The kinetic limit of a system of coagulating planar Brownian particles

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Abstract

We study a model of mass-bearing coagulating planar Brownian particles. Coagulation occurs when two particles are within a distance of order ε. We assume that the initial number of particles N is of order $|\log \varepsilon|$. Under suitable assumptions of the initial distribution of particles and the microscopic coagulation propensities, we show that the macroscopic particle densities satisfy a Smoluchowski-type equation.

1 Introduction

A colloid consists of a large number of small particles that are suspended in an environment of far smaller and more numerous molecules. Large numbers of molecules bombard each particle, and random fluctuations among these collisions tend to give rise to an Ornstein-Uhlenbeck motion of the particle, in which its velocity is forced by a Brownian motion, with a drag force acting in the direction opposite to its velocity. On a long time scale, the colloidal particles move according to Brownian motions, because an Ornstein-Uhlenbeck process approximates such a motion over a long period of time. The particles of a colloid may also be liable to interact. In [2], we studied a model of a colloid in which this means of interaction took the form of a coagulation, this reaction being liable to take place between a pair of particles if they come to lie close enough to one another. The density of particles at the initial time was chosen so that the dynamics occur in a regime of mean free path, wherein a typical particle meets a bounded number of other particles in a unit of time. Speaking in rough terms, this choice of scaling causes the effects of diffusion and interaction

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on the macroscopic evolution of the system to be comparable. In common with much of non-equilibrium statistical mechanics, we interpret the macroscopic behaviour of the system in terms of the evolution of a small number of thermodynamic parameters, in this case, the density of particles of a given mass, as a function of macroscopic space and time. In [2], we proved that, when the initial number of particles is chosen to be high, this density typically evolves as the solution of the Smoluchowski system of PDE,

$$\frac{\partial f_n}{\partial t}(x,t) = d(n)\Delta f_n(x,t) + Q_1^n(f)(x,t) - Q_2^n(f)(x,t). \qquad n = 1, 2, \ldots \tag{1.1}$$

The first term on the right-hand-side of (1.1) corresponds to the diffusion among particles of mass n, with $d(n)$ being one-half of the diffusion rate of such particles. The terms in (1.1) corresponding to the interaction of pairs of particles are given by the gain term

$$Q_1^n(f)(x,t) = \frac{1}{2}\sum_{m=1}^{n} \beta(m, n-m) f_m(x,t) f_{n-m}(x,t), \tag{1.2}$$

and the loss term

$$Q_2^n(f) = f_n(x,t)\sum_{m=1}^{\infty} \beta(m,n) f_m(x,t). \tag{1.3}$$

Here, the collection of constants $\beta : \mathbb{N}^2 \to (0, \infty)$ quantify the macroscopic propensity of mass at a pair of values to combine.

We will be concerned with weak solutions of the system (1.1), defined by the equality of the left- and right-hand sides of (1.1) after multiplication by $J_n : \mathbb{R}^d \times [0,\infty) \to [0,\infty)$, integration in space-time, and integration by parts. The equality is demanded over all choices of sequences of compactly supported smooth functions $\{J_n : n \in \mathbb{N}\}$ such that only finitely many terms in the sequence are not identically zero.

The arguments of [2] were valid in the case where the dimension d of the system was assumed to be at least three. The question of the behaviour of such a system in two dimensions is significantly different, and it is this topic that we address in this paper. We now turn to describe the model in more detail, after which, we will discuss the ways in which the two-dimensional case differs from that of higher dimensions.

We will be working with a collection of microscopic models, each model carrying an index $N \in \mathbb{N}$, this being the total number of particles present in the system at the initial time. Each of these N particles is independently assigned a random integer mass and placed at the initial time at a random location whose law depends on that mass. More precisely, we will be describing the state of the system at any given moment in time by a configuration, by which we mean a map $q : I_q \to \mathbb{R}^2 \times \mathbb{N}$, whose domain I_q is some finite set of a countable index set I. That is, if $i \in I_q$ has $q(i) = (x_i, m_i)$, then the system currently contains a particle of mass m_i at $x_i \in \mathbb{R}^2$. To define the initial configuration, we choose a sequence of continuous functions $\big\{h_n : \mathbb{R}^2 \to [0,\infty),\, n \in \mathbb{N}\big\}$ that must satisfy some conditions that we will shortly specify. We set $Z = \sum_{n=1}^{\infty} \int_{\mathbb{R}^2} h_n \in (0,\infty)$, and choose N points in $\mathbb{N} \times \mathbb{R}^2$ indepedently according to a law whose density at (x,n) is equal to $h_n(x)/Z$. Selecting arbitrarily a set of N symbols $\{i_j : j \in \{1,\ldots,N\}\}$ from I, we define the initial configuration q_0 by insisting that $q_0(i_j)$ is equal to the j-th of the randomly chosen members of $\mathbb{N} \times \mathbb{R}^2$.

Each particle moves according to an independent Brownian motion whose diffusion rate $2d(m)$ depends on its mass $m \in \mathbb{N}$. As we will explain later, we require some conditions on the choice of the function $d : \mathbb{N} \to \mathbb{R}$, although the restriction imposed by these conditions is far from prohibiting the physically reasonable choice where d is decreasing. Any pair of particles that approach to within a certain range of interaction are liable to coagulate, at which time, they disappear from the system, to be replaced by a particle whose mass is equal to the sum of the colliding particles, and whose location is at some point nearby the place where the collision took place. This range of interaction is taken to be equal to a parameter ϵ, whose dependence on the total particle number N must be stipulated. We make the choice $N = |\log \epsilon| Z$. This will ensure that a particle randomly chosen from those initally present experiences an expected number of collisions in a given unit of time that remains bounded away from zero and ∞ as N is taken to be high. The effects of motion and reaction determine the macroscopic evolution of the system to comparable extents in this scaling.

We now describe the mathematical details of these dynamics. Let $F : \{\mathbb{R}^2 \times \mathbb{N}\}^I \to \mathbb{R}$ denote a smooth function, whose domain is given the product topology. The dynamics is such that the action on F of the model's infinitesimal generator $\mathbb{L}$ is given by

$$(\mathbb{L}F)(q) = \mathbb{A}_0 F(q) + \mathbb{A}_C F(q),$$

where the diffusion and collision operators are given by

$$\mathbb{A}_0 F(q) = \sum_{i \in I_q} d(m_i) \Delta_{x_i} F \tag{1.4}$$

and

$$\mathbb{A}_C F(q) \;=\; \frac{1}{2} \sum_{i,j \in I_q} \epsilon^{-2} \big| \log \epsilon \big|^{-1} V\Big(\frac{x_i - x_j}{\epsilon}\Big) \alpha(m_i, m_j) \tag{1.5}$$

$$\bigg[\frac{m_i}{m_i + m_j} F\big(S^1_{i,j} q\big) + \frac{m_j}{m_i + m_j} F\big(S^2_{i,j} q\big) - F(q) \bigg].$$

Note that:

- the function $V:\mathbb{R}^2 \to [0,\infty)$ is assumed to be Hölder continuous of compact support, and with $\int_{\mathbb{R}^2} V(x)dx = 1$.

- we denote by $S^1_{i,j}q$ that configuration formed from q by removing the indices i and j from I_q, and adding a new index from I to which $S^1_{i,j}q$ assigns the value (x_i, m_i+m_j). The configuration $S^2_{i,j}q$ is defined in the same way, except that it assigns the value (x_j, m_i+m_j) to the new index. The specifics of the collision event then are that the new particle appears in one of the locations of the two particles being removed, with the choice being made randomly with weights proportional to the mass of the two colliding particles.

We will denote by $\mathbb{P}_N$ the measure on functions from $t \in [0,\infty)$ to the configurations determined by the process at time t. Its expectation will be denoted $\mathbb{E}_N$.

The form of the collision term in (1.5) differs from that used in the case of higher dimensions, in that the factor of $\big|\log\epsilon\big|^{-1}$ is absent in the latter case. To explain why we make this change, we firstly recall the reason for the form of the collision operator in the case when $d \geq 3$. Suppose that, for some such choice of the dimension, two particles (x_i, m_i) and (x_j, m_j) have, at some time t_0, just become liable to interact, in the sense that the difference $x_i - x_j$ has become of order ϵ. This state of affairs is liable to persist for a time of order ϵ^2, but not much longer: for $d \geq 3$ and C a large constant, the Brownian displacement $x_i - x_j$ would return a distance of ϵ from the origin with only a small probability after a time of $C\epsilon^2$ after the moment t_0. This means that, by choosing a form of collision dynamics in which the factor of $|\log\epsilon|^{-1}$ is absent from (1.5), we ensure that the integral

$$I_T = \int_{t=t_0}^{T} \alpha(m_i, m_j)\epsilon^{-2} V\Big(\frac{x_i - x_j}{\epsilon}\Big)dt,$$

reaches its eventual value after a time of order ϵ^2. Other particles are unlikely to interfere with this pair in such a short period of time, and, as such, we may neglect their influence. The probability of collision between the pair before time T is equal to $1 - \exp\big\{ - I_T\big\}$. Thus, for $d \geq 3$, our choice of dynamics is such that, among all the pairs of particles that at some moment lie within ϵ of each other, the fraction that eventually coagulate is bounded in N away from 0 and 1, with this fraction being close to 0 or 1 depending on whether the relevant constant $\alpha(m_i, m_j)$ is high or low.

Turning to the planar case, note firstly that, in order that the probability of pair collision may remain equal to $1 - \exp\big\{ - I_T\big\}$, we alter the definition of I_T by introducing a factor of $\big|\log\epsilon\big|^{-1}$. The two-dimensional case differs, because a planar Brownian motion returns almost surely to any open set at indefinitely later times. As such, the difference $x_i - x_j$ will endlessly re-enter the ϵ-ball centred at the origin, ensuring that $I_T \to \infty$ as $T \to \infty$. In a system of two particles, their coagulation is inevitable. In the system that we consider, where the regime of constant mean free path has been selected, a pair of particles at ϵ distance may find that their ongoing efforts to coagulate, as measured by the increase of I_T, are interrupted by the arrival of a third particle, the probability of appearance of such an intruder becoming appreciable at a small time, independent of ϵ, after that at which the pair in question first came close to each other. The factor of $\big|\log \epsilon\big|^{-1}$ that appears in (1.5) ensures that, during this short fixed time, the probability of coagulation between the pair is of unit order, with the constant $\alpha(m_i, m_j)$ determining whether this probability is high or low. To write a statement analogous to that for the higher dimensional case: among the set of pairs of particles that are at some moment at a distance of order ϵ, the fraction that combine with each other, rather than with some other particles, is bounded in N away from 0 and 1, with the value of the constant α determining whether this fraction is high or low, similarly to the earlier case.

