aligned} \mathbb{P}_t \left( \mathbf{P}^-, \mathcal{M}_\phi = \emptyset, \mathbf{C} \cap \mathbf{BN}^c \right) &= \mathbb{P}_t \left( \mathbf{P}^-, \#S \geq 1, \mathcal{M}_\phi = \emptyset, \mathbf{C} \cap \mathbf{BN}^c \right) \\ &\leq \left( 1 + \frac{2}{5} \right) p_n e^{3\tau} \left( (\tau(1+\tau)d^{-1/2} + r\tau d^{-1/2} \log d + d^{-r(\log \log d + \log r - \log 5)}) \right) \\ &\quad \times \mathbb{P}_t(\mathcal{M}_\phi = \emptyset, \mathbf{C} \cap \mathbf{BN}^c), \end{aligned}$$

while in case (2) the second term in the large bracket on the right-hand side of the inequality is replaced by  $d^{-r \log 2}$ . In case (2), note that (4.32) is satisfied if  $d$  is high enough, so that we obtain (4.9) for this case. For case (1), note that with  $r = 3/4$  and for  $d \geq 40000$ ,  $d^{-r(\log \log d + \log r - \log 5)} \leq 6d^{-1/2}$ ; in this

case  $\tau \leq 1 + 2d^{-1}$ , so that (4.32) is satisfied. This yields (4.9) for the first set of parameter choices.

**4.8. The loss term with  $\#\mathcal{M}_\phi \geq n_1$ .** From Lemma 4.9, (4.8) and  $p_n(t) \geq \varepsilon$  follows (4.12).

### 5. THE BOTTLENECK FAR FROM THE BOUNDARY: DERIVING (3.7)

We will verify (3.7) by showing that, for each  $(e, h) \in \text{High} \times [0, 1)$ ,

$$\mathbb{P}_t(\mathbf{P}_n^+ | \mathbf{C} \cap \text{NoEsc} \cap \{b_{\text{BN}} = (e, h)\}) \geq \mathbb{P}_t(\mathbf{P}_n^- | \mathbf{C} \cap \text{NoEsc} \cap \{b_{\text{BN}} = (e, h)\}). \quad (5.1)$$

Note that we have displayed the  $n$ -dependence of  $\mathbf{P}^+$  and  $\mathbf{P}^-$ . We will reexpress the condition (5.1) by another condition involving a different value of  $n$ ; for this reason, we indicate the  $n$ -dependence of such quantities as  $\mathbf{C}$  and  $\mathcal{A}$  in this section.

**Definition 5.1.** Define the ordered addition operation on  $V(\mathcal{T})$  that corresponds to concatenation of labels for the natural labelling of  $V(\mathcal{T})$  by finite strings of symbols in  $\{0, \dots, d-1\}$ . Extend the operation by setting  $e + v = (e^+ + v, e^- + v) \in E(\mathcal{T})$  for  $e \in E(\mathcal{T})$  and  $v \in V(\mathcal{T})$ . For  $(v, h) \in V(\mathcal{T}) \times [0, 1)$  and for  $b = (e', s) \in E(\mathcal{T}_{[v]}) \times [0, 1)$ , define the  $(v, h)$ -shift  $b^{(v, h)}$  of  $b$  to be the bar  $(e' - v, (s - h) \bmod 1) \in E(\mathcal{T}) \times [0, 1)$ . For a given bar set  $\mathcal{B}_0 \subseteq E(\mathcal{T}_{[v]}) \times [0, 1)$ , define its  $(v, h)$ -shift  $\mathcal{B}_0^{(v, h)} \subseteq E(\mathcal{T}) \times [0, 1)$  to be  $\{b^{(v, h)} : b \in \mathcal{B}_0\}$ .

For  $m \in \mathbb{N}^+$ , recall from Lemma 4.2 that  $\mathbb{P}_{t, \mathcal{B}}^{\mathbf{C}_m \cap \text{BN}^c}$  denotes the marginal distribution of  $\mathcal{B}$  under  $\mathbb{P}_t(\cdot | \mathbf{C}_m \cap \text{BN}^c)$ , where the law of  $\mathcal{A}_m$  is normalized Lebesgue measure on  $E(\mathcal{T}_m) \times [0, 1)$  and  $\mathbf{C}_m$  is the associated crossing event.

**Lemma 5.2.** Let  $(e, h) \in \text{High} \times [0, 1)$ . For given  $\mathcal{B}' \subseteq E(\mathcal{T}) \times [0, 1)$ , write  $\mathcal{B}'_{e^-} = \mathcal{B}' \cap (E(\mathcal{T}_{[e^-]}) \times [0, 1))$ . Under  $\mathbb{P}_t$ , write  $\mathcal{B}'_{e^-}$  in place of  $(\mathcal{B}'_{e^-})^{(e^-, h)}$ . Then the conditional law of  $\mathcal{B}'_{e^-}$  under  $\mathbb{P}_t(\cdot | \mathbf{C}_n \cap \text{NoEsc} \cap \{b_{\text{BN}} = (e, h)\})$  equals  $\mathbb{P}_{t, \mathcal{B}}^{\mathbf{C}_{n-d(\phi, e^-)} \cap \text{BN}^c}$ .

The proof needs a definition.

**Definition 5.3.** Let  $(e, h) \in E(\mathcal{T}_n) \times [0, 1)$ . Let  $\text{ViLoc}_{(e, h)}$  denote the set of  $b \in (E(\mathcal{T}_n) \cap E(\mathcal{T}_{[e^-]}) \times [0, 1)$  such that both of the following conditions apply:

- $X_{(e^-, h)}^{\mathcal{B}'_{e^-}}$  visits at least one joint of  $b$  before visiting  $\mathcal{V}_n \times [0, 1)$ ;
- every edge in the path  $P_{e^-, E(b)^+}$  supports at least two bars in  $\mathcal{B}'_{e^-}$ .

**Proof of Lemma 5.2.** Note that the following conditions are each necessary for  $\mathbf{C}_n \cap \text{NoEsc} \cap \{b_{\text{BN}} = (e, h)\}$ :

- $e$  supports exactly one bar in  $\mathcal{B}$ , this being  $(e, h)$ ;
- $X_{(e^+, h)}^{\mathcal{B}}$  visits  $(\phi, 0)$  before  $\mathcal{V}_n \times [0, 1)$ ;
- $\mathcal{A}_n \in E(\mathcal{T}_{[e^-]}) \times [0, 1)$ .

Condition  $\mathbb{P}_t$  on the intersection of these events. Note that the event  $\mathbf{C}_n \cap \mathbf{NoEsc} \cap \{b_{\mathbf{BN}} = (e, h)\}$  is conditionally equal to  $\mathcal{A}_n \in \mathbf{ViLoc}_{(e^-, h)}$ . Thus, under  $\mathbb{P}_t$  given  $\mathbf{C}_n \cap \mathbf{NoEsc} \cap \{b_{\mathbf{BN}} = (e, h)\}$ , the conditional distribution of  $(\mathcal{B}_{e^-}, \mathcal{A}_n)$  is equal to an independent Poisson- $t$  law on  $E(\mathcal{T}_{[e^-]}) \times [0, 1)$  and a Lebesgue-distributed element of  $(E(\mathcal{T}_{[e^-]}) \cap E(\mathcal{T}_n)) \times [0, 1)$  conditioned on  $\mathcal{A}_n \in \mathbf{ViLoc}_{(e^-, h)}$ . Given the conditioning, the event  $\mathcal{A}_n \in \mathbf{ViLoc}_{(e^-, h)}$  coincides with  $\mathcal{A}_n^{(e^-, h)} \in \mathbf{ViLoc}(\mathcal{B}_{e^-}^{(e^-, h)})$ . That is, under  $\mathbb{P}_t(\cdot | \mathbf{C}_n \cap \mathbf{NoEsc} \cap \{b_{\mathbf{BN}} = (e, h)\})$ ,  $(\mathcal{B}_{e^-}^{(e^-, h)}, \mathcal{A}_n^{(e^-, h)})$  is distributed as  $(\mathcal{B}, \mathcal{A}_{n-d(\phi, e^-)})$  under  $\mathbb{P}_t$  given  $\mathcal{A}_{n-d(\phi, e^-)} \in \mathbf{ViLoc}(\mathcal{B})$ ; thus, the conditional law of  $\mathcal{B}_{e^-}^{(e^-, h)}$  is  $\mathbb{P}_{t, \mathcal{B}}^{\mathbf{C}_{n-d(\phi, e^-)} \cap \mathbf{BN}^c}$ .  $\square$

**Lemma 5.4.** *Let  $(e, h) \in \text{High} \times [0, 1)$ . Under  $\mathbb{P}_t(\cdot | \mathbf{C}_n \cap \mathbf{NoEsc} \cap \{b_{\mathbf{BN}} = (e, h)\})$ , let  $\eta^{\mathcal{B}} = \inf \{s > 0 : X_{(e^-, h)}^{\mathcal{B}} \in \{(e^-, h)\} \cup (\mathcal{V}_n \times [0, 1))\}$ ; let  $\eta^{\mathcal{B} \cup \mathcal{A}}$  denote the analogous stopping time for  $X_{(e^-, h)}^{\mathcal{B} \cup \mathcal{A}}$ . Then, given the conditioning,  $H_n^{\mathcal{B}} = \infty$  if and only if  $X_{(e^-, h)}^{\mathcal{B}}(\eta) = (e^-, h)$ , and  $H_n^{\mathcal{B} \cup \mathcal{A}} = \infty$  if and only if  $X_{(e^-, h)}^{\mathcal{B} \cup \mathcal{A}}(\eta) = (e^-, h)$ .*

**Proof.** Under the law in question,  $X^{\mathcal{B}}$  crosses  $b_{\mathbf{BN}}$  without having reached  $\mathcal{V}_n \times [0, 1)$  (because this crossing must happen before  $H_{\mathcal{A}}^{\mathcal{B}}$  which itself is before  $H_n^{\mathcal{B}}$ ). After the crossing,  $X^{\mathcal{B}}$  follows the trajectory of  $X_{(e^-, h)}^{\mathcal{B}}$  for the duration  $\eta^{\mathcal{B}}$ , either ending up in  $\mathcal{V}_n \times [0, 1)$  and thus realizing  $H_n^{\mathcal{B}} < \infty$ , or returning to  $(e^-, h)$  and then pursuing the trajectory of  $X_{b_{\mathbf{BN}}^+}^{\mathcal{B}}$ . In the latter case,  $\mathbf{NoEsc}$  ensures that  $X^{\mathcal{B}}$  returns to  $(\phi, 0)$  before time  $H_n^{\mathcal{B}}$ , forcing this meander into a periodic trajectory and ensuring that  $H_n^{\mathcal{B}} = \infty$ . Likewise for  $X^{\mathcal{B} \cup \mathcal{A}}$ .  $\square$

Lemmas 5.2 and 5.4 may be applied to reformulate (5.1) in the form

$$\mathbb{P}_t\left(\mathbf{P}_{n-d(\phi, e^-)}^+ \mid \mathbf{C}_{n-d(\phi, e^-)} \cap \mathbf{BN}^c\right) \geq \mathbb{P}_t\left(\mathbf{P}_{n-d(\phi, e^-)}^- \mid \mathbf{C}_{n-d(\phi, e^-)} \cap \mathbf{BN}^c\right).$$

Note that  $e \in \text{High}$  implies that  $n - d(\phi, e^-) \geq 2n_1$ . Hence, (3.6) implies (5.1) and thus (3.7).