Our main result is conveniently expressed in terms of the empirical measures on the locations of particles of a given mass. For each $n \in \mathbb{N}$ and $t \in [0,\infty)$, we write $g_n(dx,t)$ for the measure on $\mathbb{R}^2$ given by

$$g_n(dx,t) = |\log \varepsilon|^{-1} \sum_{i \in I_{q(t)}} \delta_{x_i(t)}(dx) \mathbb{1}\big(m_i(t) = n\big).$$

We write g for the random measure on space-mass-time $\mathbb{R}^2 \times \mathbb{N} \times [0,\infty)$ such that, for each $t \geq 0$, its time-t marginal $g(\cdot,t)$ is given by

$$g(\cdot,t) = |\log \varepsilon|^{-1} \sum_{i \in I_{q(t)}} \delta_{\big(x_i(t), m_i(t)\big)}.$$

We also require a mild hypothesis on the diffusion coefficients $d : \mathbb{N} \to (0,\infty)$ (see the first remark after Theorem 1.1 below). Namely, we suppose that there exists a function $\gamma : \mathbb{N}^2 \to (0,\infty)$ such that $\alpha \leq \gamma$, with γ satisfying

$$n_2 \gamma\big(n_1, n_2 + n_3\big) \max\Big\{1, \Big[\frac{d(n_2+n_3)}{d(n_2)}\Big]^3\Big\} \leq \big(n_2+n_3\big)\gamma(n_1,n_2). \tag{1.6}$$

The initial random configuration of N particles is formed by scattering particles of numerous masses independently in $\mathbb{R}^2$ according to densities that are prescribed for each mass. These densities will be chosen as continuous functions $\{h_n : \mathbb{R}^2 \to [0,\infty), n \in \mathbb{N}\}$, and should satisfy some fairly weak bounds. To be specific, we insist that

- $k \in L^1(\mathbb{R}^2)$ and $\bar{k} \in L^\infty_{loc}(\mathbb{R}^2)$ where $k := \sum_{n=1}^\infty nh_n$ and

$$\bar{k}(x) = \iint_{|x_1-x_2-y|\le 1} \big|\log|x_1-x_2-y|\big| k(x_1)k(x_2)dx_1dx_2.$$

- For every m, $\sum_{n=1}^\infty d(n)^{2/3}\gamma(n,m)\hat{h}_n \in L^\infty_{loc}(\mathbb{R}^2)$ for $\hat{h}_n(x) = \int h_n(y)|x-y|^{-4/3}dy$.
- For every m, $\sum_{n=1}^\infty d(n)^{3/4}\gamma(n,m)\tilde{h}_n \in L^\infty_{loc}(\mathbb{R}^2)$ for $\tilde{h}_n(x) = \int h_n(y)|x-y|^{-3/2}dy$.

We then set $Z = \sum_{n=1}^\infty \int_{\mathbb{R}^2} h_n \in (0,\infty)$ and choose N points in $\mathbb{N}\times\mathbb{R}^2$ independently according to a law whose density at (x,n) is equal to $h_n(x)/Z$. Selecting arbitrarily a set of N symbols $\{i_j : j \in \{1,\dots,N\}\}$ from I, we define the initial configuration q_0 by insisting that $q_0(i_j)$ is equal to the j-th of the randomly chosen members of $\mathbb{N}\times\mathbb{R}^2$.
Remark. It is not hard to show that our assumptions on the initial data $\{h_n\}$ are satisfied if k is bounded, has a bounded support, $d(\cdot)$ is bounded and $\gamma(n,m) \le C(m)n$ for a function $C(\cdot)$. Indeed if k is bounded and has a bounded support, then $\bar{k}L^\infty_{loc}$ and $\hat{h}_n, \tilde{h}_n \in L^\infty_{loc}$ for every n. It is worth mentioning that if k belongs to the negative Sobolev Space $H^{-1} = W^{-1,2}$, then $\bar{k} \in L^\infty$.

The main theorem is now stated.

Theorem 1.1 *Let $\mathcal{P}_N$ denote the law on measures on $\mathbb{R}^2\times\mathbb{N}\times[0,\infty)$ given by the law of g under $\mathbb{P}_N$; recall that ϵ is related to N by means of the formula $N|\log\varepsilon|^{-1} = Z$, with the constant $Z \in (0,\infty)$ being given by the expression $Z = \sum_{n\in\mathbb{N}}\int_{\mathbb{R}^2} h_n$.*

The sequence $\{\mathcal{P}_N\}$ is tight. Moreover, any limit point $\mathcal{P}$ of the sequence $\{\mathcal{P}_N\}$ is concentrated on the space of measures taking the form $\sum_{n=0}^\infty f_n(x,t)dx \times \delta_n \times dt$ where $\{f_n : n\in\mathbb{N}\}$ ranges over weak solutions of (1.1) that satisfy the initial condition $f_n(\cdot,0) = h_n(\cdot)$; recall that the collection of constants $\beta : \mathbb{N}^2 \to [0,\infty)$ is given by

$$\beta(n,m) = \frac{2\pi\big(d(n)+d(m)\big)\alpha(n,m)}{2\pi\big(d(n)+d(m)\big)+\alpha(n,m)}. \tag{1.7}$$

Note that convergence in Theorem 1.1 is asserted only subsequentially and to a limiting object which may be a random superposition of weak solutions of (1.1). The need for such a weak statement of convergence disappears in the case that uniqueness of the weak solution of (1.1) are known. Some such conditions are provided by Proposition 2.6 of [6]. Shortly after the present paper originally appeared, we proved uniqueness in a reasonably general setting. The next result is a consequence of the main theorems of [3] as explained in Remark 1.2 of that paper.

Proposition 1 *Let the dimension satisfy* $d \geq 1$. *For* $a, b > 0$ *such that* $a + b < 1$, *and for positive constants* c_1 *and* c_2, *assume that* $\beta(n,m) \leq c_1(n^a + m^a)$ *and* $d(n) \geq c_2 n^{-b}$ *for all* $n, m \in \mathbb{N}$. *Also assume that* $d : \mathbb{N} \to (0, \infty)$ *is non-increasing. There exists* $e > 0$ *such that* $\sum_n n^e \|h_n\|_{L^\infty(\mathbb{R}^d)} < \infty$ *and* $\|\sum_n n^e h_n\|_{L^1(\mathbb{R}^d)} < \infty$ *imply that (1.1) has a unique weak solution. (In fact, this solution conserves mass, in the sense that* $I : [0, \infty) \to [0, \infty)$ *given by* $I(t) = \sum_{m \in \mathbb{N}} m \int_{\mathbb{R}^d} f_m(x,t) dx$ *satisfies* $I(t) = I(0)$ *for all* $t \in [0, \infty)$.*)*

Theorem 1.1 and Proposition 1 permit convergence of the empirical measures to be asserted in a more satisfying sense:

Corollary 1 *Suppose that the assumptions of Theorem 1.1 and Proposition 1 are in force.*

Let $J : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}$ *be a bounded and continuous test function. Then, for each* $n \in \mathbb{N}$ *and* $t \in (0, \infty)$,

$$\limsup_{N \to \infty} \mathbb{E}_N \left| \int_{\mathbb{R}^2} J(x,n,t) \big(g_n(dx,t) - f_n(x,t) dx \big) \right| = 0, \tag{1.8}$$

where again $N |\log \varepsilon|^{-1} = Z$, *with* $Z = \sum_{n \in \mathbb{N}} \int_{\mathbb{R}^2} h_n$. *In (1.8),* $\{ f_n : \mathbb{R}^2 \times [0, \infty) \to [0, \infty), n \in \mathbb{N} \}$ *denotes the unique weak solution to the system of partial differential equations (1.1) with the initial data* $f_n(\cdot, 0) = h_n(\cdot)$.

Remarks

- Included in the space of parameter values that satisfy (1.6) is the case where the diffusion rate d is a decreasing function of the mass, and the coagulation propensities α satisfy $\alpha\big(n,m\big) \leq Cnm$. In fact for a nonincreasing $d(\cdot)$, the condition (1.6) is equivalent to saying that $\alpha(n,m) \leq C(n)m$ for a function $C(n)$. Also, if the microscopic coagulation rate α is identically constant, then the condition (1.6) is equivalent to saying that the function $d(n)n^{-1/4}$ is nonincreasing.

- Note that the macroscopic coagulation propensities β depend only on the total integral of V that is assumed to be 1 for convenience. However when the dimension is 3 or more the propensity $\beta(n,m)$ does depend on V in a nontrivial way and is given as $\alpha(n,m) \int (1+u) V dx$, where u solves the PDE $\Delta u = \tau(1+u)V$ with $\tau = \alpha(n,m)/(d(n) + d(m))$.

- Our technique of proof also yields a kinetic limit derivation for the model in which particles are assumed to have a range of interaction that is mass-dependent. To give an example of such a variant, suppose that each particle of mass m has a radius $r(m)$, where $r(m) = \sqrt{m}$. We stipulate that particles of mass m and n are liable to react when their displacement reaches the order of $\big(r(m)+r(n)\big)\epsilon$. More precisely, we modify the definition (1.5) of the collision operator $\mathbb{A}_C$ by replacing the appearance of V by $\big(r(n)+r(m)\big)^{-2}V\big(\cdot/(r(n)+r(m))\big)$, (the factor that multiplies V being introduced so that, roughly speaking, the altered collision mechanism respects the spatial-temporal scaling of Brownian motion). Theorem 1.1 is still valid for this modified model with the same macroscopic coagulation propensities β. This is in sharp contrast with the case $d \geq 3$ for which the mass dependence affects the macroscopic coagulation propensities β.

In common with the proof for $d \geq 3$, a central element in deriving Theorem 1.1 is establishing that, at any given moment after the initial time, the presence of a particle of some given mass at some fixed point in space significantly affects the likelihood of a particle being at some other point in space only if that other point is at a short distance from the first particle. That is, on distances of short order, the presence of a particle makes it less likely to find another nearby, because the pair would have been liable to coagulate shortly beforehand. However, the distribution of particle at a given time is similar to one in which they were scattered independently, except for this short-range repulsion. The following proposition, whose form differs from that in the case $d \geq 3$ only in its scaling factor, formalises this assertion.

Proposition 2 *Set*

$$Q = |\log\varepsilon|^{-2} \sum_{(i,j)\in I_q} \alpha(m_i,m_j)V_\epsilon(x_i-x_j)J(x_i,m_i,t)\overline{J}(x_j,m_j,t), \tag{1.9}$$

where $J,\overline{J}:\mathbb{R}^2\times\mathbb{N}\times[0,\infty)\to[0,\infty)$ *are test functions satisfying the same conditions as those stated in Theorem 1.1. We also assume that* $J(x,m,t)=0$ *unless* $m=M_1$ *and* $\overline{J}(x,m,t)=0$ *unless* $m=M_2$. *Let* $\eta:\mathbb{R}^2\to[0,\infty)$ *denote a smooth function of compact support for which* $\int_{\mathbb{R}^2}\eta(x)dx=1$. *We have that*

$$\begin{aligned} &\int_0^T Q(t)dt \qquad\qquad (1.10)\\ =\ &\int_0^T dt\int_{\mathbb{R}^2} d\omega\,\beta(M_1,M_2)J(\omega,M_1,t)\overline{J}(\omega,M_2,t)\Big[|\log\varepsilon|^{-1}\sum_{i\in I_q:m_i=M_1}\delta^{-2}\eta\Big(\frac{x_i-\omega}{\delta}\Big)\Big]\\ &\Big[|\log\varepsilon|^{-1}\sum_{j\in I_q;m_j=M_2}\delta^{-2}\eta\Big(\frac{x_j-\omega}{\delta}\Big)\Big] + \mathrm{Err}\big(\epsilon,\delta\big), \end{aligned}$$

where the constants $\beta : \mathbb{N}^2 \to [0,\infty)$ *were defined in (1.7), and where the function* Err *satisfies*

$$\lim_{\delta \downarrow 0} \limsup_{\epsilon \downarrow 0} \mathbb{E}_N \big| Err\big(\epsilon, \delta\big) \big| = 0.$$

Why is this statement a mathematical rendering of the claim discussed before it was made? The quantity $\int_0^T Q(t)dt$ can be thought of as the total propensity of particles to combine during the interval of time $[0,T]$. Proposition 1 asserts it may be approximated by a time-averaged product of empiricial approximations to the density of particles (of the appropriate mass). That is, particles are arranged independently enough near most of the collision events that the rate of these collisions is roughly proportional to that arising in a system in which particles are scattered indepedently at random according to densities given by measuring the system in question on scale δ that is much larger than the reaction range ϵ. There is, however, a constant of proportion corresponding to the change from microscopic reaction propensity α appearing in the definition of Q to its macroscopic counterpart β. Its presence may be explained by the negative short-range correlation between particles discussed before the statement of Proposition 2.

The analogue of Theorem 1.1 that appears in [2] for the case $d \geq 3$ is derived as a consequence of Proposition 2. In Section 2 of [2], a sketch of the proof of Theorem 1.1 may be found. The details of the derivation of Theorem 1.1 from Proposition 2 do not differ in the two-dimensional case, so that we do not present these arguments again in this paper. Our task here is rather to present a detailed derivation of Proposition 2 in the case when $d = 2$. Before reading further, however, the reader may wish to consult Section 2 of [2]. We refer the reader to [2] also for a discussion of previous work related to the problem. Here, we mention only Sznitman [5], in which a model of Brownian spheres that annihilate as soon as they touch is studied. The partial differential equation by which the density of particles evolves was derived for the kinetic limit, in each dimension $d \geq 2$. In this work, the macroscopic annihilation rate is exactly 2π when the dimension is 2. This is compatible with our main results because if $d(\cdot)$ is identically $1/2$ and $\alpha \to \infty$, the macroscopic coagulation rate β approches 2π. Note that our model approximates the *hard core* model as α gets large.

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2 Establishing the Stosszahlansatz

For any given pair $(n,m) \in \mathbb{N}^2$, $u^\epsilon = u^\epsilon_{n,m} : \mathbb{R}^2 \to [0,\infty)$ is a function whose existence is ensured by Theorem 3.1 that lies in $C^2\big(\mathbb{R}^2\big)$ satisfying

$$u^\epsilon_{n,m}\big(x\big) = \frac{1}{2\pi}\tau\big(n,m\big)\int \log|x-y|\Big[V_\epsilon\big(y\big)u^\epsilon_{n,m}\big(y\big) + V^\epsilon\big(y\big)\Big], \tag{2.1}$$

where $\tau(n,m) = \alpha(n,m)/(d(n)+d(m))$. As a consequence,

$$\big(d(n)+d(m)\big)\Delta u^\epsilon_{n,m}\big(x\big) = \alpha\big(n,m\big)\Big[V_\epsilon\big(x\big)u^\epsilon_{n,m}\big(x\big) + V^\epsilon\big(x\big)\Big].$$

We are using the notations

$$V^\epsilon(x) = \epsilon^{-2}V\Big(\frac{x}{\epsilon}\Big)$$

and

$$V_\epsilon(x) = \epsilon^{-2}|\log\epsilon|^{-1}V\Big(\frac{x}{\epsilon}\Big).$$

We present the conditions on the two test functions $J,\overline{J} : \mathbb{R}^2 \times \mathbb{N} \times [0,\infty) \to \mathbb{R}$ that appear in Proposition 1. It suffices to work with functions that take non-zero values for only one value in the second argument, such functions measuring the presence of particles of a given mass. By a temporary abuse of notation, we write

$$\begin{aligned} J(x,M_1,t) &= J(x,t)\mathbb{1}\big\{m=M_1\big\} \\ \overline{J}(x,M_1,t) &= \overline{J}(x,t)\mathbb{1}\big\{m=M_2\big\}, \end{aligned}$$

where on the right-hand-side, J and $\overline{J}$ denote smooth maps from $\mathbb{R}^2\times[0,\infty)$ to $\mathbb{R}$ of compact support. We will suppress the appearance of the t-variable when writing the arguments of J and $\overline{J}$.

In seeking to verify Stosszahlansatz, we define

$$X_z(q) = |\log\epsilon|^{-2}\sum_{i,j\in I_q} u^\epsilon_{M_1,M_2}(x_i - x_j + z)J(x_i,M_1,t)\overline{J}(x_j,M_2,t)\mathbb{1}\big\{m_i = M_1, m_j = M_2\big\}. \tag{2.2}$$

The relevance of the expression (2.2) for our purposes is that the term Q and its variations appear as we apply the infinitesimal generaor on the expression $X_z - X_0$. We refer the reader to Section 2 of [2] for some heuristic justification of the special form of X_z.

Numerous terms arise when the operators $\mathbb{A}_0$ and $\mathbb{A}_C$ act on the expression $X_z - X_0$ (recall that the functions of configurations X_z, indexed by $z \in \mathbb{R}^2$, were defined in (2.2)). We now label these terms. Unless stated otherwise, we will adopt a notation whereby all the index labels appearing in sums should be taken to be distinct. This includes the case of multiple sums. For example, $\sum_{k,l\in I_q}\sum_{i\in I_q} f(x_k,x_l,x_i)$ denotes the sum of the evaluation of the function f over all arguments that are triples (x_k,x_l,x_i) where k, l and i are distinct indices in I. Note also that, unless otherwise stated, whenever the symbol u^ϵ appears in a summand, we mean $u^\epsilon_{M_1,M_2}$.

Firstly, we label those terms arising from the action of the diffusion operator. To do so, note that, for a time-dependent functional F of the configuration space, this action is given by

$$\Big(\frac{\partial}{\partial t}+\mathbb{A}_0\Big)F=\frac{\partial}{\partial t}F+\sum_{i\in I_q} d\big(m_i\big)\Delta_{x_i}F.$$

Thus, we label as follows:

$$\left(\frac{\partial}{\partial t}+\mathbb{A}_0\right)(X_z-X_0)(q(t))=H_{11}+H_{12}+H_{13}+H_{14}+H_2+H_3+H_4,$$

with

$$\begin{aligned}
H_{11} &= \big|\log\epsilon\big|^{-2}\sum_{i,j\in I_q}\alpha(m_i,m_j)\Big[V^\epsilon\big(x_i-x_j+z\big)-V^\epsilon\big(x_i-x_j\big)\Big]J(x_i,m_i,t)\overline{J}(x_j,m_j,t)\\
H_{12} &= -\big|\log\epsilon\big|^{-2}\sum_{i,j\in I_q}\alpha(m_i,m_j)V_\epsilon\big(x_i-x_j\big)u^\epsilon\big(x_i-x_j\big)J(x_i,m_i,t)\overline{J}(x_j,m_j,t)\\
H_{13} &= \big|\log\epsilon\big|^{-2}\sum_{i,j\in I_q}\alpha(m_i,m_j)V_\epsilon\big(x_i-x_j+z\big)u^\epsilon\big(x_i-x_j+z\big)J(x_i,m_i,t)\overline{J}(x_j,m_j,t),\\
H_{14} &= \big|\log\epsilon\big|^{-2}\sum_{i,j\in I_q}\Big[u^\epsilon\big(x_i-x_j+z\big)-u^\epsilon\big(x_i-x_j\big)\Big]\\
&\qquad\Big[J_t\big(x_i,m_i,t\big)\overline{J}(x_j,m_j,t)+J\big(x_i,m_i,t\big)\overline{J}_t(x_j,m_j,t)\Big],
\end{aligned}$$

along with

$$\begin{aligned}
H_2 &= 2\big|\log\epsilon\big|^{-2}\sum_{i,j\in I_q} d(m_i)\overline{J}(x_j,m_j,t)\\
&\qquad\Big[u^\epsilon_x(x_i-x_j+z)-u^\epsilon_x(x_i-x_j)\Big]\cdot J_x(x_i,m_i,t),\\
H_3 &= -2\big|\log\epsilon\big|^{-2}\sum_{i,j\in I_q} d(m_j)J(x_i,m_i,t)\\
&\qquad\Big[u^\epsilon_x(x_i-x_j+z)-u^\epsilon_x(x_i-x_j)\Big]\cdot\overline{J}_x(x_j,m_j,t),
\end{aligned}$$

and

$$H_4 = \big|\log\epsilon\big|^{-2}\sum_{i,j\in I_q}\Big[u^\epsilon(x_i-x_j+z)-u^\epsilon(x_i-x_j)\Big]$$
$$\Big[d(m_i)\Delta_x J(x_i,m_i,t)\overline{J}(x_j,m_j,t)+d(m_j)J(x_i,m_i,t)\Delta_x\overline{J}(x_j,m_j,t)\Big],$$

where f_x denotes the gradient of f, and $\cdot$ the scalar product. As for those terms arising from the action of the collision operator,

$$\mathbb{A}_C(X_z-X_0)(q)=G_z(1)+G_z(2)-G_0(1)-G_0(2),$$

where $G_z(1)$ is set equal to

$$\frac{1}{2}\sum_{k,l\in I_q}\alpha(m_k,m_l)V_\epsilon(x_k-x_l)\big|\log\epsilon\big|^{-2}\sum_{i\in I_q}$$
$$\Big\{\frac{m_k}{m_k+m_l}\Big[u^\epsilon(x_k-x_i+z)J(x_k,m_k+m_l,t)\overline{J}(x_i,m_i,t)$$
$$+u^\epsilon(x_i-x_k+z)J(x_i,m_i,t)\overline{J}(x_k,m_k+m_l,t)\Big]$$
$$+\frac{m_l}{m_k+m_l}\Big[u^\epsilon(x_l-x_i+z)J(x_l,m_k+m_l,t)\overline{J}(x_i,m_i,t)$$
$$+u^\epsilon(x_i-x_l+z)J(x_i,m_i,t)\overline{J}(x_l,m_k+m_l,t)\Big]$$
$$-\Big[u^\epsilon(x_k-x_i+z)J(x_k,m_k,t)\overline{J}(x_i,m_i,t)$$
$$+u^\epsilon(x_i-x_k+z)J(x_i,m_i,t)\overline{J}(x_k,m_k,t)\Big]$$
$$-\Big[u^\epsilon(x_l-x_i+z)J(x_l,m_l,t)\overline{J}(x_i,m_i,t)$$
$$+u^\epsilon(x_i-x_l+z)J(x_i,m_i,t)\overline{J}(x_l,m_l,t)\Big]\Big\},$$

and where

$$G_z(2) = -\big|\log\epsilon\big|^{-2}\sum_{k,l\in I_q}\alpha(m_k,m_l)V_\epsilon(x_k-x_l)\quad (2.3)$$
$$u^\epsilon(x_k-x_l+z)J(x_k,m_k,t)\overline{J}(x_l,m_l,t).$$

The terms in $G_z(1)$ arise from the changes in the functional X_z when a collision occurs due to the influence of the appearance and disppearance of particles on other particles that are not directly involved. Those in $G_z(2)$ are due to the absence after collision of the summand in X_z indexed by the colliding particles.