## 6. THE BOTTLENECK CLOSE TO THE BOUNDARY: DERIVING (3.8)

Here we prove (3.8) and Lemma 3.1. We will show (3.8) by exploiting an effect which thus far we have neglected. We will argue that  $\mathbf{NoEsc}$ , the event that  $X_{b_{\mathbf{BN}}^+}^{\mathcal{B}}$  reaches  $(\phi, 0)$  before  $\mathcal{V}_n \times [0, 1)$ , has a probability which decays exponentially in the journey distance  $d(\phi, e_{\mathbf{BN}}^+)$ . We will do this by

identifying a positive proportion of edges in the path  $P_{\phi, e_{\text{BN}}^+}$  over which the passage of  $X_{b_{\text{BN}}^+}^{\mathcal{B}}$  incurs a positive chance that this meander is diverted from further progress towards  $(\phi, 0)$  by falling into a nearby and unvisited subtree.

Let  $\text{CB}'$  denote the event that  $d(\phi, E(\mathcal{A})^-) \geq n - 2n_1 + 1$ . By definition,  $d(\phi, e_{\text{BN}}^-) \leq d(\phi, E(\mathcal{A})^-)$ . Hence,  $\text{CB} \subseteq \text{CB}'$ . For this reason, the next proposition is sufficient to prove (3.8).

**Proposition 6.1.** *Let  $\varepsilon > 0$ . Suppose that  $d \in \mathbb{N}$  and  $t \in (0, \infty)$  satisfy  $p_n = p_n(t) \geq \varepsilon$  and  $d \geq 2^6 \tau^6 (6\tau + 1)$ . Then  $n \geq 4n_1$  implies that*

$$\begin{aligned} \mathbb{P}_t(\text{NoEsc} \cap \text{C} \cap \text{CB}') &\leq (d-1)^{-2} d^{2n_1+2} \varepsilon^{-2} e^{2t} (\#E(\mathcal{T}_n))^{-1} \\ &\quad \left( \left(1 - \varepsilon e^{-\tau d^{-1}} (d+1)^{-1} (1 - e^{-\tau d^{-2}})\right)^{n/2} \right. \\ &\quad \quad \left. + 3 \left(2^{1/2} \tau^{1/2} (6\tau + 1)^{1/12} d^{-1/12}\right)^n \right. \\ &\quad \quad \left. + 8\varepsilon^{-3} e^{(d+1)t} n e^{2t} (1 - \varepsilon e^{-\tau d^{-1}})^{n-1} \right). \end{aligned}$$

We write  $\text{CB}' = \cup_{i=0}^{2n_1-1} \text{CB}'_i$ , where  $\text{CB}'_i = \{d(\phi, E(\mathcal{A})^-) = n - i\}$ . Two steps will yield Proposition 6.1.

**Lemma 6.2.** *For  $0 \leq i \leq n - 1$ ,*

$$\mathbb{P}_t(\text{C} | \text{CB}'_i) \leq \frac{d^{i+2}}{d-1} p_n^{-2} e^{2t} (\#E(\mathcal{T}_n))^{-1}.$$

**Lemma 6.3.** *Assume that  $d \geq 2^6 \tau^6 (6\tau + 1)$ . Let  $0 \leq i \leq 2n_1 - 1$ . Then  $n \geq 4n_1$  implies that*

$$\begin{aligned} &\mathbb{P}_t(\text{NoEsc} | \text{C} \cap \text{CB}'_i \cap \{H_n^{\mathcal{B}} < \infty\}) \\ &\leq \left(1 - p_n e^{-t} (d+1)^{-1} (1 - e^{-td^{-1}})\right)^{n/2} + 3 \left(2^{1/2} \tau^{1/2} (6\tau + 1)^{1/12} d^{-1/12}\right)^n \\ &\quad + 8p_n^{-3} e^{(d+1)t} n e^{2t} (1 - p_n e^{-t})^{n-1}. \end{aligned} \tag{6.1}$$

**Proof of Proposition 6.1.** The proposition follows from Lemmas 6.2 and 6.3 applied for  $0 \leq i \leq 2n_1 - 1$ .  $\square$

The next two subsections provide the proofs of these two lemmas.

**6.1. Crossing probability: deriving Lemma 6.2.** Recall that  $Y^{\mathcal{B}} : [0, \infty) \rightarrow V(\mathcal{T})$  is the vertex component of  $X^{\mathcal{B}} : [0, \infty) \rightarrow V(\mathcal{T}) \times [0, 1)$ .

**Lemma 6.4.** *For  $i \in \mathbb{N}$  and  $k \in \mathbb{N}^+$ ,*

$$\mathbb{P}_t(|\mathcal{V}_i \cap Y^{\mathcal{B}}[0, H_n^{\mathcal{B}}]| \geq k) \leq (1 - p_{n-i} e^{-t})^{k-1}.$$

**Remark.** The quantity  $|\mathcal{V}_i \cap Y^{\mathcal{B}}[0, H_n^{\mathcal{B}}]|$  is the cardinality of a set of vertices; we mention that this use is slightly at odds with the overall convention from Definition 1.6.

**Proof of Lemma 6.4.** Let  $v \in \mathcal{V}_i$ , and suppose that at a certain time  $s \in (0, H_n^{\mathcal{B}})$ ,  $X^{\mathcal{B}}$  reaches the pole at  $v$  for the first time. At time  $s$ ,  $X^{\mathcal{B}}$  crosses a bar  $b$  in  $\mathcal{B}$  of the form  $b = ((v^+, v), r)$ ,  $r \in [0, 1)$ . By Lemma 1.13, given  $X^{\mathcal{B}}$  until time  $s$ , the conditional probability that  $b$  is the only bar in  $\mathcal{B}$  supported on  $(v^+, v)$  is  $e^{-c}$ ,  $c = |\text{UnTouch}_s \cap ((v^+, v) \times [0, 1))|$ , which is at least  $e^{-t}$ . By Lemma 1.13 and the strong Markov property, given that this  $b$  is indeed the only such bar, the subsequent trajectory of  $X^{\mathcal{B}}$  in  $E(\mathcal{T}_{[v]}) \times [0, 1)$  stopped on return to  $(v, r)$  has the law of  $X^{\hat{\mathcal{B}}}$  in  $\mathcal{T}$  begun at  $(v, r)$  until the same stopping time, where  $\hat{\mathcal{B}}$  denotes a Poisson- $t$  bar collection on  $E(\mathcal{T}_{[v]}) \times [0, 1)$ . Conditionally on  $b$  being the only bar supported on  $(v^+, v)$ , Lemma 1.12 implies that there is probability  $p_{n-i}$  that  $X^{\mathcal{B}}$  visits  $\mathcal{V}_n \times [0, 1)$  before it returns to  $(v, r)$ , an event which entails that the non-decreasing process  $[0, H_n^{\mathcal{B}}] \rightarrow \mathbb{N} : t \rightarrow |\mathcal{V}_i \cap Y^{\mathcal{B}}[0, t]|$  assumes its eventual value at time  $s$  when  $v$  joins this set. That is, with each singleton that joins this growing set, there is conditional probability at least  $p_{n-i}e^{-t}$  that the set grows no more.  $\square$

**Proof of Lemma 6.2.** For  $j \geq 1$ , set  $q_j = \#\mathcal{E}_j$ . For  $i \in \{0, \dots, n-1\}$  and  $k \geq 0$ , we note that

$$\mathbb{P}_t\left(\mathbf{C} \mid |\mathcal{V}_{n-1-i} \cap Y^{\mathcal{B}}[0, H_n^{\mathcal{B}}]| = k, \mathbf{CB}'_i\right) \leq dkq_{n-1-i}^{-1},$$

because, given  $\{|\mathcal{V}_{n-1-i} \cap Y^{\mathcal{B}}[0, H_n^{\mathcal{B}}]| = k\} \cap \mathbf{CB}'_i$ , the conditional law of  $\mathcal{A}$  is normalized Lebesgue measure on  $\mathcal{E}_{n-1-i}$ , and  $E(\mathcal{A})^+ \in Y^{\mathcal{B}}[0, H_n^{\mathcal{B}}]$  is necessary for  $\mathbf{C}$  to occur. It follows that

$$\begin{aligned} & \mathbb{P}_t(\mathbf{C} \mid \mathbf{CB}'_i) \\ &= \sum_{k \geq 0} \mathbb{P}_t\left(\mathbf{C} \mid |\mathcal{V}_{n-1-i} \cap Y^{\mathcal{B}}[0, H_n^{\mathcal{B}}]| = k, \mathbf{CB}'_i\right) \mathbb{P}_t\left(|\mathcal{V}_{n-1-i} \cap Y^{\mathcal{B}}[0, H_n^{\mathcal{B}}]| = k \mid \mathbf{CB}'_i\right) \\ &\leq \sum_{k \geq 1} dkq_{n-1-i}^{-1} (1 - p_n e^{-t})^{k-1} = dq_{n-1-i}^{-1} p_n^{-2} e^{2t}; \end{aligned} \tag{6.2}$$

we also used the independence of  $\mathcal{B}$  and  $\mathcal{A}$  in applying

$$\mathbb{P}_t\left(|\mathcal{V}_{n-1-i} \cap Y^{\mathcal{B}}[0, H_n^{\mathcal{B}}]| = k \mid \mathbf{CB}'_i\right) = \mathbb{P}_t\left(|\mathcal{V}_{n-1-i} \cap Y^{\mathcal{B}}[0, H_n^{\mathcal{B}}]| = k\right),$$

and made use of Lemma 6.4 and that  $\{p_m : m \in \mathbb{N}\}$  is non-increasing. Note that  $q_j = d^{j+1}$  while  $\#E(\mathcal{T}_n) = \frac{d}{d-1}(d^n - 1)$ , so that  $q_{n-1-i} \geq (1 - d^{-1})d^{-i}\#E(\mathcal{T}_n)$ . Thus, (6.2) implies Lemma 6.2.  $\square$

## 6.2. Probable escape: deriving Lemma 6.3.

**Definition 6.5.** Let  $t > 0$  be such that  $X^{\mathcal{B}}$  has not revisited  $(\phi, 0)$  during  $(0, t]$ . An edge  $e \in E(P_{\phi, Y^{\mathcal{B}}(t)})$  with  $e^+ \neq \phi$  is said to witness a potential escape at time  $t$  if the following conditions hold:

- the edge  $e$  supports exactly one bar in  $\mathcal{B}$ , which we denote by  $(e, s_e)$  for some  $s_e \in [0, 1)$  (note that  $X$  has necessarily crossed  $(e, s_e)$  exactly once before time  $t$ );
- $X^{\mathcal{B}}[0, t]$  has not visited the pole at  $e^+$  at any point of the form  $(e^+, r)$  for  $r \in (s_e, s_e + d^{-1})$ ;
- writing  $e^{++}$  for the parent of  $e^+$ , there is no bar in  $\mathcal{B}$  of the form  $(e^{++}, e^+) \times (s_e, s_e + d^{-1})$ ;
- there exists an offspring vertex  $v$  of  $e^+$  such that  $X^{\mathcal{B}}[0, t]$  has empty intersection with the pole at  $v$ .

Let  $\text{WPE}_t \subseteq E(P_{\phi, Y^{\mathcal{B}}(t)})$  denote the set of edges that witness a potential escape at time  $t$ . For  $e \in \text{WPE}_t$ , let the escape vertex  $v_{\text{esc}}(e)$  denote the lowest labelled vertex  $v$  satisfying the fourth of the above conditions; necessarily,  $v_{\text{esc}}(e) \neq e^-$ .

Recalling that  $H_{\mathcal{A}} = H_{\mathcal{A}}^{\mathcal{B}} \in [0, \infty]$  denotes the time at which  $X^{\mathcal{B}}$  first visits a joint of  $\mathcal{A}$ , suppose given  $\mathcal{C}$  and the trajectory  $X^{\mathcal{B}} : [0, H_{\mathcal{A}}] \rightarrow V(\mathcal{T}) \times [0, 1)$ . Then  $\text{WPE}_{H_{\mathcal{A}}}$  is a random set of edges in the path  $P_{\phi, Y_{H_{\mathcal{A}}}^{\mathcal{B}}}$  whose elements are totally ordered according to their location on this path; list them  $e_1, \dots, e_{\ell}$ ,  $\ell = \#\text{WPE}_{H_{\mathcal{A}}}$ , so that  $d(\phi, e_i^+)_{\{1 \leq i \leq \ell\}}$  is decreasing; also abbreviate  $v_{\text{esc}}^i = v_{\text{esc}}(e_i)$  and set  $h_i = s_{e_i}$ .

The next lemma shows that if  $\mathcal{A}$  is close to the boundary and  $\mathcal{C}$  occurs then, at the moment when  $X^{\mathcal{B}}$  meets  $\mathcal{A}$ , there are typically of the order of  $n$  edges on the path from  $\phi$  to  $Y^{\mathcal{B}}$  that witness a potential escape.

**Lemma 6.6.** Assume that  $d \geq 2^6 \tau^6 (6\tau + 1)$ . Let  $0 \leq i \leq 2n_1 - 1$ . Then  $n \geq 4n_1$  implies that

$$\begin{aligned} & \mathbb{P}_t \left( \#\text{WPE}_{H_{\mathcal{A}}} \leq n/2 \mid \mathcal{C} \cap \text{CB}'_i \right) \\ & \leq 3 \left( 2^{1/2} \tau^{1/2} (6\tau + 1)^{1/12} d^{-1/12} \right)^n + 8p_n^{-3} e^{(d+3)t} n (1 - p_n e^{-t})^{n-1}. \end{aligned}$$

We now prove Lemma 6.3 using Lemma 6.6, and then prove Lemma 6.6.

**Proof of Lemma 6.3.** Let  $\text{Grand}$  denote the event that

- $\mathcal{C} \cap \text{CB}'_i \cap \{H_n^{\mathcal{B}} < \infty\}$  occurs, and
- after crossing the bottleneck bar  $b_{\text{BN}}$  at some moment before  $H_n^{\mathcal{B}}$ ,  $X^{\mathcal{B}}$  does not recross  $b_{\text{BN}}$  before this time.

Note that

$$\mathcal{C} \cap \text{CB}' \cap \{H_n^{\mathcal{B}} < \infty\} \cap \text{NoEsc} \subseteq \text{Grand} \subseteq \mathcal{C} \cap \text{CB}' \cap \{H_n^{\mathcal{B}} < \infty\}. \quad (6.3)$$

The latter inclusion is by definition. In regard to the former, were the left-hand event to occur without **Grand** also happening,  $X^{\mathcal{B}}$  would recross  $b_{\text{BN}}$  before time  $H_n^{\mathcal{B}}$ ; by the definition of **NoEsc**,  $X^{\mathcal{B}}$  would then return to  $(\phi, 0)$  before time  $H_n^{\mathcal{B}}$ , thereby forming a periodic orbit and forcing the contradictory  $H_n^{\mathcal{B}} = \infty$ .

Note then that, under the assumptions of Lemma 6.6,

$$\begin{aligned} & \mathbb{P}_t \left( \text{NoEsc}, H_n^{\mathcal{B}} < \infty \mid \mathcal{C} \cap \text{CB}'_i \right) \\ & \leq \mathbb{P}_t \left( \text{NoEsc} \mid \mathcal{C} \cap \text{CB}'_i, \#\text{WPE}_{H_{\mathcal{A}}} > n/2, H_n^{\mathcal{B}} < \infty \right) + \hat{c}_n \\ & \leq \mathbb{P}_t \left( \text{NoEsc} \mid \text{Grand}, \#\text{WPE}_{H_{\mathcal{A}}} > n/2 \right) + \hat{c}_n; \end{aligned} \quad (6.4)$$

here,  $\hat{c}_n$  is the right-hand side in Lemma 6.6 and the latter inequality is due to (6.3).

The role of the next lemma is to take account of the weighting effect caused by conditioning involving  $\mathcal{A}$ . The conditioning in the lemma includes  $b_{\text{BN}} \in E(\mathcal{T}_n) \times [0, 1)$  and  $\mathcal{B} \cap (E(P_{\phi, e_{\text{BN}}^-}) \times [0, 1))$  because doing so renders  $\text{WPE}_{H_{\mathcal{A}}}$  a deterministic object – a known system of potential escape locations for the meander  $X_{b_{\text{BN}}}^{\mathcal{B}}$  journeying towards  $(\phi, 0)$ .

**Lemma 6.7.** *Consider the law  $\mathbb{P}_t$  conditional on **Grand** and on the data  $X^{\mathcal{B}} : [0, H_n^{\mathcal{B}}] \rightarrow V(\mathcal{T}_n) \times [0, 1)$ ,  $b_{\text{BN}} \in E(\mathcal{T}_n) \times [0, 1)$  and  $\mathcal{B} \cap (E(P_{\phi, e_{\text{BN}}^-}) \times [0, 1))$ . Set  $\mathcal{B}_{[\phi, e_{\text{BN}}^-]} = \mathcal{B} \cap (E(P_{\phi, e_{\text{BN}}^-}) \times [0, 1))$ . Then the conditional distribution of  $\mathcal{B}$  is given by  $\text{Found}_{H_n^{\mathcal{B}}} \cup \mathcal{B}_{[\phi, e_{\text{BN}}^-]} \cup \mathcal{B}'$ , where  $\mathcal{B}'$  is a random bar collection with Poisson law of intensity  $t$  with respect to Lebesgue measure on  $\text{UnTouch}_{H_n^{\mathcal{B}}} \cap (E(P_{\phi, e_{\text{BN}}^-}) \times [0, 1))^c$ .*

In brief, the reason that Lemma 6.7 is true is as follows:

- certain bars in  $\mathcal{B}$  must be present due to the conditioning. These are the elements of  $\text{Found}_{H_n^{\mathcal{B}}} \cup \mathcal{B}_{[\phi, e_{\text{BN}}^-]}$ . No further bars in  $\mathcal{B}$  lie in  $E(P_{\phi, e_{\text{BN}}^-}) \times [0, 1)$ . Were the conditioning not to involve the location of  $\mathcal{A}$ , we would be done by arguing in the style of Lemma 1.13, since the set of conditionally unexplored bar locations is  $\text{UnTouch}_{H_n^{\mathcal{B}}} \cap (E(P_{\phi, e_{\text{BN}}^-}) \times [0, 1))^c$ ;
- by conditioning on **Grand** and on  $b_{\text{BN}}$ , there is a reweighting given by the Lebesgue measure of the set of bar locations at which the placement of  $\mathcal{A}$  would realize the conditioning; however, this set of admissible locations is determined by  $X_{[0, H_n^{\mathcal{B}}]}^{\mathcal{B}}$ , and thus the law of  $\mathcal{B}$ 's intersection with  $\text{UnTouch}_{H_n^{\mathcal{B}}} \cap (E(P_{\phi, e_{\text{BN}}^-}) \times [0, 1))^c$  is in fact not perturbed from the Poisson- $t$  law suggested in the first point.

We later present a formal proof of Lemma 6.7, which is in essence a symbolic rendering of the above, in Subsection 6.4.