Note that

$$H_{12} + G_0\big(2\big) = 0. \tag{2.4}$$

The process $\big\{\big(X_z - X_0\big)(t) : t \geq 0\big\}$ satisfies

$$\begin{aligned} \big(X_z - X_0\big)\big(T\big) &= \big(X_z - X_0\big)\big(0\big) + \int_0^T \Big(\frac{\partial}{\partial t} + \mathbb{A}_0\Big)\big(X_z - X_0\big)(t)dt \\ &\quad + \int_0^T \mathbb{A}_C(X_z - X_0)(t)dt \, + \, M(T), \end{aligned} \tag{2.5}$$

with $\big\{M(t) : t \geq 0\big\}$ being a martingale. By using the labels for the various terms that we just introduced, we find from (2.5) by use of (2.4) that

$$\begin{aligned} &\Big| \int_0^T H_{11}\big(t\big)dt + \int_0^T H_{13}\big(t\big)dt \Big| \\ \leq\ & \big|X_z - X_0\big|\big(q(T)\big) + \big|X_z - X_0\big|\big(q(0)\big) \\ &+ \int_0^T \big|H_{14}\big|(t)dt + \int_0^T \big|H_2\big|(t)dt + \int_0^T \big|H_3\big|(t)dt + \int_0^T \big|H_4\big|(t)dt \\ &+ \int_0^T \big|G_z(1) - G_0(1)\big|(t)dt + \int_0^T \big|G_z(2)\big|(t)dt \, + \, \big|M(T)\big|. \end{aligned} \tag{2.6}$$

Since J is of compact support, we have that $X_z(q(T)) = 0$ for T sufficiently large. We aim to prove the following estimates: for each $T > 0$,

$$\begin{aligned} &\int_0^T \mathbb{E}_N|H_{14}|(t)dt \leq C|z|^{1/2}\big|\log|z|\big|, \\ &\int_0^T \mathbb{E}_N|H_2|(t)dt \leq C|z|^{1/9}\big|\log|z|\big|, \\ &\int_0^T \mathbb{E}_N|H_3|(t)dt \leq C|z|^{1/9}\big|\log|z|\big|, \\ &\int_0^T \mathbb{E}_N|H_4|(t)dt \leq C|z|^{1/2}\big|\log|z|\big|, \\ &\int_0^T \mathbb{E}_N|G_z(1) - G_0(1)|(t)dt \leq C|z|^{1/2}\big|\log|z|\big|\big|, \\ &\int_0^T \mathbb{E}_N|G_z(2)|(t)dt \leq C\big|\log|z|\big|\big|\log\varepsilon\big|^{-1}, \\ &\mathbb{E}_N|X_z - X_0(0)| \leq C|z|. \end{aligned} \tag{2.7}$$

Later, we apply the limit $|z| \to 0$ after sending ε to 0. We will also show that, for each $T \in (0, \infty)$,

$$\mathbb{E}_N \Big[M\big(T\big)^2 \Big] \le C \big| \log \epsilon \big|^{-1}. \tag{2.8}$$

2.1 Lemmas bounding collision propensity

In this subsection, we discuss three lemmas that in essence serve as the backbone of the proof of the various inequalities that appear in (2.7). These lemmas allow us to reduce the proof to a calculation involving the initial configuarions for which the independence of particles and our assumptions on the initial densities can be used. In fact the proof of Lemmas 2.1 and 2.3 is very similar to the corresponding Lemmas 3.1 and 3.3 of [2]. For this reason, their proofs are omitted. It is Lemma 2.2 that is somewhat different from what we have in [2] as Lemma 3.2 and we provide a detailed proof for it. In fact this difference explains to some extent a major technical difficulty that is two dimensional and is not encountered when the dimension is 3 or more. To explain this further, let us observe that if the dimension d is 3 or more and J is a nonnegative function, then we can find a solution to the Poisson equation $-\Delta H = J$ that satisfies $H \ge 0$. Indeed the solution H is defined by

$$c_0(d) \int |x - y|^{2-d} J(y) dy,$$

where $c_0(d) = (d(d-2)\omega(d))^{-1}$ with $\omega(d)$ denoting the volume of the unit ball in $\mathbb{R}^d$. This is no longer true in dimension 2 because the solution is given by

$$-\frac{1}{2\pi} \int \log |x - y| \; J(y) dy.$$

This causes some difficulty in treating various terms that appear in (2.7). To get around this, let us define

$$H(x) = -\frac{1}{2\pi} \int_{|x-y| \le 1} \log |x - y| \; J(y) dy. \tag{2.9}$$

We now have that $-\Delta H = J - \tilde{J}$ where

$$\tilde{J}(x) = \frac{1}{2\pi} \int_{|z|=1} J(x - z) dS(z), \tag{2.10}$$

where dS denotes the 1-Lebesgue measure on the unit circle S^1. The point is that by using Lemma 2.2, we reduce bounding an expression involving J to an expression involving H at time $t = 0$, and a similar expression involving $\tilde{J}$. Since the funcion $\tilde{J}$ is an average of J, we have an easier task to bound the expression involving $\tilde{J}$. In the case of the terms H_2 and H_3, we need to apply this process three times so that the final $\tilde{J}$ has a simple pointwise bound. Our three lemmas are:

Lemma 2.1 *For any $T \in [0, \infty)$,*

$$\big|\log\epsilon\big|^{-1}\mathbb{E}_N\int_0^T dt\sum_{i,j\in I_q}\alpha(m_i,m_j)V_\epsilon\big(x_i-x_j\big)\le 2Z.$$

Lemma 2.2 *Let $J:\mathbb{R}^2\to[0,\infty)$ be continuous, and let $H:\mathbb{R}^2\to[0,\infty)$ be given by (2.9) We also define $\tilde{J}:\mathbb{R}^2\to\mathbb{R}$ according to (2.10). Then we have the following inequality,*

$$\begin{aligned}
&\big|\log\epsilon\big|^{-2}\mathbb{E}_N\int_0^T\sum_{i,j\in I_q}J(x_i-x_j)m_im_j(d(m_i)+d(m_j))dt\\
&+\big|\log\epsilon\big|^{-2}\mathbb{E}_N\int_0^T\sum_{i,j\in I_q}V_\varepsilon(x_i-x_j)\alpha(m_i,m_j)H(x_i-x_j)m_im_jdt\\
\le\;&\big|\log\epsilon\big|^{-2}\mathbb{E}_N\sum_{i,j\in I_{q(0)}}H(x_i-x_j)m_im_j\\
&+\big|\log\epsilon\big|^{-2}\mathbb{E}_N\int_0^T\sum_{i,j\in I_q}\tilde{J}(x_i-x_j)m_im_j(d(m_i)+d(m_j))dt.
\end{aligned}$$

Lemma 2.3 *Assume that the function $\gamma:\mathbb{N}^2\to(0,\infty)$ satisfies*

$$n_2\gamma\big(n_1,n_2+n_3\big)\max\Big\{1,\Big[\frac{d(n_2+n_3)}{d(n_2)}\Big]^2\Big\}\le\big(n_2+n_3\big)\gamma(n_1,n_2),\tag{2.11}$$

There exists a collection of constants $C:\mathbb{N}^2\to(0,\infty)$, such that, for any smooth function $J:\mathbb{R}^4\to[0,\infty)$, and any given $n_1,n_3\in\mathbb{N}$,

$$\begin{aligned}
&\mathbb{E}_N\int_0^T dt\sum_{k,l,i\in I_{q(t)}}\gamma(m_i,m_j)V_\epsilon\big(x_i-x_j\big)J(x_i,x_k)1\!\!1\{m_i=n_1,m_k=n_3\}\\
\le\;&C_{n_1,n_3}\big|\log\epsilon\big|^3\sum_{n_2\in\mathbb{N}}\int A^\epsilon_{n_1,n_2,n_3}\big(x_1,x_2,x_3\big)h_{n_1}(x_1)h_{n_2}(x_2)h_{n_3}(x_3)dx_1dx_2dx_3,
\end{aligned}\tag{2.12}$$

where, also given $\epsilon>0$ and $n_2\in\mathbb{N}$, the function $A^\epsilon_{n_1,n_2,n_3}:\mathbb{R}^6\to[0,\infty)$ is defined by

$$\begin{aligned}
&\big(d(n_1)\Delta_{x_1}+d(n_2)\Delta_{x_2}+d(n_3)\Delta_{x_3}\big)A^\epsilon_{n_1,n_2,n_3}(x_1,x_2,x_3)\\
&=-\gamma(n_1,n_2)V_\epsilon(x_1-x_2)J(x_1,x_3).
\end{aligned}\tag{2.13}$$

It is worth mentioning that the function $A^{\epsilon}_{n_1,n_2,n_3}(x_1,x_2,x_3)$ of Lemma 2.3 is given by

$$c_0(6)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(\frac{|x_1-z|^2}{d(n_1)}+\frac{|x_2-y|^2}{d(n_2)}+\frac{|x_3-y'|^2}{d(n_3)}\right)^{-2}\gamma(n_1,n_2)J(z,y')V_\epsilon(z-y)dzdydy',$$

where $c_0(d)=(d(d-2)\omega(d))^{-1}$ with $\omega(d)$ denoting the volume of the unit ball in $\mathbb{R}^d$. Note that for Lemma 2.3 we are dealing with a solution to a Laplace type equation in $\mathbb{R}^6$ as opposed to Lemma 2.2 for which the pecularity of the Laplace equation in $\mathbb{R}^2$ played a role. This is why the proof of [2] in the case of Lemma 2.3 can be repeated line by line.