**Lemma 6.8.** *Consider the law  $\mathbb{P}_t$  conditional on **Grand** and on the data  $X^{\mathcal{B}} : [0, H_n^{\mathcal{B}}] \rightarrow V(\mathcal{T}_n) \times [0, 1)$ ,  $b_{\text{BN}} \in E(\mathcal{T}_n) \times [0, 1)$  and  $\mathcal{B} \cap (E(P_{\phi, e_{\text{BN}}^-}) \times [0, 1))$ . Then the conditional probability of **NoEsc** is almost surely at most*

$$\left(1 - p_n e^{-t} (d+1)^{-1} (1 - e^{-td^{-1}})\right)^{\#\text{WPE}_{H_A}^{-1}}.$$

**Proof.** Note that  $e_1$  is an edge in the path  $P_{\phi, Y_{H_A}^{\mathcal{B}}}$  supporting exactly one bar in  $\mathcal{B}$ ; by the definition of  $e_{\text{BN}}$ ,  $e_{\text{BN}}$  lies on this same path and is either equal to  $e_1$  or is further from  $\phi$  than  $e_1$ . For each  $1 \leq i \leq \#\text{WPE}_{H_A}$ , let  $\sigma_i \in [0, \infty]$  denote the first time at which  $X_{b_{\text{BN}}}^{\mathcal{B}}$  visits the parent joint of the unique bar on the edge  $e_i$ . It may be that  $e_1 = e_{\text{BN}}$ , in which case, this meander is already at the parent joint of the bar on  $e_1$  at time zero; in this event, then,  $\sigma_1 = 0$ .

Each of the edges in  $\text{WPE}_{H_A}$  represents a distinct obstacle to the arrival of  $X_{b_{\text{BN}}}^{\mathcal{B}}$  at  $(\phi, 0)$ . That is, the stopping times  $\sigma_i$  increase strictly (until the absorbing state  $\infty$ ), and all must be finite if this meander is to reach  $(\phi, 0)$ . We now provide a bound on the conditional probability that the crossing of the bar on  $e_i$  by  $X_{b_{\text{BN}}}^{\mathcal{B}}$  leads this process to escape from its journey towards  $(\phi, 0)$ .

**Lemma 6.9.** *Consider the law  $\mathbb{P}_t$  conditional on **Grand**, on the data  $X^{\mathcal{B}} : [0, H_n^{\mathcal{B}}] \rightarrow V(\mathcal{T}_n) \times [0, 1)$ ,  $b_{\text{BN}} \in E(\mathcal{T}_n) \times [0, 1)$ ,  $\mathcal{B} \cap (E(P_{\phi, e_{\text{BN}}^-}) \times [0, 1))$ , and, for some  $1 \leq i \leq \#\text{WPE}_{H_A} - 1$ , on  $\sigma_i < \infty$  and the trajectory  $X_{b_{\text{BN}}}^{\mathcal{B}} : [0, \sigma_i] \rightarrow V(\mathcal{T}_n) \times [0, 1)$ . Then the conditional probability that  $\sigma_{i+1} < \infty$  is at most*

$$1 - p_n e^{-t} (d+1)^{-1} (1 - e^{-td^{-1}}). \quad (6.5)$$

An analogue of Lemma 1.13 is needed for the proof.

**Lemma 6.10.** *Consider the law  $\mathbb{P}_t$  conditional on **Grand** and on the data  $X^{\mathcal{B}} : [0, H_n^{\mathcal{B}}] \rightarrow V(\mathcal{T}_n) \times [0, 1)$ ,  $b_{\text{BN}} \in E(\mathcal{T}_n) \times [0, 1)$  and  $\mathcal{B} \cap (E(P_{\phi, e_{\text{BN}}^-}) \times [0, 1))$ .*

*Let  $\sigma \in [0, \infty]$  be a random variable, which, given the above data, is a stopping time for the process  $X_{b_{\text{BN}}}^{\mathcal{B}}$ . Let  $\text{Found}_{[\sigma]} \subseteq E(\mathcal{T}_n) \times [0, 1)$  denote the set of bars in  $\mathcal{B}$  that have been crossed either by  $X^{\mathcal{B}}$  during  $[0, H_n^{\mathcal{B}}]$  or by  $X_{b_{\text{BN}}}^{\mathcal{B}}$  during  $[0, \sigma]$ . Let  $\text{UnTouch}_{[\sigma]} \subseteq E(\mathcal{T}_n) \times [0, 1)$  denote the set of bar locations neither of whose joints lies in  $X^{\mathcal{B}}[0, H_n^{\mathcal{B}}] \cup X_{b_{\text{BN}}}^{\mathcal{B}}[0, \sigma]$ .*

*Consider the conditioning in Lemma 6.9 with  $\sigma_i$  replaced by  $\sigma$ . Then the conditional distribution of  $\mathcal{B}$  is given by  $\text{Found}_{[\sigma]} \cup \mathcal{B}_{[\phi, e_{\text{BN}}^-]} \cup \mathcal{B}_{(\sigma, \infty)}$ , where again we abbreviate  $\mathcal{B}_{[\phi, e_{\text{BN}}^-]} = \mathcal{B} \cap (E(P_{\phi, e_{\text{BN}}^-}) \times [0, 1))$ . Here,  $\mathcal{B}_{(\sigma, \infty)}$  is a random bar*



**Proof of Lemma 6.9.** Employing the notation in the paragraph after Definition 6.5, note that  $e_i \in \text{WPE}_{H_A}$  implies that

$$\text{no bar in Found}_{[\sigma_i]} \text{ has a joint in } e_i^+ \times (h_i, h_i + d^{-1}); \quad (6.6)$$

writing  $e_i^{++}$  for the parent of  $e_i^+$ ,

$$\mathcal{B} \text{ is disjoint from } (e_i^{++}, e_i^+) \times (h_i, h_i + d^{-1}); \quad (6.7)$$

and that

$$(e_i^+, v_{\text{esc}}^i) \times (h_i, h_i + d^{-1}) \subseteq \text{UnTouch}_{[\sigma_i]} \cap (E(P_{\phi, e_{\text{BN}}^-}) \times [0, 1))^c. \quad (6.8)$$

Consider the *escape* event that, subsequently to  $\sigma_i$ ,

- $X_{b_{\text{BN}}^+}^{\mathcal{B}}$  exits the pole at  $e_i^+$  by crossing a bar supported on  $(e_i^+, v_{\text{esc}}^i)$  at some time  $\chi \leq \sigma_i + d^{-1}$ ;
- after time  $\chi$ ,  $X_{b_{\text{BN}}^+}^{\mathcal{B}}$  visits  $\mathcal{V}_n \times [0, 1)$  before returning to the pole at  $e_i^+$ .

The escape event is sufficient to ensure that  $\sigma_{i+1} = \infty$ , so that it is enough to argue that its conditional probability is at least the four-term product appearing in (6.5). To verify this, note that the following conditions ensure that the event takes place:

- there is a bar in  $\mathcal{B}$  with a joint in  $\{e_i^+\} \times (h_i, h_i + d^{-1})$ ;
- among such bars, the one with the lowest height is supported on the edge  $(e_i^+, v_{\text{esc}}^i)$  (call this bar  $b$ );
- the edge  $(e_i^+, v_{\text{esc}}^i)$  supports no bar in  $\mathcal{B}$  except for  $b$ ;
- the meander  $X_b^{\mathcal{B}}$  visits  $\mathcal{V}_n \times [0, 1)$  before recrossing  $b$ .

The conditional probability of each of these requirements (given the existing conditions and the earlier requirements) is respectively:

- at least  $1 - e^{-td^{-1}}$ ;
- at least  $(d + 1)^{-1}$ ;
- at least  $e^{-t}$ ;
- equal to  $p_{n-d(\phi, v_{\text{esc}}^i)}$  and thus at least  $p_n$ .

These bounds are derived by applying Lemma 6.10 for  $\sigma = \sigma_i$  with (6.6), (6.7) and (6.8) being used; note that  $\sigma_i$  is indeed a conditional stopping time in the sense of this lemma.

This completes the proof of Lemma 6.9. □

The proof of Lemma 6.8 is concluded by noting that the event that  $X_{b_{\text{BN}}^+}^{\mathcal{B}}$  returns to  $(\phi, 0)$  before visiting  $\mathcal{V}_n \times [0, 1)$  is contained in the intersection of the events that  $\sigma_{i+1} < \infty$  for each  $1 \leq i \leq \#\text{WPE}_{H_A} - 1$ , and by multiplying the conclusion of Lemma 6.9 as  $i$  ranges over these values. □

Lemma 6.3 follows from (6.4) and Lemma 6.8. □

### 6.3. Deriving Lemma 6.6.

**Definition 6.11.** Let  $e \in E(\mathcal{T})$ . We say that the edge  $e$  is good if

- $e$  supports exactly one bar in  $\mathcal{B}$ , (which we denote by  $(e, s)$  for some  $s \in [0, 1)$ );
- there exists an offspring  $v$  of  $e^+$  (with  $v \neq e^-$ ) such that the edge  $(e^+, v)$  supports no bar in  $\mathcal{B}$ ;
- there is no bar in  $\mathcal{B}$  with a joint in  $\{e^+\} \times (s, s + d^{-1})$ .

Let  $\text{Good} \subseteq E(\mathcal{T})$  denote the set of good edges.

Let  $m \in \mathbb{N}$ . Let  $T_m$  denote the set of times before  $X^{\mathcal{B}} : [0, \infty) \rightarrow V(\mathcal{T}) \times [0, 1)$  returns to  $(\phi, 0)$  at which  $X^{\mathcal{B}}$  visits  $\mathcal{V}_m \times [0, 1)$ .

**Lemma 6.12.** Let  $m \in \mathbb{N}$ . For all  $t \in T_m$ ,  $E(P_{\phi, Y^{\mathcal{B}}(t)}) \cap \text{Good} \subseteq \text{WPE}_t$ .

**Proof.** Let  $e \in E(P_{\phi, Y^{\mathcal{B}}(t)}) \cap \text{Good}$ . Note that  $Y^{\mathcal{B}}(t)$  is a descendent of  $e^-$ , so that, at some time before  $t$ ,  $X^{\mathcal{B}}$  has crossed the only bar  $(e, s)$  in  $\mathcal{B}$  supported on  $e$ . Note that, once  $X^{\mathcal{B}}$  recrosses this bar in the opposite direction, it may only cross back again into  $V(\mathcal{T}_{[e^-]}) \times [0, 1)$  after it returns to  $(\phi, 0)$ . From  $t \in T_m$ , we find that  $X^{\mathcal{B}}$  has during  $[0, t]$  crossed the bar  $(e, s)$  and has not recrossed it. The third condition satisfied by  $e$  being good implies that the only way that  $X^{\mathcal{B}}$  may reach  $\{e^+\} \times (s, s + d^{-1})$  is by recrossing the bar  $(e, s)$  to arrive at  $(e^+, s)$ . This recrossing not having happened by time  $t$ , we see that  $X^{\mathcal{B}}[0, t] \cap (\{e^+\} \times (s, s + d^{-1})) = \emptyset$ . If  $X^{\mathcal{B}}$  is ever to reach the pole of a given offspring  $w$  of  $e^+$ , then clearly it must do so by crossing a bar in  $\mathcal{B}$  supported on  $(e^+, w)$ ; we see then that the vertex  $v$  specified by  $e$  being good is such that  $X^{\mathcal{B}}[0, t] \cap (\{v\} \times [0, 1)) = \emptyset$ . Finally, the third condition required for  $e \in \text{WPE}_t$  is a consequence of the third condition implied by  $e \in \text{Good}$ .  $\square$

**Lemma 6.13.** Let  $v \in V(\mathcal{T})$ . The path  $P_{\phi, v}$  is called occupied if each of its edges supports a bar in  $\mathcal{B}$ . Let  $m \in \mathbb{N}$  and suppose now that  $d(\phi, v) = m$ . The path  $P_{\phi, v}$  is said to offer little prospect of escape if  $\#(E(P_{\phi, v}) \cap \text{Good}) \leq m/2$ .

Suppose that  $t \leq 1$  and  $d \geq 2$ . Then, for each  $m \in \mathbb{N}$ ,

$$\begin{aligned} & \mathbb{P}_t \left( \exists v \in \mathcal{V}_m : P_{\phi, v} \text{ is occupied and offers little prospect of escape} \right) \\ & \leq 3 \cdot 2^m \tau^m (6\tau + 1)^{m/6} d^{-m/6}. \end{aligned}$$

**Proof.** Let  $v \in \mathcal{V}(\mathcal{T})$  and  $e \in E(P_{\phi, v})$ . We begin by claiming that if  $t \leq 1$  then

$$\mathbb{P}_t \left( e \text{ is not good} \mid P_{\phi, v} \text{ is occupied} \right) \leq 6t + d^{-1}. \quad (6.9)$$

Indeed, note that  $e$  supports a bar in  $\mathcal{B}$  and is not good if and only if at least one of the following alternatives applies:

- $e$  supports at least two bars in  $\mathcal{B}$ ;
- $e$  supports exactly one bar  $(e, s)$  in  $\mathcal{B}$ , and, for every offspring  $v$  of  $e^+$ , the edge  $(e^+, v)$  supports a bar in  $\mathcal{B}$ ;
- $e$  supports exactly one bar  $(e, s)$  in  $\mathcal{B}$ , and  $\{e^+\} \times (s, s+d^{-1})$  contains the joint of some bar in  $\mathcal{B}$ .

Assume that  $e^+ \neq \phi$ ; the other case follows with trivial changes. Let  $\bar{e}$  denote the edge that connects  $e^+$  to its parent. Under the conditional law in (6.9),  $\mathcal{B}$  on  $e \times [0, 1)$  has the Poisson- $t$  law given at least one bar; likewise on  $\bar{e} \times [0, 1)$ ; while on edges  $(e^+, v)$ , where  $v$  is an offspring of  $e^+$  other than  $e^-$ , it has the Poisson- $t$  law. These imply that the probabilities of the first two listed alternatives under the conditional law are  $\frac{1-(1+t)e^{-t}}{1-e^{-t}} \leq \frac{t^2}{\min\{t/2, 1/2\}} \leq 2t+2t^2$  and  $\frac{te^{-t}}{1-e^{-t}} \cdot (1-e^{-t})^{d-1} \leq t^{d-1}$ . As for the third, note that, under the conditioning, the distribution of  $\mathcal{B}$  on edges other than  $e$  incident to  $e^+$  is stochastically dominated by the union of a bar with  $U[0, 1]$  height supported on  $\bar{e}$  and an independent Poisson- $t$  random variable; thus, the third eventuality has conditional probability at most  $\frac{te^{-t}}{1-e^{-t}} \cdot (d^{-1} + (1 - e^{-t})) \leq d^{-1} + t$ . From  $t \leq 1$  and  $d \geq 2$  follows (6.9).

Let  $\text{Atyp}$  denote the set of edges  $e$  that support a bar in  $\mathcal{B}$  and for which  $e$  is not good. For  $1 \leq i \leq 3$ , say that an edge  $e \in E(\mathcal{T})$  is of type- $i$  if the modulo-3 remainder of  $d(\phi, e^+)$  is  $i - 1$ . Note that, for each  $v \in \mathcal{V}(\mathcal{T})$ , and for any subset  $E \subseteq E(P_{\phi, v})$  all of whose elements have given type, the collection of events  $\{e \in \text{Atyp} : e \in E\}$  is independent under  $\mathbb{P}_t$  given that  $P_{\phi, v}$  is occupied: indeed, for each  $e \in E(\mathcal{T})$ ,  $e \in \text{Atyp}$  is measurable with respect to  $\mathcal{B}$  over the edges incident to  $e^+$ , and any two edges of given type are at distance at least two.

Let  $v \in \mathcal{V}_m$ . If  $P_{\phi, v}$  is occupied and offers little prospect of escape, then, for some  $i \in \{1, 2, 3\}$ , there are at least  $m/6$  elements in  $E(P_{\phi, v}) \cap \text{Atyp}$  of type  $i$ ; the remaining elements of  $E(P_{\phi, v})$  each support a bar in  $\mathcal{B}$ . For given such  $i$ , this event has probability is

$$\mathbb{P}_t(P_{\phi, v} \text{ is occupied}) \sum_{E \subseteq E(P_{\phi, v}) : |E| \geq m/6} q_E,$$

where  $q_E$  is the conditional probability that  $E \subseteq \text{Atyp}$  given that  $P_{\phi, v}$  is occupied. By (6.9) and the claimed conditional independence,  $q_E \leq (6t + d^{-1})^{|E|}$ . Hence,

$$\begin{aligned} & \mathbb{P}_t\left(P_{\phi, v} \text{ is occupied and offers little prospect of escape}\right) \\ & \leq 3 \cdot (1 - e^{-t})^m \cdot 2^m \cdot (6t + d^{-1})^{m/6}. \end{aligned}$$

Note that  $t = \tau d^{-1}$  and  $|\mathcal{V}_m| = d^m$  to complete the proof of Lemma 6.13.  $\square$

**Lemma 6.14.** *Suppose that  $t \leq 1$ . We have that*

$$\mathbb{P}_t\left(\#(\text{Good} \cap E(P_{\phi, Y_s^{\mathcal{B}}})) \leq m/2 \exists s \in \mathbb{T}_m\right) \leq 3 \cdot 2^m \tau^m (6\tau + 1)^{m/6} d^{-m/6}.$$

**Proof.** The statement is trivial in the case that  $H_m^{\mathcal{B}} = \infty$  because then  $\mathbb{T}_m = \emptyset$ . Note that, for all  $s \geq 0$ , every element in  $E(P_{\phi, Y_s^{\mathcal{B}}})$  supports at least one bar in  $\mathcal{B}$ . Therefore, if for some  $s \in \mathbb{T}_m$ ,  $\#(\text{Good} \cap E(P_{\phi, Y_s^{\mathcal{B}}})) \leq m/2$ , the path  $P_{\phi, Y_s^{\mathcal{B}}}$  is occupied and offers little prospect of escape. For such  $s$ ,  $Y_s^{\mathcal{B}} \in \mathcal{V}_m$ , so that Lemma 6.13 provides the required inequality.  $\square$

We need a variant of Lemma 4.2.

**Lemma 6.15.** *For  $0 \leq i \leq n-1$ , let  $\mathbb{P}_t^{(i)} := \mathbb{P}_t(\cdot | \mathbb{C} \cap \mathbb{C}\mathcal{B}'_i)$ . Let  $\mathbb{P}_{t, \mathcal{B}}$  and  $\mathbb{P}_{t, \mathcal{B}}^{(i)}$  denote the marginal distribution of  $\mathcal{B} \cap (E(\mathcal{T}_n) \times [0, 1))$  under  $\mathbb{P}_t$  and  $\mathbb{P}_t^{(i)}$ . Then, for any given  $\mathcal{B}' \subseteq E(\mathcal{T}_n) \times [0, 1)$ ,*

$$\frac{d\mathbb{P}_{t, \mathcal{B}}^{(i)}}{d\mathbb{P}_{t, \mathcal{B}}}(\mathcal{B}') = Z_i^{-1} \left| \text{Crossed}(\mathcal{B}') \cap (\mathcal{E}_{n-1-i} \times [0, 1)) \right|,$$

where  $\text{Crossed}(\mathcal{B}') = \{b \in E(\mathcal{T}_n) \times [0, 1) : X^{B'}[0, H_n^{\mathcal{B}}] \cap \{b^+, b^-\} \neq \emptyset\}$ . We have that  $Z_i = \int |\text{Crossed}(\mathcal{B}') \cap (\mathcal{E}_{n-1-i} \times [0, 1))| d\mathbb{P}_t(\mathcal{B}')$ .