Proof of Lemma 2.2 Set

$$X_q=\big|\log\epsilon\big|^{-2}\sum_{i,j\in I_q}H(x_i-x_j)m_im_j.$$

Recall the mechanism of the dynamics at collision: the location of the newly created particle is one of the two locations of the colliding particles, with weights proportional to the masses of the incident particles. We see that when $\mathbb{A}_C$ acts on X_q, all those terms indexed by pairs of particles one of which is not involved in the collision cancel. Thus,

$$\mathbb{A}_C X=-\big|\log\epsilon\big|^{-2}\sum_{i,j\in I_q}V_\epsilon(x_i-x_j)\alpha(m_i,m_j)m_im_jH(x_i-x_j). \tag{2.14}$$

By $\Delta H=-J+\tilde{J}$,

$$\begin{aligned}\mathbb{A}_0X &= \big|\log\epsilon\big|^{-2}\sum_{i,j\in I_q}\Delta H(x_i-x_j)m_im_j(d(m_i)+d(m_j)) & (2.15)\\ &= -\big|\log\epsilon\big|^{-2}\sum_{i,j\in I_q}J(x_i-x_j)m_im_j(d(m_i)+d(m_j))\\ &\quad +\big|\log\epsilon\big|^{-2}\sum_{i,j\in I_q}\tilde{J}(x_i-x_j)m_im_j(d(m_i)+d(m_j))\\ &= :\mathbb{A}_0^1X+\mathbb{A}_0^2X & (2.16)\end{aligned}$$

From the non-positivity of $\mathbb{A}_0^1X$, the non-positivity of $\mathbb{A}_CX$, apparent from (2.14), and the non-negativity of X, follows

$$-\mathbb{E}_N\int_0^T\mathbb{A}_0^1X(t)dt-\mathbb{E}_N\int_0^T\mathbb{A}_CX(t)dt\le\mathbb{E}_NX(0)+\mathbb{E}_N\int_0^T\mathbb{A}_0^2X(t)dt. \tag{2.17}$$

□

2.2 Bounds on functionals of $u_{n,m}$

We will verify the assertions presented in (2.7). The following lemma provides the bounds on the behaviour of the functions $\{u^{\epsilon}_{n,m} : \mathbb{R}^2 \to [0,\infty) : (n,m) \in \mathbb{N}\}$ and other functions that will be used in this section. We choose the constant R_0 so that $V(x) = 0$ whenever $|x| \geq R_0$. Recall that $k = \sum_n n h_n$.

Lemma 2.4 *There exists a collection of constants $C : \mathbb{N}^2 \to (0,\infty)$ for which the following bounds hold.*

- *for $x \in \mathbb{R}^2$ satisfying $|x| \leq 2R_0\varepsilon$, $\big|u^{\epsilon}_{n,m}(x)\big| \leq C_{n,m}|\log\varepsilon|$, and for all $x \in \mathbb{R}^2$, $\big|u^{\epsilon}_{n,m}(x)\big| \leq C_{n,m}\big|\log|x|\big|$.*
- *for $x \in \mathbb{R}^2$, $\big|\nabla u^{\epsilon}_{n,m}(x)\big| \leq C_{n,m}\min\left\{\frac{1}{|x|}, \frac{1}{\varepsilon}\right\}$.*
- *for $x \in \mathbb{R}^2$,*
$$\Big|u^{\epsilon}_{n,m}\big(x+z\big) - u^{\epsilon}_{n,m}\big(x\big)\Big| \leq C_{n,m}|z|\min\left\{\frac{1}{|x|}, \frac{1}{\varepsilon}\right\} \tag{2.18}$$
- *for $x \in \mathbb{R}^2$ satisfying $|x| \geq \max\big\{2|z| + R_0\epsilon, 2R_0\epsilon\big\}$,*
$$\Big|\nabla u^{\epsilon}_{n,m}\big(x+z\big) - \nabla u^{\epsilon}_{n,m}\big(x\big)\Big| \leq \frac{C_{n,m}|z|}{|x|^2}. \tag{2.19}$$
- *let $H = H_{n,m} : \mathbb{R}^2 \times \mathbb{R}^2 \to [0,\infty)$ be given by*
$$H(x;z) = \frac{-1}{2\pi}\int_{|x-y|\leq 1} \log|x-y| u^{\epsilon}_{n,m}\big(y+z\big)\mathbb{1}\big\{|y| \leq \rho\big\}dy.$$
Then,
$$\sup_{|z|\leq 1}\int H(x_1 - x_2; z)k(x_1)k(x_2)dx_1dx_2 \leq C_{n,m}\rho^2|\log\rho|.$$
- *Let $\hat{H}\big(= \hat{H}_{n,m}\big) : \mathbb{R}^2 \times \mathbb{R}^2 \to [0,\infty)$ be given by*
$$\hat{H}(x;z) = \frac{-1}{2\pi}\int_{|x-y|\leq 1} \log|x-y| \,\Big|\nabla u^{\epsilon}_{n,m}\big(y+z\big)\Big|\mathbb{1}\big\{|y| \leq \rho\big\}dy.$$
Then, for every z with $|z| \leq 1$,
$$\int \hat{H}(x_1 - x_2; z)k(x_1)k(x_2)dx_1dx_2 \leq C_{n,m}\big(\rho + |z|\big). \tag{2.20}$$

- *for* $(x,z)\in\mathbb{R}^2\times\mathbb{R}^2$, *let* $L(x;z)$ *be given by*

$$\frac{-1}{2\pi}\int_{|x-y|\le 1}\log|x-y|\left[\big|\log|1-|y+z||\big|+1\right]1\!\!1\big\{1-\rho\le|y+z|\le 1+\rho\big\}dy.$$

Then we have the bound

$$L(x;z)\le C\rho\big(\log\rho\big)^2. \tag{2.21}$$

- *for any positive integers* n *and* m *and a nonnegative smooth function* $\overline{J}$ *of compact support, there exists a constant* $C_{n,m}(\overline{J})$ *such that, for any given* $z\in\mathbb{R}^2$, *the function* $A^{\epsilon}_{n_1,n_2,n_3}:\mathbb{R}^6\to[0,\infty)$ *defined by*

$$\begin{aligned}&\big(d(n_1)\Delta_{x_1}+d(n_2)\Delta_{x_2}+d(n_3)\Delta_{x_3}\big)A^{\epsilon}_{n_1,n_2,n_3}(x_1,x_2,x_3)\\&\quad=-u^{\varepsilon}_{n,m}(x_1-x_3+z)V_\epsilon\big(x_1-x_2\big)1\!\!1\big\{|x_1-x_3|\le\rho\big\}\overline{J}(x_3)\end{aligned}$$

satisfies

$$\begin{aligned}\sum_{n_1,n_2,n_3}\int_{\mathbb{R}^6}&A^{\epsilon}_{n_1,n_2,n_3}\big(x_1,x_2,x_3\big)h_{n_1}(x_1)h_{n_2}(x_2)h_{n_3}(x_3)dx_1dx_2dx_3\\&\le C_{n,m}(\overline{J})\big|\log\epsilon\big|^{-1}(\rho+|z|)\log\big(\rho+|z|\big).\end{aligned} \tag{2.22}$$

Proof Throughout the proof, we write u^ε for the function $u^\varepsilon_{n,m}$ and τ for the constant $\alpha(n,m)/(d(n)+d(m))$. The dependence of the constants on n and m arises from that of τ, and is also omitted. The first part of the Lemma is a straightforward consequence of our results in Section 3. As a consequence of Theorems 3.1-3.2 and Lemma 3.1 we know that there exists a constant $c_1\in(0,1)$ such that for small ε and y satisfying $|y|\le 2R_0$,

$$\log\varepsilon\le u^\varepsilon(\varepsilon y)\le c_1\log\varepsilon,$$

or equivalently,

$$0\le u^\varepsilon(\varepsilon y)|\log\varepsilon|^{-1}+1\le 1-c_1. \tag{2.23}$$

From (2.1), we learn that for x satisfying $|x|\ge 2\varepsilon R_0$,

$$\Lambda_\varepsilon\log(|x|/2)\le u^\varepsilon(x)\le\log(2|x|)\Lambda_\varepsilon,$$

where

$$\Lambda_\varepsilon=\frac{1}{2\pi}\tau\int\Big(V_\epsilon(y)u^\epsilon_{n,m}(y)+V^\epsilon(y)\Big)dy.$$

From this and (2.22) we learn that there are two positive constants k_1 and k_2 such that if $|x| \geq 2\varepsilon R_0$, then

$$k_1 \log(|x|/2)) \leq u^\varepsilon(x) \leq k_2 \log(2|x|).$$

To prove the second part of the lemma, recall firstly that

$$u^\epsilon(x) = \frac{\tau}{2\pi} \int_{\mathbb{R}^2} \log|x-y| \Big(u^\epsilon(y) V_\epsilon(y) + V^\epsilon(y) \Big) dy.$$

As a result,

$$\nabla u^\epsilon(x) = \frac{\tau}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{\big|x-y\big|^2} \Big(u^\epsilon(y) V_\epsilon(y) + V^\epsilon(y) \Big) dy. \tag{2.24}$$

If $|x| \geq 2R_0\varepsilon$, then $\big|x-y\big| \geq |x|/2$, and

$$\left| \frac{x-y}{|x-y|^2} \right| = \frac{1}{|x-y|} \leq \frac{2}{|x|},$$

implying that

$$\big|\nabla u^\epsilon(x)\big| \leq \frac{2\Lambda_\varepsilon}{|x|} \leq \frac{3}{|x|},$$

for small ε. If $|x| \leq 2R_0\varepsilon$, then we use (2.22) to deduce

$$\big|\nabla u^\epsilon(x)\big| \leq c_1 \int_{|x-y| \leq 3R_0\varepsilon} \frac{1}{|x-y|} V^\varepsilon(y) dy \leq c_2 \epsilon^{-2} \int_0^{3R_0\epsilon} dr \leq c_3 \varepsilon^{-1}.$$

Thus,

$$\big|\nabla u^\epsilon(x)\big| \leq C \min\left\{ \frac{1}{\varepsilon}, \frac{1}{|x|} \right\},$$

as claimed in the second part of the lemma.

To prove the third part of the lemma, note that

$$\begin{aligned} &\Big| u^\epsilon(x+z) - u^\epsilon(x) \Big| \\ \leq\ & \frac{1}{2\pi} \int_{\mathbb{R}^2} \left| \log \frac{|x-y+z|}{|x-y|} \right| \Big(u^\epsilon(y) V_\epsilon(y) + V^\epsilon(y) \Big) dy \\ \leq\ & \frac{1}{2\pi} \int_{\mathbb{R}^2} \left| \log \frac{|x-y+z|}{|x-y|} \right| V^\epsilon(y) dy, \end{aligned} \tag{2.25}$$

the latter inequality by means of (2.22). From this and the elementary inequalities

$$\log \frac{|x-y+z|}{|x-y|} - 1 \leq \frac{|x-y+z|}{|x-y|} \leq \frac{|z|}{|x-y|}, \tag{2.26}$$

we deduce that

$$\Big|u^{\epsilon}\big(x+z\big)-u^{\epsilon}\big(x\big)\Big| \le \frac{1}{2\pi}|z|\int_{\mathbb{R}^2}\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{V^{\epsilon}(y)}{|x-y|}dy$$

We now use this and argue as in the proof of the second part of the lemma to deduce the third part of the lemma.

In seeking to prove the fourth part of the lemma, note that

$$\frac{x+z-y}{\big|x+z-y\big|^2}-\frac{x-y}{\big|x-y\big|^2}=\frac{\big|x-y\big|^2\big(x+z-y\big)-\big|x+z-y\big|^2\big(x-y\big)}{\big|x+z-y\big|^2\big|x-y\big|^2}.$$

Note that, for any $a\in\mathbb{R}^2$,

$$\begin{aligned}\Big||a|^2\big(a+z\big)-|a+z|^2a\Big| &\le |a|\Big||a+z|^2-|a|^2\Big|+|z||a|^2\\ &\le c_1|z||a|^2,\end{aligned}\tag{2.27}$$

so long as $|z|\le|a|$. Note that since by our assumption $|x|\ge 2|z|+R_0\varepsilon$, we have that $|x-y|\ge 2|z|$. We may apply (2.27) with the choice $a=x-y$ to the formula (2.24), hereby obtaining

$$\Big|\nabla u^{\epsilon}(x+z)-\nabla u^{\epsilon}(x)\Big|\le C|z|\int_{\mathbb{R}^2}\frac{V^{\varepsilon}(y)}{\big|x+z-y\big|^2}dy,$$

where we used (2.22). From the inequality $|x|\ge\max\big\{2|z|+R_0\epsilon,2R_0\epsilon\big\}$, we deduce that $\big|x+z-y\big|\ge|x-y|/2$ and $|x-y|\ge|x|/2$. We conclude that

$$\Big|\nabla u^{\epsilon}(x+z)-\nabla u^{\epsilon}(x)\Big|\le\frac{C|z|}{|x|^2},$$

as required.

To prove the fifth part of the lemma, note that

$$|H(x;z)|\le C\int_{|x-y|\le 1}|\log|x-y||\log|y+z||\mathbb{1}\big\{|y|\le\rho\big\}dy\,,$$

by the first part of the lemma. Hence,

$$\begin{aligned}&\int|H(x_1-x_2;z)|k(x_1)k(x_2)dx_1dx_2\\ \le\;& C\int_{|y|\le\rho}\big|\log|y+z|\big|\int_{|x_1-x_2-y|\le 1}\big|\log|x_1-x_2-y|\big|k(x_1)k(x_2)dx_1dx_2dy\\ \le\;& C\int_{|y|\le\rho}\big|\log|y|\big|dy\\ \le\;& C\int_0^{\rho}r|\log r|dr\le C\rho^2|\log\rho|,\end{aligned}$$

where in the second inequality, we used our first assumption on the initial data and the fact that if $\rho + |z| \le 1$, then the expression $\int_{|y|\le\rho} \big|\log|y+z|\big|dy|$ is maximized as a function of $z \in \mathbb{R}^2$ when $z = 0$. This establishes the third part of the lemma.

To prove the sixth part of the lemma, note that, by the first part,

$$\hat{H}(x;z) \le \frac{1}{2\pi} \int_{|x-y|\le 1} \big|\log|x-y|\big| \frac{1}{|y+z|} \mathbb{1}\big\{|y| \le \rho\big\} dy.$$

It follows that

$$\begin{aligned}
&\int |\hat{H}(x_1-x_2;z)| k(x_1)k(x_2)dx_1dx_2 \\
\le\ & c_1 \int_{|y|\le\rho} \frac{1}{|y+z|} \int_{|x_1-x_2-y|\le 1} \big|\log|x_1-x_2-y|\big| k(x_1)k(x_2)dx_1dx_2dy \\
\le\ & c_2 \int_{|y|\le\rho} \frac{1}{|y+z|} dy \le c_2 \int_{|y+z|\le\rho+|z|} \frac{1}{|y+z|} dy \\
\le\ & c_3\big(\rho + |z|\big).
\end{aligned} \tag{2.28}$$

We have deduced (2.20).

As for the seventh part of the lemma, first observe that $L(x;z) = L(x+z;0)$. Hence we only need to verify (2.21) when $z = 0$. In this case we divide the domain of integration into the sets $|x-y| \le |1-|y||$ and $|1-|y|| \le |x-y| \le 1$. Hence,

$$\begin{aligned}
L(x;0) \le\ & \frac{1}{2\pi} \int_{|x-y|\le\rho} \Big[\big|\log|x-y|\big| + \big|\log|x-y|\big|^2\Big] dy \\
& + \frac{1}{2\pi} \int_{\left|1-|y|\right|\le|x-y|\le 1} \Big[\big|\log|1-|y||\big| + \big|\log|1-|y||\big|^2\Big] \mathbb{1}\big(|y| \in (1-\rho, 1+\rho)\big) dy \\
\le\ & \int_0^\rho r\left[|\log r| + |\log r|^2\right] dr + \int_{1-\rho}^{1+\rho} \Big[\big|\log|1-r|\big| + \big|\log|1-r|\big|^2\Big] rdr \\
\le\ & C\rho(\log\rho)^2,
\end{aligned}$$

establishing (2.21).

As for the eighth part of the lemma, let us write $J(a)$ for $\mathbb{1}\big\{|a| \le \rho\big\}$ and define the quantity I according to

$$\begin{aligned}
I \ =\ & c_0(6) \sum_{n_1,n_2,n_3} \gamma(n_1,n_2) \int dx_1dx_2dx_3 \int_{\mathbb{R}^6} \Big(\frac{|x_1-z'|^2}{d(n_1)} + \frac{|x_2-y|^2}{d(n_2)} + \frac{|x_3-y'|^2}{d(n_3)}\Big)^{-2} \\
& h_{n_1}(x_1)h_{n_2}(x_2)h_{n_3}(x_3)u^\varepsilon(z'-y'+z)V_\epsilon(z'-y)J(z'-y')\overline{J}(y')dz'dydy'.
\end{aligned}$$

We write

$$I = c_0(6)\int_{\mathbb{R}^6} u^\varepsilon(z'-y'+z)V_\epsilon(z'-y)J(z'-y')\overline{J}(y')G(z',y,y')dz'dydy',$$

where $G(z',y,y')$ is given by

$$\sum_{n_1,n_2,n_3\gamma(n_1,n_2)\in\mathbb{N}} \int \Big(\frac{|x_1-z'|^2}{d(n_1)}+\frac{|x_2-y|^2}{d(n_2)}+\frac{|x_3-y'|^2}{d(n_3)}\Big)^{-2} h_{n_1}(x_1)h_{n_2}(x_2)h_{n_3}(x_3)dx_1dx_2dx_3.$$

Using the elementary inequality $abc \le (a^2+b^2+c^2)^{3/2}$ we deduce that $G(z',y,y')$ is at most

$$\sum_{n_1,n_2,n_3\in\mathbb{N}} \gamma(n_1,n_2)(d(n_1)d(n_2)d(n_3))^{2/3}\int |x_1-z'|^{-4/3}\ |x_2-y|^{-4/3}|x_3-y'|^{-4/3}$$
$$h_{n_1}(x_1)h_{n_2}(x_2)h_{n_3}(x_3)dx_1dx_2dx_3.$$

From our assumptions on h_n we deduce that $G\in L^\infty_{loc}$. Hence,

$$I \le C\int_{\mathbb{R}^6} u^\varepsilon(z'-y'+z)V_\epsilon(z'-y)J(z'-y')\overline{J}(y')dz'dydy'.$$

Note that, for fixed $z'\in\mathbb{R}^3$,

$$\int_{\mathbb{R}^2} V\Big(\frac{z'-y}{\epsilon}\Big)dy = \epsilon^2.$$

Thus,

$$\begin{aligned}
I &\le C\big|\log\epsilon\big|^{-1}\int_{\mathbb{R}^4} u^\varepsilon(z'-y'+z)J(z'-y')\overline{J}(y')dz'dy' \\
&\le C\big|\log\epsilon\big|^{-1}\int_K\int_{\mathbb{R}^2} u^\varepsilon(z'-y'+z)J(z'-y')dz'dy' \\
&\le C\big|\log\epsilon\big|^{-1}\int_K dy'\int_{\mathbb{R}^2}\big|\log|z'-y'+z|\big|J(z'-y')dz' \qquad (2.29)\\
&\le C\big|\log\epsilon\big|^{-1}\int_{|a|\le\rho+|z|}\big|\log|a|\big|da \le C\big|\log\epsilon\big|^{-1}(\rho+|z|)^2\big|\log\big(\rho+|z|\big)\big|,
\end{aligned}$$

where $K\subseteq\mathbb{R}^2$ denotes a compact set containing the support $\overline{J}$, and where we made use of the first part of the lemma in the third inequality. This is the bound stated in (2.22). □

2.3 Estimating the terms

2.3.1 The case of H_{14} and H_4

The estimate of $\mathbb{E}_N \int_0^T |H_4(t)|dt$ is derived in an identical fashion to that of $\mathbb{E}_N \int_0^T |H_{14}(t)|dt$. Note that

$$\mathbb{E}_N \Big| \int_0^T H_{14}(t)dt \Big| \leq C \big| \log \epsilon \big|^{-2} \mathbb{E}_N \int_0^T dt \sum_{i,j \in I_q} \Big| u^\epsilon\big(x_i - x_j + z\big) - u^\epsilon\big(x_i - x_j\big) \Big| \mathbb{1}\Big\{ m_i = M_1, m_j = M_2 \Big\},$$

where the constant C depends on the L^∞ bounds satisifed by $J, \overline{J}$ and their time derivatives. Hence

$$\mathbb{E}_N \Big| \int_0^T H_{14}(t)dt \Big| \leq K_1 + K_2,$$

where K_1 is given by

$$C \big| \log \epsilon \big|^{-2} \mathbb{E}_N \int_0^T dt \sum_{i,j \in I_q : |x_i - x_j| > \rho} \Big| u^\epsilon(x_i - x_j + z) - u^\epsilon(x_i - x_j) \Big| \mathbb{1}\{m_i = M_1\} \mathbb{1}\{m_j = M_2\}$$

and K_2 is given by

$$\big| \log \epsilon \big|^{-2} \mathbb{E}_N \int_0^T dt \sum_{i,j \in I_q : |x_i - x_j| \leq \rho} \Big| u^\epsilon(x_i - x_j + z) - u^\epsilon(x_i - x_j) \Big| \mathbb{1}\{m_i = M_1\} \mathbb{1}\{m_j = M_2\}.$$

Firstly, we treat K_1. Note that

$$K_1 \leq \frac{C|z| \big| \log \epsilon \big|^{-2}}{\rho} \mathbb{E}_N \int_0^T dt \sum_{i,j \in I_q : |x_i - x_j| > \rho} \mathbb{1}\{m_i = M_1\} \mathbb{1}\{m_j = M_2\} \leq \frac{C|z|Z^2}{\rho},$$

where the first inequality follows from the third part of Lemma 2.4, and the final one from the initial number of particles N equals $Z|\log \varepsilon|$.