**Proof.** Given  $\mathcal{B} = \mathcal{B}'$ , the events  $\mathbb{C} \cap \mathbb{C}\mathcal{B}'_i$  and  $\mathcal{A} \in \text{Crossed}(\mathcal{B}') \cap (\mathcal{E}_{n-1-i} \times [0, 1))$  are equal. The proof follows that of Lemma 4.2.  $\square$

**Lemma 6.16.** *Set  $\psi = \frac{d\mathbb{P}_{t, \mathcal{B}}^{(i)}}{d\mathbb{P}_{t, \mathcal{B}}}$ , and note that  $\psi = \psi_{\mathcal{B}}$  is well defined under  $\mathbb{P}_t$ . For each  $n \in \mathbb{N}$ ,  $i \in \{0, \dots, n-1\}$  and  $k \in \mathbb{N}$ ,*

$$\mathbb{P}_t\left(\psi \geq (d+1)k(p_{n-1-i}d)^{-1}e^{(d+1)t}\right) \leq 2(1 - p_n e^{-t})^{k-1}.$$

**Proof.** Write  $\Theta_i = \text{Crossed}(\mathcal{B}) \cap (\mathcal{E}_{n-1-i} \times [0, 1))$ . By Lemma 6.15, it suffices to argue that

$$Z_i \geq p_{n-1-i} d e^{-(d+1)t}. \quad (6.10)$$

and that

$$\mathbb{P}_t\left(|\Theta_i| \geq (d+1)k\right) \leq 2(1 - p_n e^{-t})^{k-1}. \quad (6.11)$$

To derive (6.10), if  $H_{n-1-i}^{\mathcal{B}} < \infty$  occurs and  $X^{\mathcal{B}}$ , after crossing a bar at time  $H_{n-1-i}^{\mathcal{B}}$  to arrive at the pole of some vertex  $v \in \mathcal{V}_{n-1-i}$ , remains at that pole for an entire unit of time, then  $\{e \in E(\mathcal{T}) : e^+ = v\} \times [0, 1) \subseteq \Theta_i$ , so that  $|\Theta_i| \geq d$ . Note that  $\mathbb{P}_t(H_{n-1-i}^{\mathcal{B}} < \infty) = p_{n-1-i}$ . By Lemma 1.13, note that, given  $X^{\mathcal{B}} : [0, H_{n-1-i}^{\mathcal{B}}] \rightarrow V(\mathcal{T}) \times [0, 1)$ , the conditional probability of no bar crossing during  $[H_{n-1-i}^{\mathcal{B}}, H_{n-1-i}^{\mathcal{B}} + 1)$  is  $\exp\{-|\mathcal{B}_v|t\}$  where  $\mathcal{B}_v$  is the set of bars in  $\text{UnTouch}_{H_{n-1-i}^{\mathcal{B}}}$  one of whose joints lies in the pole at  $v$ ;  $|\mathcal{B}_v| \leq d+1$ , whence (6.10).

We now derive (6.11). Let  $\Theta'_i \subseteq [0, H_n^{\mathcal{B}}]$  denote the set of times  $t \in [0, H_n^{\mathcal{B}}]$  such that there exists  $b \in \mathcal{E}_{n-1-i} \times [0, 1)$  for which  $X^{\mathcal{B}}$  first visits  $\{b^+, b^-\}$  at time  $t$ . The set  $\Theta'_i$  is naturally partitioned as  $\Theta'_i = \Theta_i^+ \cup \Theta_i^-$  with  $\Theta_i^+$  being the subset of  $t \in \Theta'_i$  such that  $Y^{\mathcal{B}}(t) \in \mathcal{V}_{n-1-i}$  while  $\Theta_i^-$  is the subset of  $t \in \Theta'_i$  such that  $Y^{\mathcal{B}}(t) \in \mathcal{V}_{n-i}$ . Note that

$$|\Theta_i| \leq d|\Theta_i^+| + |\Theta_i^-|,$$

because, during times  $s \in \Theta_i^+$ ,  $X^{\mathcal{B}}(s)$  is the joint of at most  $d$  elements of  $\Theta_i$ , while, for  $s \in \Theta_i^-$ ,  $X^{\mathcal{B}}(s)$  is the joint of at most one such element.

Note then that  $|\Theta_i| \geq (d+1)k$  forces either  $|\Theta_i^+| \geq k$  or  $|\Theta_i^-| \geq k$ . The first alternative implies that  $|\mathcal{V}_{n-1-i} \cap Y^{\mathcal{B}}[0, H_n^{\mathcal{B}}]| \geq k$  and the second that  $|\mathcal{V}_{n-i} \cap Y^{\mathcal{B}}[0, H_n^{\mathcal{B}}]| \geq k$ . Hence, (6.11) follows from Lemma 6.4.  $\square$

**Lemma 6.17.** *Set  $\hat{C}_{d,t} = (d+1)(p_n d)^{-1} e^{(d+1)t}$ . Assume that  $d \geq 2^6 \tau^6 (6\tau+1)$ . Let  $i \in \{1, \dots, 2n_1 - 1\}$ . Then  $n \geq 4n_1$  implies that*

$$\begin{aligned} & \mathbb{P}_t \left( \#(\text{Good} \cap E(P_{\phi, Y_{H_A}})) \leq n/2 \mid \mathbf{C} \cap \mathbf{CB}'_i \right) \\ & \leq 3 \left( 2^{1/2} \tau^{1/2} (6\tau+1)^{1/12} d^{-1/12} \right)^n \\ & \quad + 4 \hat{C}_{d,t} n p_n^{-2} e^{2t} (1 - p_n e^{-t})^{n-1}. \end{aligned}$$

**Proof.** Recall that  $\psi = \frac{d\mathbb{P}_{t,\mathcal{B}}^{(i)}}{d\mathbb{P}_{t,\mathcal{B}}}$ . Note that

$$\begin{aligned} & \mathbb{P}_t \left( \#(\text{Good} \cap E(P_{\phi, Y_{H_A}})) \leq \frac{n-i}{2} \mid \mathbf{C} \cap \mathbf{CB}'_i \right) \\ & \leq \hat{C}_{d,t} n \mathbb{P}_t \left( \#(\text{Good} \cap E(P_{\phi, Y_{H_A}})) \leq \frac{n-i}{2} \text{ occurs for some } s \in T_{n-i} \right) \\ & \quad + \mathbb{P}_t(\psi \geq \hat{C}_{d,t} n \mid \mathbf{C} \cap \mathbf{CB}'_i). \end{aligned} \tag{6.12}$$

Let  $\nu$  denote the law of  $\psi$  under  $\mathbb{P}_t$ . Note that  $\mathbb{P}_t(\psi \geq \hat{C}_{d,t} n \mid \mathbf{C} \cap \mathbf{CB}'_i) = \int_{\hat{C}_{d,t} n}^{\infty} \psi d\nu$ , which is at most

$$\begin{aligned} & \sum_{k=n}^{\infty} \int_{\hat{C}_{d,t} k}^{\hat{C}_{d,t} (k+1)} \psi d\nu \leq \sum_{k=n}^{\infty} \hat{C}_{d,t} (k+1) \mathbb{P}_t(\psi \geq \hat{C}_{d,t} k) \tag{6.13} \\ & \leq 2 \hat{C}_{d,t} \sum_{k=n}^{\infty} (k+1) (1 - p_n e^{-t})^{k-1} \\ & \leq 2 \hat{C}_{d,t} (n+1) \sum_{k=n}^{\infty} (k-n+1) (1 - p_n e^{-t})^{k-1} \\ & = 2 \hat{C}_{d,t} (n+1) p_n^{-2} e^{2t} (1 - p_n e^{-t})^{n-1}, \end{aligned}$$

the first inequality by Lemma 6.16 and the third due to  $k + 1 \leq (n + 1)(k - n + 1)$  for all  $k \geq n$ .

Applying Lemma 6.14 with  $m = n - i$ , (6.13),  $2\tau(6\tau + 1)^{1/6}d^{-1/6} \leq 1$  and  $i < 2n_1 \leq n/2$  to (6.12) yields the statement of the lemma.  $\square$

**Proof of Lemma 6.6.** This is implied by Lemmas 6.12 and 6.17.  $\square$

**6.4. Proof of Lemma 6.7.** We start with an analogue of Lemma 4.2 suitable in the present context. Recall that we write  $\mathcal{B}_{[\phi, e_{\text{BN}}^-]} = \mathcal{B} \cap (E(P_{\phi, e_{\text{BN}}^-}) \times [0, 1))$ .

**Lemma 6.18.** *Let  $(e, s) \in E(\mathcal{T}_n) \times [0, 1)$ . Let  $\mathcal{B}^0 \subseteq E(\mathcal{T}_n) \times [0, 1)$  satisfy:*

- *$e$  supports exactly one bar in  $\mathcal{B}^0$ , and this bar is  $(e, s)$ ;*
- *$\text{Found}_{H_n^{\mathcal{B}}} \cup \mathcal{B}_{[\phi, e_{\text{BN}}^-]} = \mathcal{B}^0$  implies that  $X^{\mathcal{B}} : [0, H_n^{\mathcal{B}}] \rightarrow V(\mathcal{T}_n) \times [0, 1)$  crosses  $(e, s)$  exactly once.*

*Under  $\mathbb{P}_t$ , let  $\mathcal{B}^{\text{UnTo}}$  denote  $\mathcal{B} \cap \text{UnTouch}_{H_n^{\mathcal{B}}} \cap (E(P_{\phi, e_{\text{BN}}^-}) \times [0, 1))^c$ . Let  $\mathbb{P}_{\mathcal{B}^0}$  denote the conditional law of  $\mathcal{B}^{\text{UnTo}}$  under  $\mathbb{P}_t$  given  $\text{Found}_{H_n^{\mathcal{B}}} \cup \mathcal{B}_{[\phi, e_{\text{BN}}^-]} = \mathcal{B}^0$ . Let  $\mathbb{P}_{\mathcal{B}^0}^{\text{C} \cap \text{CB}'_i}$  denote the conditional law of  $\mathcal{B}^{\text{UnTo}}$  under  $\mathbb{P}_t$  given  $\text{Found}_{H_n^{\mathcal{B}}} \cup \mathcal{B}_{[\phi, e_{\text{BN}}^-]} = \mathcal{B}^0$ ,  $b_{\text{BN}} = (e, s)$ , and  $\text{C} \cap \text{CB}'_i$ . Then  $\mathbb{P}_{\mathcal{B}^0}^{\text{C} \cap \text{CB}'_i} = \mathbb{P}_{\mathcal{B}^0}$ .*

**Proof.** The event that  $\text{Found}_{H_n^{\mathcal{B}}} \cup \mathcal{B}_{[\phi, e_{\text{BN}}^-]} = \mathcal{B}^0$  being equivalent to  $X^{\mathcal{B}}[0, H_n^{\mathcal{B}}] \rightarrow V(\mathcal{T}_n) \times [0, 1)$  taking some given form and a specification of  $\mathcal{B} \cap (E(P_{\phi, e_{\text{BN}}^-}) \times [0, 1))$ , Lemma 1.13 implies that  $\mathbb{P}_{\mathcal{B}^0}$  is simply the Poisson- $t$  law on  $\text{UnTouch}_{H_n^{\mathcal{B}}} \cap (E(P_{\phi, e_{\text{BN}}^-}) \times [0, 1))^c$ . Under  $\mathbb{P}_t(\cdot | \text{Found}_{H_n^{\mathcal{B}}} \cup \mathcal{B}_{[\phi, e_{\text{BN}}^-]} = \mathcal{B}^0)$ , the randomness remains in  $\mathcal{B}^{\text{UnTo}}$  and in  $\mathcal{A}$ ; the event  $\text{C} \cap \text{CB}'_i \cap \{b_{\text{BN}} = (e, s)\}$  happens if and only if these two random variables take values such that each of the following occurs:

- $\mathcal{A} \in \mathcal{E}_{n-1-i} \times [0, 1)$ ;
- $X^{\mathcal{B}}$  meets a joint of  $\mathcal{A}$  after crossing  $(e, s)$  and before time  $H_n^{\mathcal{B}}$ ;
- each element in  $E(P_{e^-, E(\mathcal{A})^+})$  supports at least two bars in  $\mathcal{B}$ .