We now treat the term K_2. By writing,

$$K_2 \leq C \big| \log \epsilon \big|^{-2} \mathbb{E}_N \int_0^T dt \sum_{i,j \in I_q : |x_i - x_j| \leq \rho} \Big[\big| u^\epsilon\big(x_i - x_j + z\big) \big| + \big| u^\epsilon\big(x_i - x_j\big) \big| \Big] \mathbb{1}\{m_i = M_1\} \mathbb{1}\{m_j = M_2\},$$

we obtain an expression on the right-hand-side which may be bounded by applying Lemma 2.2. As a result we can write $K_2 \le K_{21} + K_{22}$ where K_{21} and K_{22} represent the first and the second term on the right-hand-side in Lemma 2.2. For K_{21}, the relevant estimate is provided by the fifth part of Lemma 2.4, with a bound of $C\rho^2|\log\rho|$. To bound the term K_{22}, note that, with the function J in Lemma 2.2 chosen to be $J(x) = u^\epsilon(x+z)1\!\!1(|x| \le \rho)$, we have that

$$
\begin{aligned}
|\tilde{J}(x)| &= \left| \int_{|y|=1} u^\epsilon\big(x+y+z\big) 1\!\!1\big\{|x+y| \le \rho\big\} S(dy) \right| \\
&\le C \int_{|y|=1} \big|\log|x+y+z|\big| 1\!\!1\big\{|x+y| \le \rho\big\} S(dy)
\end{aligned}
$$

Let us assume that $|z| \le \rho$ ans set $a = x + z$. We then have

$$
\begin{aligned}
|\tilde{J}(x)| &\le C \int_{|y|=1} |\log|a+y|| 1\!\!1\big\{|a+y| \le 2\rho\big\} S(dy) \\
&\le 2C \int_0^{c_2\rho} |\log(c_1 r)| dr \le c_3 \rho |\log\rho|,
\end{aligned}
$$

where for the second inequality we used that fact that the conditions $|a+y| \le \rho$, $|y| = 1$ mean that the point y belongs to an arc on the unit circle of center $\sigma = a/|a|$ and length $2c_2\rho$, and if the lenght of the arc σy is r, then $|a+y| \ge c_1 r$ for positive constants c_1 and c_2.

In this way, we find that $\tilde{J}$ is uniformly bounded by $C\rho|\log\rho|$ and this in turn implies that the term K_{22} is bounded above by

$$
C\big|\log\epsilon\big|^{-2}\rho|\log\rho|\mathbb{E}_N \int_0^T dt \Big| \Big\{(i,j) \in I_q^2 : m_i = M_1, m_j = M_2\Big\} \Big| \le CZ^2\rho|\log\rho|,
$$

the latter inequality following from $N \le Z\big|\log\epsilon\big|$. We find that

$$
K_2 \le C\rho|\log\rho|\,.
$$

Hence,

$$
\mathbb{E}_N \left| \int_0^T H_{13}(t) \right| dt \le K_1 + K_2 \le \frac{C|z|}{\rho} + C\rho|\log\rho|.
$$

Setting $\rho = |z|^{1/2}$, we find that

$$
\mathbb{E}_N \left| \int_0^T H_{13}(t) \right| dt \le C|z|^{\frac{1}{2}} \big|\log|z|\big|.
$$

2.3.2 The cases of H_2 and H_3

The estimate of $\mathbb{E}_N \int_0^T |H_3(t)|dt$ is derived in an identical fashion to that of $\mathbb{E}_N \int_0^T |H_2(t)|dt$. Picking $\rho \in \mathbb{R}$ that satisfies $\rho \geq \max\{2|z| + R_0\epsilon, 2R_0\epsilon\}$, we write

$$\int_0^T \mathbb{E}_N |H_2(t)|dt \leq R_1 + R_2, \tag{2.30}$$

where

$$\begin{aligned} R_1 &= \big|\log\epsilon\big|^{-2}\mathbb{E}_N \int_0^T \sum_{i,j\in I_q:|x_i-x_j|>\rho} d(m_i)\,\big|\overline{J}(x_j,m_j,t)\big| \\ &\qquad \Big|u_x^\epsilon(x_i-x_j+z) - u_x^\epsilon(x_i-x_j)\Big|\Big|J_x\big(x_i,m_i,t\big)\Big|dt, \end{aligned}$$

and

$$\begin{aligned} R_2 &= \big|\log\epsilon\big|^{-2}\mathbb{E}_N \int_0^T \sum_{i,j\in I_q:|x_i-x_j|\leq\rho} d(m_i)\big|\overline{J}(x_j,m_j,t)\big| \\ &\qquad \Big|u_x^\epsilon(x_i-x_j+z) - u_x^\epsilon(x_i-x_j)\Big|\Big|J_x\big(x_i,m_i,t\big)\Big|dt. \end{aligned}$$

Firstly, we examine the sum R_1. Recalling that we consider test functions J and $\overline{J}$ respectively supported on particles of mass M_1 and M_2,

$$R_1 \leq C\big|\log\epsilon\big|^{-2}\frac{|z|}{\rho^2}d(M_1)\Big|\{(i,j)\in I_q^2 : m_i = M_1, m_j = M_2\}\Big|,$$

where the lower bound on ρ allowed us to apply the fourth part of Lemma 2.4. Thus,

$$R_1 \leq C\frac{|z|}{\rho^2}. \tag{2.31}$$

Secondly, we bound the sum R_2. Note that

$$\begin{aligned} R_2 &\leq \big|\log\epsilon\big|^{-2}||J_x||\;||\overline{J}||\;d(M_1)\;\mathbb{E}_N \int_0^T dt \sum_{i,j\in I_q:|x_i-x_j|\leq\rho} \\ &\qquad \Big|u_x^\epsilon(x_i-x_j+z) - u_x^\epsilon(x_i-x_j)\Big|\mathbb{1}\{m_i = M_1, m_j = M_2\} \\ &\leq C\big|\log\epsilon\big|^{-2}\;\mathbb{E}_N \int_0^T dt \sum_{i,j\in I_q} m_i m_j\big(d(m_i)+d(m_j)\big) \\ &\qquad \Big[\Big|u_x^\epsilon(x_i-x_j+z)\Big| + \Big|u_x^\epsilon(x_i-x_j)\Big|\Big]\mathbb{1}\Big\{\big|x_i-x_j\big|\leq\rho\Big\}, \end{aligned} \tag{2.32}$$

where $\|\cdot\|$ denotes the L^∞ norm and the constant C depends on the test functions J and $\overline{J}$. The expression (2.32) is written in a form to which Lemma 2.2 may be applied. Doing so yields

$$\begin{aligned} R_2 &\leq C\big|\log\epsilon\big|^{-2}\mathbb{E}_N \sum_{i,j\in I_q} m_i m_j\Big(\hat{H}(x_i-x_j;0)+\hat{H}(x_i-x_j;z)\Big) \\ &\quad + C\big|\log\epsilon\big|^{-2}\ \mathbb{E}_N\int_0^T dt \sum_{i,j\in I_q} m_i m_j\big(d(m_i)+d(m_j)\big) \\ &\qquad \Big[\tilde{J}\big(x_i-x_j;z\big)+\tilde{J}\big(x_i-x_j;0\big)\Big] \\ &=: R_{21}+R_{22} \end{aligned} \tag{2.33}$$

where the function $\hat{H}$ appears in the sixth part of Lemma 2.4, and where $\tilde{J}:\mathbb{R}^2\times\mathbb{R}^2\to[0,\infty)$ in this case is given by

$$\tilde{J}(x;z)=\frac{1}{2\pi}\int_{|y|=1}\big|u_x^\epsilon(x+y+z)\big|1\!\!1\big\{|x+y|\leq\rho\big\}dS(y).$$

From the sixth part of Lemma 2.4 and our assumptions on the initial data, we deduce that

$$R_{21}\leq C(\rho+|z|). \tag{2.34}$$

It follows from the second part of Lemma 2.4 that the function $\tilde{J}$ satisfies the bound

$$|\tilde{J}(x;z)|\leq C\int_{|y|=1}\frac{1}{\big|x+y+z\big|}1\!\!1\big\{|x+y|\leq\rho\big\}dS(y).$$

By our assumption, we certainly have $|z|\leq\rho/2$. Hence,

$$|\tilde{J}(x;z)|\leq C\int_{|y|=1}\frac{1}{\big|a+y\big|}1\!\!1\big\{|a+y|\leq 2\rho\big\}dS(y),$$

for $a=x+z$. Note that there exists a positive constant c_1 such that the conditions $|a+y|\leq 2\rho,\ |y|=1$ mean that $y\in\Gamma$, where Γ is an arc of the unit circle with the center $\sigma=-a/|a|$. It is not hard to show that there exist positive constants c_1 and c_2 such that

$$c_1\big(\ell+\big|1-|a|\big|\big)\leq|a+y|\leq c_2\big(\ell+\big|1-|a|\big|\big),$$

where ℓ denotes the length of the arc from $\sigma=-a/|a|$ to y on the unit circle. From this we deduce

$$\begin{aligned} |\tilde{J}(x;z)| &\leq 1\!\!1\big\{1-\rho\leq|a|\leq 1+\rho\big\}\int 1\!\!1\left(c_1c_2^{-1}\big|1-|a|\big|\leq\ell\leq 2c_1^{-1}\rho\right)\frac{c_3}{\ell}d\ell \\ &\leq c_3 1\!\!1\big\{1-\rho\leq|x+z|\leq 1+\rho\big\}\left(\big|\log|1-|x+z||\big|+\log\left(c_2c_1^{-2}\right)\right) \end{aligned}$$

Using this bound on $\tilde{J}$, and then applying Lemma 2.2, we learn that the term R_{22} is bounded above by $R_{221} + R_{222}$, where

$$R_{221} = C\big|\log\epsilon\big|^{-2}\mathbb{E}_N \sum_{i,j\in I_{q(0)}} \Big(L\big(x_i - x_j; 0\big) + L\big(x_i - x_j; z\big)\Big) m_i m_j,$$

and,

$$R_{222} = 2C\big|\log\epsilon\big|^{-2}\mathbb{E}_N \int_0^T \Big(\tilde{L}\big(x_i - x_j\big) + \tilde{L}\big(x_i - x_j + z\big)\Big) m_i m_j \big(d(m_i) + d(m_j)\big) dt,$$

where the function $L : \mathbb{R}^2 \to [0,\infty)$ appears in the seventh part of Lemma 2.4, and where the function $\tilde{L} : \mathbb{R}^2 \to [0,\infty)$ is given by

$$\tilde{L}(x) = \frac{1}{2\pi}\int_{|y|=1} 1\!\!1\big\{1-\rho \le |x+y| \le 1+\rho\big\} \Big[\big|\log|1-|x+y||\big| + 1\Big]\, dS(y)$$