In terms of the data  $(\mathcal{B}^{\text{UnTo}}, \mathcal{A})$ , these conditions are equivalent to the occurrence of each of the following:

- $\mathcal{A} \in \mathcal{E}_{n-1-i} \times [0, 1)$ ;
- $\mathcal{A} \in \text{ViLoc}_{(e, s)}(\mathcal{B}_{e^-}^0)$ ,

where the set  $\text{ViLoc}_{(e, s)}$  is specified in Definition 5.3 and  $\mathcal{B}_{e^-}^0 = \mathcal{B}^0 \cap (E(\mathcal{T}_{[e^-]}) \times [0, 1))$ . However, these conditions are in fact independent of  $\mathcal{B}^{\text{UnTo}}$ , because they make no reference to  $\mathcal{B}^{\text{UnTo}}$  which is independent of  $\mathcal{A}$ . Hence,  $\mathbb{P}_{\mathcal{B}^0}^{\text{C} \cap \text{CB}'_i} = \mathbb{P}_{\mathcal{B}^0}$ .  $\square$

To conclude that Lemma 6.7 holds, note that, under the conditioning in the lemma's statement,  $\mathcal{B}$  has the law of  $\text{Found}_{H_n^{\mathcal{B}}} \cup \mathcal{B}_{[\phi, e_{\text{BN}}^-]} \cup \mathcal{B}^{\text{UnTo}}$ , where  $\mathcal{B}^{\text{UnTo}}$  has the law of  $\mathbb{P}_{\mathcal{B}^0}^{\text{C} \cap \text{CB}'_i}$ , with the value of  $\mathcal{B}^0$  being determined by

the conditioning. By Lemma 6.7,  $\mathbb{P}_{\mathcal{B}^0}^{\mathcal{C} \cap \mathcal{CB}'}$  equals  $\mathbb{P}_{\mathcal{B}^0}$ , the Poisson- $t$  law on  $\text{UnTouch}_{H_n^{\mathcal{B}}} \cap (E(P_{\phi, e_{\mathcal{BN}}^-}) \times [0, 1))^c$ , and we are done.  $\square$

**6.5. Deriving Lemma 3.1.** The argument for Lemma 3.1 is in essence a specialization of those made in this section with some changes in notation.

**Definition 6.19.** Let  $m \in \mathbb{N}$ . Let  $\overline{\mathcal{BN}}_m \subseteq \{H_m^{\mathcal{B}} < \infty\}$  denote the event that some element in  $E(P_{\phi, Y_{H_m^{\mathcal{B}}}})$  supports exactly one bar in  $\mathcal{B}$ . On  $\overline{\mathcal{BN}}_m$ , let  $e_m \in E(P_{\phi, Y_{H_m^{\mathcal{B}}}})$  denote the furthest such element from  $\phi$ , and let  $b_m$  denote the unique bar in  $\mathcal{B}$  supported on  $e_m$ . On  $\overline{\mathcal{BN}}_m$ , let  $\text{NoEsc}'_m$  denote the event that  $X_{b_m^+}^{\mathcal{B}}$  visits  $(\phi, 0)$  before  $\mathcal{V}_n \times [0, 1)$ .

**Lemma 6.20.** Consider the law  $\mathbb{P}_t$  given  $H_m^{\mathcal{B}} < \infty$  and the data  $X^{\mathcal{B}} : [0, H_m^{\mathcal{B}}] \rightarrow V(\mathcal{T}_n) \times [0, 1)$  and  $\text{WPE}_{H_m^{\mathcal{B}}} \subseteq E(P_{\phi, Y_{H_m^{\mathcal{B}}}})$ . Then the conditional probability of  $\text{NoEsc}'_m$  is almost surely at most

$$\left(1 - p_n e^{-t} (d+1)^{-1} (1 - e^{-td^{-1}})\right)^{|\text{WPE}_{H_m^{\mathcal{B}}}|^{-1}}.$$

**Proof.** The proof is that of Lemma 6.8 after straightforward notational changes. The analogue of Lemma 6.7 is simpler, because weighting effects due to conditioning on events concerning  $\mathcal{A}$  do not appear.  $\square$

**Lemma 6.21.** Suppose that  $t \leq 1$ . Then

$$\mathbb{P}_t(H_m^{\mathcal{B}} < \infty, |\text{WPE}_{H_m^{\mathcal{B}}}| \leq m/2) \leq 3 \cdot 2^m \tau^m (6\tau + 1)^{m/6} d^{-m/6}.$$

**Proof.** This is implied by Lemmas 6.12 and 6.14.  $\square$

**Proof of Lemma 3.1.** Noting that  $\{H_m^{\mathcal{B}} < \infty\} \cap \{|\text{WPE}_{H_m^{\mathcal{B}}}| \geq 1\} \subseteq \overline{\mathcal{BN}}_m$ , we find that

$$\begin{aligned} p_m - p_n &= \mathbb{P}_t(H_m^{\mathcal{B}} < \infty, H_n^{\mathcal{B}} = \infty) \\ &\leq \mathbb{P}_t(H_m^{\mathcal{B}} < \infty, |\text{WPE}_{H_m^{\mathcal{B}}}| \leq m/2) + \mathbb{P}_t(H_m^{\mathcal{B}} < \infty, |\text{WPE}_{H_m^{\mathcal{B}}}| > m/2, \text{NoEsc}'_m). \end{aligned}$$

Estimating the right-hand side by means of Lemmas 6.20 and 6.21, we see that, if  $n, m \in \mathbb{N}$  satisfy  $n \geq m$ ,  $p_m > (1 + \frac{1}{25})p_n$  and  $p_n \geq \varepsilon$ , then

$$3 \cdot 2^m \tau^m (6\tau + 1)^{m/6} d^{-m/6} + \left(1 - \varepsilon e^{-t} (d+1)^{-1} (1 - e^{-td^{-1}})\right)^{m/2} \geq \frac{\varepsilon}{25}.$$

Given that  $\tau \leq 1 + 2d^{-1} \leq 2$ , this condition is violated if  $m$  exceeds

$$\max \left\{ \frac{\log(150\varepsilon^{-1})}{\log(4^{-1} \cdot 13^{11/6} d^{1/6})}, \frac{2 \log(50\varepsilon^{-1})}{-\log(1 - \varepsilon e^{-2} (d+1)^{-1} (1 - e^{-2d^{-2}}))} \right\}. \quad (6.14)$$

This proves Lemma 3.1 with  $n_1$  equal to (6.14) in the case that  $0 \leq t \leq d^{-1} + 2d^{-1}$ ; in the case that  $0 \leq t \leq \frac{1}{7}d^{-1} \log d$ , an explicit value for  $n_1$  is similarly computed.  $\square$

## APPENDIX A. SOME TECHNICAL REFINEMENTS

In this appendix, we revisit a few aspects of the derivation, providing stronger versions of some of the tools which we have developed. These permit Theorem 1.1 to be derived for  $d \geq 1640$ .

**A.1. Statements of the stronger lemmas.** The plan is to replace the use of the event **NoBar** in for example Lemma 4.5, which provides a lower bound on the normalization  $Z$  from Lemma 4.2, with the event **NewEv** :=  $\{\mathcal{M}_\phi = \emptyset\} \cap \{H_n^\infty = \infty\}$ . We remark that this event shares a key property with **NoBar**, namely that

$$\text{NewEv} \subseteq \{\text{ViLoc}(\mathcal{B}) = \mathcal{E}_0 \times [0, 1)\}. \quad (\text{A.1})$$

Indeed, under the event on the left-hand side, the meander  $X^{\mathcal{B}}$  travels upwards along the pole at  $\phi$  from  $(\phi, 0)$ , departing across any bar in  $\mathcal{B}$  that it may encounter, and then returning via the same bar to continue its upward progress at the pole at  $\phi$ . In this way, it travels over the whole of the pole at  $\phi$ , returning  $(\phi, 0)$  without having visited  $\mathcal{V}_n \times [0, 1)$ . The definition of  $\text{ViLoc}(\mathcal{B})$  thus indeed entails that  $\text{ViLoc}(\mathcal{B}) = \mathcal{E}_0 \times [0, 1)$ .

The use of **NewEv** in place of **NoBar** makes a difference when  $t$  is close to the transition, with  $t \leq (1+\varepsilon)d^{-1}$ , say. In this regime,  $X^{\mathcal{B}}$  is unlikely to reach  $\mathcal{V}_n \times [0, 1)$ , making **NewEv** likely if  $d$  is high. This change in the argument means that we dispense with all factors of the form  $e^\tau \approx e$  that appear in the loss estimates of Section 4.4.

We now present strengthenings of Lemma 4.5, 4.8 and 4.10.

**Lemma A.1.** *Let  $t \leq d^{-1} + 2d^{-2}$ . Assume that  $d \geq 150$ . Then*

$$Z \geq d\left(1 - \frac{1}{16}\right).$$

**Lemma A.2.** *If  $d \geq 22\tau^2$ , then*

$$\mathbb{P}_t(\text{NewEv} \mid \mathcal{C} \cap \text{BN}^c) \geq 1 - \frac{1}{6}.$$

**Lemma A.3.** *Lemma 4.10 holds with the final expression in the statement equal to  $(1 + \frac{1}{25})(1 + \frac{1}{15})p_n\ell$ .*

**A.2. Applying the new estimates.** When the derivation is undertaken with Lemmas A.1, A.2 and A.3 substituting for Lemmas 4.5, 4.8 and 4.10, some changes arise in the estimates (4.9), (4.10), (4.11) and (4.12). We now state the revised estimates. As before, they concern the four cases  $m = 0$ ,  $m = 1$ ,  $2 \leq m \leq n_1 - 1$  and  $m \geq n_1$ . In addition to the restrictions on parameters stated below, assume that  $p_n(t) \geq \varepsilon$  and that  $n \geq 2n_1$ .

- for  $d \geq 1000$ ,  $1 \leq \tau \leq 1 + 2d^{-1}$ ,

$$\mathbb{P}_t\left(\text{P}^-, \#\mathcal{M}_\phi = 0 \mid \mathcal{C} \cap \text{BN}^c\right) \leq \left(1 + \frac{2}{5}\right)p_n \left(1 + \frac{1}{15}\right)^3 \left((\tau(1+\tau)+4)d^{-1/2} + \tau d^{-1/2} \log d\right); \quad (\text{A.2})$$

- for  $d \geq 11\tau^2$ ,

$$\mathbb{P}_t\left(\mathbf{P}^-, \#\mathcal{M}_\phi = 1 \mid \mathbf{C} \cap \mathbf{BN}^c\right) \leq 4\left(1 + \frac{1}{5}\right)p_n\tau^3\left(1 + \frac{1}{15}\right)^2d^{-1}; \quad (\text{A.3})$$

- for  $m \geq 0$  (and in particular for  $2 \leq m \leq n_1 - 1$ ),  $d \geq 10e\tau^2$  and  $\tau \geq 1$ ,

$$\mathbb{P}_t\left(\mathbf{P}^-, \#\mathcal{M}_\phi = m \mid \mathbf{C} \cap \mathbf{BN}^c\right) \leq 6\left(1 + \frac{1}{25}\right)dp_n\left(1 + \frac{1}{15}\right)e^{\tau-1}(m+1)(2e\tau^2d^{-1})^m; \quad (\text{A.4})$$

- and, for  $d \geq 10e\tau^2$  and  $\tau \geq 1$ ,

$$\mathbb{P}_t\left(\mathbf{P}^-, \#\mathcal{M}_\phi \geq n_1 \mid \mathbf{C} \cap \mathbf{BN}^c\right) \leq \frac{1}{50}p_n. \quad (\text{A.5})$$

We now find an explicit bound on  $d$  as before. If, alongside the conditions required for the four preceding bounds, we have that  $d \geq \max\{40e\tau^2, 1000\}$  and  $d^{-1} \leq t \leq d^{-1} + 2d^{-2}$ , then (4.9), (4.10), (4.11) and (4.12) show that  $\mathbb{P}_t(\mathbf{P}^- \mid \mathbf{C} \cap \mathbf{BN}^c)$  is at most the product of  $p_n$  and

$$\begin{aligned} & \left(1 + \frac{2}{5}\right)\left(1 + \frac{1}{15}\right)^3\left(\left(1 + \frac{1}{500}\right)\left(2 + \frac{1}{500}\right) + 4 + \left(1 + \frac{1}{500}\right)\log d\right)d^{-1/2} \\ & + 4\left(1 + \frac{1}{5}\right)\left(1 + \frac{1}{500}\right)^3\left(1 + \frac{1}{15}\right)^2d^{-1} \\ & + 6 \cdot 16\left(1 + \frac{1}{25}\right)\left(1 + \frac{1}{9}\right)\left(1 + \frac{1}{500}\right)^4\left(1 + \frac{1}{15}\right)^2ed^{-1} + \frac{1}{50} \\ & \leq 10.21d^{-1/2} + 1.71d^{-1/2}\log d + (5.5 + 346)d^{-1} + \frac{1}{50}. \end{aligned} \quad (\text{A.6})$$

If  $t \leq d^{-1} + 2d^{-2}$  and  $d \geq 1000$  then  $1 - \tau \geq -\frac{1}{500}$ , so that, in this case, if (A.6) is at most  $\left(1 - \frac{1}{5}\right) = 0.8$  then (4.6) holds. Given that  $t \rightarrow t^{-1/2}\log t$  is decreasing on  $[e^{e^2}, \infty) \supseteq [1619, \infty)$  and that (A.6) at  $d = 1640$  is at most 0.7991, we confirm (4.6) for the first of the two stated parameter choices.

This completes the derivation of Proposition 1.9 and thus of Theorem 1.1 for  $d \geq 1640$ , subject to proving the new versions of the lemmas.

### A.3. Proving the stronger lemmas.

**Proof of Lemma A.1.** It follows from (A.1) that

$$\mathbf{NewEv} \subseteq \{|\mathbf{ViLoc}(\mathcal{B})| = d\} \quad (\text{A.7})$$

and thus that  $Z \geq d\mathbb{P}_t(\mathbf{NewEv})$ .

Hence, Lemma A.1 follows from the claim that, if  $n \geq n_1$ ,

$$\mathbb{P}_t(\mathbf{NewEv}) \geq 1 - \frac{1}{16}. \quad (\text{A.8})$$

To verify this, note that if  $n \geq n_1$  then  $p_n \leq \left(1 + \frac{1}{25}\right)p_\infty$  by Lemma 3.1. Thus,

$$\begin{aligned} \mathbb{P}_t(\mathbf{NewEv}) & \geq 1 - p_n - \mathbb{P}_t(\mathcal{M}_\phi \neq \emptyset) \\ & \geq 1 - \left(1 + \frac{1}{25}\right)p_\infty - \left(1 + \frac{1}{10}\right)\tau^2d^{-1} \geq 1 - \frac{1}{20}\left(1 + \frac{1}{25}\right) - \frac{1}{96}, \end{aligned}$$

where we also used Lemma 4.7 and the upcoming Lemma A.5 (and  $d \geq 150$ ). (Note that Lemma 4.7's hypothesis  $d \geq 11\tau^2$  applies because  $\tau \leq 1 + 2d^{-1}$  and  $d \geq 50$ .) By  $\tau \leq 1 + 2d^{-1}$  and  $d \geq 110$ ,  $(1 + \frac{1}{10})\tau^2 d^{-1} \leq \frac{1}{96}$ . Hence, we obtain (A.8).  $\square$

**Proof of Lemma A.2.** We state a variant of Lemma 4.6; the result has the same proof.

**Lemma A.4.** *Let  $d \geq 22\tau^2$ . Then  $Z \leq (1 + \frac{1}{8})d$ .*

By Lemma 4.2 and (A.7),

$$\mathbb{P}_t(\text{NewEv} | \mathbf{C} \cap \text{BN}^c) = dZ^{-1}\mathbb{P}_t(\text{NewEv}).$$

Hence, the result follows from Lemma A.4 and (A.8).  $\square$

**Proof of Lemma A.3.** In the proof of Lemma 4.10, replace (4.19) by

$$\mathbb{P}_{t,\mathcal{B}}^{v, \mathbf{C} \cap \text{BN}^c}(\text{Esc}_{b_0}) \leq (1 + \frac{1}{15})\mathbb{P}_{t,\mathcal{B}}^v(\text{Esc}_{b_0}). \quad (\text{A.9})$$

Note that an event  $\text{NoBar}_v = \text{NoBar}_v(\mathcal{B}')$  is introduced in order to derive (4.19). Replace this event by  $\text{NewEv}_v := \{\mathcal{M}_v = \emptyset\} \cap \text{Esc}_{b_0}^c$ . The proof of (A.9) is identical to that of (4.19) except that the use of  $\mathbb{P}_{t,\mathcal{B}}^v(\text{NoBar}_v) = e^{-\tau}$  is replaced by  $\mathbb{P}_{t,\mathcal{B}}^v(\text{NewEv}_v) \geq 1 - \frac{1}{16}$ ; the latter bound is due to (A.8) (and Lemma 1.12); note that the hypothesis of (A.8) is satisfied because indeed we have that  $n - d(\phi, e_0^-) \geq n_1$ .  $\square$

**Lemma A.5.** *Suppose that  $d \geq 6$ . If  $t \leq d^{-1} + 2d^{-2}$  then  $p_\infty(t) \leq 6d^{-1}$ .*

**Proof.** Consider the branching process  $\Theta$  defined under  $\mathbb{P}_t$  with seed  $\phi \in V(\mathcal{T})$  such that for each  $e \in E(\mathcal{T})$ ,  $e^-$  is an offspring of  $e^+$  (for  $\Theta$ ) if  $e$  supports at least one bar of  $\mathcal{B}$ . Note that it is necessary for  $(\phi, 0) \notin X^{\mathcal{B}}(0, \infty)$  that  $\Theta$  have infinitely many individuals. Thus,  $p_\infty(t) \leq 1 - q_{\text{ext}}$ , where  $q_{\text{ext}}$  is the extinction probability of  $\Theta$ .

It suffices then to show that, if  $d \geq 6$ ,

$$q_{\text{ext}} > 1 - 6d^{-1}. \quad (\text{A.10})$$

Note that  $q_{\text{ext}}$  is non-increasing in  $t$ . Thus, we need establish (A.10) only for  $t = d^{-1} + 2d^{-2}$ . Let  $f : [0, 1] \rightarrow [0, 1]$ ,  $f(s) = \sum_{i=0}^d q_i s^i$ , with  $q_i = \binom{d}{i} (1 - e^{-t})^i e^{-t(d-i)}$ , denote the moment generating function of the offspring distribution of  $\Theta$ . Recall from the theory of branching processes that  $f(q_{\text{ext}}) = q_{\text{ext}}$ , with  $f(s) > s$  implying that  $s < q_{\text{ext}}$ . Thus, if we prove that  $d \geq 6$  implies that

$$f(1 - 6d^{-1}) > 1 - 6d^{-1}, \quad (\text{A.11})$$

we will obtain (A.10). Note that  $f(s) = ((1 - e^{-t})s + e^{-t})^d$ . Set  $\varepsilon = 1 - s$  and note that  $f(s) > s$  is implied by

$$\frac{d(1 - e^{-t})\varepsilon}{1 - (1 - e^{-t})\varepsilon} < \varepsilon + \varepsilon^2/2. \quad (\text{A.12})$$

With  $t = d^{-1} + 2d^{-2}$ , note that  $1 - e^{-t} \leq d^{-1} + \frac{3}{2}d^{-2}$  if  $d \geq 2$ . Using this, it is simple to see that, if  $d \geq 6$ , then  $\varepsilon \geq 6d^{-1}$  implies (A.12). Hence (A.10) holds if  $d \geq 6$ .  $\square$

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