By the seventh part of Lemma 2.4 and our assumption on the initial total density $k = \sum_n n h_n \in L^1$ we obtain

$$R_{221} \le C\rho(\log\rho)^2. \tag{2.35}$$

We decompose $R_{222} = R_{2221} + R_{2222}$ where

$$R_{222r} = 2\big|\log\epsilon\big|^{-2}\mathbb{E}_N \int_0^T \Big(L_r\big(x_i - x_j\big) + L_r\big(x_i - x_j + z\big)\Big) m_i m_j \big(d(m_i) + d(m_j)\big) dt,$$

for $r = 1, 2$, where $L_1(x) = \tilde{L}(x) 1\!\!1(|x| \ge \rho^{1/4})$ and $L_2(x) = \tilde{L}(x) 1\!\!1(|x| \le \rho^{1/4})$. Let us now analyse the behaviour of the function $\tilde{L}$. First observe that since the Lebsegue measure on the circle is rotationally invariant, we have that the function $\tilde{L}$ is radially symmetric. Because of this, let us assume that $x = (a,0)$ and $y = (cos\theta, \sin\theta)$. Note that

$$|x+y|^2 = 1 + a^2 + 2a\cos\theta.$$

As a result, the condition $|x+y| \in (1-2\rho, 1+2\rho)$ for $\rho \le 1$ implies that $|a^2 + 2a\cos\theta| \le 8\rho$. Let us first examine the case $|a| = |x| \ge \rho^{1/4}$. In this case we have that $|a + 2\cos\theta| \le 8\rho^{3/4}$. This condition is not satisfied unless $|a| \le 2$. In that case, choose $\theta_0 \in [0,\pi]$ such that $a + 2\cos\theta_0 = 0$. Note that for smallρ, the set $\{\theta : |a + 2\cos\theta| \le 8\rho^{3/4}\}$ is a union of two disjoint θ–intervals about the points θ_0 and $\theta_1 = 2\pi - \theta_0$. We now argue that there exists a positive constant c_1 such that the length of these intervals is bounded above by $c_1\rho^{1/2}$. To see this, first observe that the condition $|a| \ge \rho^{1/4}$ implies that for a positive constant c_2 we have that $|\theta_0|, |\theta_0-\pi| \geq 2c_2\rho^{1/4}$. This and $|a+2\cos\theta| \leq 8\rho^{3/4}$ implies that we also have $|\theta|, |\theta-\pi| \geq c_2\rho^{1/4}$ provided that ρ is sufficiently small. On the other hand since

$$2\cos\theta + a = -2(\sin\tau)(\theta-\theta_0),$$

for some τ between θ and θ_0, we deduce that for some positive constant c_1, we have that $|\theta-\theta_0| \leq c_1\rho^{1/2}$. Also, there exist positive constants c_3 and c_4 such that if θ is close to θ_r, then

$$\big|1-|x+y|\big| \geq c_3|a^2+2a\cos\theta| \geq c_4\sqrt{\rho}|\theta-\theta_r|,$$

for $r=0$ or 1. From this we learn that if $|x| \geq \rho^{1/4}$, then the term $|\tilde{L}(x)|$ is bounded above by

$$\frac{1}{2\pi}\int_{\theta_0-c_1\sqrt{\rho}}^{\theta_0+c_1\sqrt{\rho}} [|\log(c_4\sqrt{\rho}|\theta-\theta_0|)|+1]\,d\theta + \frac{1}{2\pi}\int_{\theta_1-c_1\sqrt{\rho}}^{\theta_1+c_1\sqrt{\rho}} [|\log(c_4\sqrt{\rho}|\theta-\theta_1|)|+1]\,d\theta.$$

As a result, $|\tilde{L}(x)| \leq C\sqrt{\rho}|\log\rho|$. This in turn implies that

$$R_{2221} \leq C\sqrt{\rho}|\log\rho| \tag{2.36}$$

We now turn to R_{2222}. For this, observe that the support of the function L_2 is contained in the set of points x for which $|x| \leq \rho^{1/4}$. Note that if $|a| \leq \rho^{1/4}$, then we can find a positive constant c_5 such that $|\theta_0-\pi/2| \leq c_5\rho^{1/4}$, $|\theta_1-3\pi/2| \leq c_5\rho^{1/4}$, where $\theta_1 = 2\pi-\theta_0$. Forthermore, we can find a positive constant c_6 such that if $\theta \in (0,\pi)$, then

$$\big|a^2+2a\cos\theta\big| = \big|2a(\cos\theta_0-\cos\theta)\big| = 2\left|a\sin\frac{\theta+\theta_0}{2}\sin\frac{\theta-\theta_0}{2}\right| \geq c_6|a(\theta-\theta_0)|. \tag{2.37}$$

The same is true if $\theta \in [\pi, 2\pi]$ (use θ_1 in place of θ_0 in (2.37).) As a result,

$$\big|1-|x+y|\big| \geq c_3|a^2+2a\cos\theta| \geq c_3c_6|a(\theta-\theta_0)|.$$

From this we learn that indeed

$$L_2(x) \leq C\big|\log|x|\big|\mathbb{1}\big\{|x| \leq \rho^{1/4}\big\}.$$

To bound R_{2222}, let us apply Lemma 2.2 one more time to write $R_{2222} \leq R_{22221} + R_{22222}$, where

$$R_{22221} = C\big|\log\epsilon\big|^{-2}\mathbb{E}_N\sum_{i,j\in I_{q(0)}}\Big(\Gamma\big(x_i-x_j\big)+\Gamma\big(x_i-x_j+z\big)\Big)m_im_j$$

and

$$R_{22222} = \big|\log \epsilon\big|^{-2} \mathbb{E}_N \int_0^T \Big(\tilde{\Gamma}\big(x_i - x_j\big) + \tilde{\Gamma}\big(x_i - x_j + z\big)\Big) m_i m_j \big(d(m_i) + d(m_j)\big) dt,$$

where the function $\Gamma : \mathbb{R}^2 \to [0, \infty)$ is very similar to the function H that appeared in the fifth part of Lemma 2.4 (except that ρ in the definition H is replaced with $\rho^{1/4}$), and where the function $\tilde{\Gamma} : \mathbb{R}^2 \to [0, \infty)$ is given by

$$\tilde{\Gamma}(x) = \int_{|y|=1} 1\!\!1\big\{|x + y| \le \rho^{1/4}\big\} \big|\log |x + y|\big| dS(y).$$

As in the fifth part of Lemma 2.4 we show

$$R_{22221} \le C\rho |\log \rho|. \tag{2.38}$$

In just the same way that we bounded $\tilde{J}$ in the subsection 3.3.1, we can readily show that $\tilde{\Gamma}(x) \le C\sqrt{\rho}|\log \rho|$. This in turn implies

$$R_{22222} \le C\rho^{1/4} |\log \rho|. \tag{2.39}$$

Putting all the pieces together we learn from (2.30)–(2.31) and (2.33)–(2.39) that

$$\int_0^T \mathbb{E}_N |H_2(t)| dt \le C \left[\frac{|z|}{\rho^2} + \rho + |z| + \rho^{1/4}\big|\log \rho\big|,\right]$$

for $\rho \le 1$. By making the choice $\rho = |z|^{\frac{4}{9}}$, we find that

$$\int_0^T \mathbb{E}_N |H_2(t)| dt \le C|z|^{\frac{1}{9}} \big|\log |z|\big|.$$

2.3.3 The case of $G_z(1) - G_0(1)$

We now estimate the term

$$\int_0^T \mathbb{E}_N \big|G_z(1) - G_0(1)\big|(t) dt.$$

To ease the notation, we do not display the dependence of J and $\overline{J}$ on the variable t. Note that

$$\int_0^T \mathbb{E}_N \big|G_z(1) - G_0(1)\big|(t) dt \le \sum_{i=1}^{8} D_i, \tag{2.40}$$

where

$$D_1 = \frac{1}{2}\mathbb{E}_N \int_0^T dt \sum_{k,l \in I_q} \alpha(m_k, m_l) V_\epsilon(x_k - x_l) \big| J(x_k, m_k) \big|$$
$$\epsilon^{2(d-2)} \sum_{i \in I_q} \big| \overline{J}(x_i, m_i) \big| \Big| u^\epsilon(x_k - x_i + z) - u^\epsilon(x_k - x_i) \Big|,$$

each of the other seven terms on the right-hand-side of (2.40) differing from D_1 only in an inessential way. Given this. the estimates involved for each of the eight cases are in essence identical, and we examine only the case of D_1. We write $D_1 = D^1 + D^2$, where we have decomposed the inner i-indexed sum according to the respective index sets

$$\{i \in I_q, i \neq k, l, |x_k - x_i| > \rho\} \text{ and } \{i \in I_q, i \neq k, l, |x_k - x_i| \leq \rho\}.$$

By the second part of Lemma 2.4, we have that

$$D^1 \leq \frac{C|z| \big| \log \epsilon \big|^{-1}}{\rho} \mathbb{E}_N \int_0^T dt \sum_{k,l \in I_q} \alpha(m_k, m_l) V_\epsilon(x_k - x_l),$$

where we have also used the fact that the test functions J and $\overline{J}$ are each supported on the set of particles of respective masses M_1 and M_2, and the fact that the total number of particles living at any given time is bounded above by $Z\big| \log \epsilon \big|$. From the bound on the collision that is provided by Lemma 2.1, it follows that

$$D^1 \leq \frac{C|z|}{\rho}.$$

Note that D^2 is bounded above by

$$C\mathbb{E}_N \int_0^T dt \sum_{k,l \in I_q} \alpha(m_k, m_l) V_\epsilon(x_k - x_l) \mathbb{1}\{m_k = M_1\} \big| \log \epsilon \big|^{-2} \sum_{i \in I_q}$$
$$\mathbb{1}\{|x_i - x_k| \leq \rho\} \mathbb{1}\{m_i = M_2\} \Big| u^\epsilon(x_k - x_i + z) - u^\epsilon(x_k - x_i) \Big| \big| J(x_i, m_i) \big|$$
$$\leq C\mathbb{E}_N \int_0^T dt \sum_{k,l \in I_q} \alpha(m_k, m_l) V_\epsilon(x_k - x_l) \mathbb{1}\{m_k = M_1\} \big| \log \epsilon \big|^{-2} \sum_{i \in I_q}$$
$$\mathbb{1}\{|x_i - x_k| \leq \rho\} \mathbb{1}\{m_i = M_2\} \Big[|u^\epsilon(x_k - x_i + z)| + |u^\epsilon(x_k - x_i)| \Big] \big| J(x_i, m_i) \big|.$$

Note that the last expectation is bounded by Lemma 2.3 because, by our assumption on α, we can find γ such that $\alpha \leq \gamma$, with γ satisfying the assumption of Lemma 2.3. The upper bound provided by this Lemma in this particular application is computed in the last part of Lemma 2.4. We find that $D^2 \leq C(\rho+|z|)\log\big(\rho+|z|\big)$.

Combining these estimates yields

$$D_3 \leq D^1 + D^2 \leq C\frac{|z|}{\rho} + C(\rho+|z|)\Big|\log\big(\rho+|z|\big)\Big|.$$

Making the choice $\rho = |z|^{\frac{1}{2}}$ leads to the inequality $D_3 \leq |z|^{\frac{1}{2}}\big|\log|z|\big|$. Since each of the cases of $\big\{D_i : i \in \{1,\ldots,8\}\big\}$ may be treated by a nearly verbatim proof, we deduce that

$$\int_0^T \mathbb{E}_N\big|G_z(1) - G_0(1)\big|(t)dt \leq C|z|^{\frac{1}{2}}\log|z|.$$

2.3.4 The case of $G_z(2)$

Recall that

$$G_z(2) = -\big|\log\epsilon\big|^{-2}\sum_{k,l\in I_q}\alpha(m_k,m_l)V_\epsilon(x_k-x_l)u^\epsilon(x_k-x_l+z)J(x_k,m_k)\overline{J}(x_l,m_l).$$

If $k,l \in I_q$ satisfy $V_\epsilon\big(x_k-x_l\big) \neq 0$, then $|x_k-x_l| \leq R_0\epsilon$, and so

$$|x_k-x_l+z| \geq |z| - R_0\epsilon \geq |z|/2,$$

provided that $|z| \geq 2R_0\varepsilon$. This implies that

$$\big|u^\epsilon(x_k-x_l+z)\big| \leq C\Big|\log|x_k-x_l+z|\Big| \leq C\Big|\log|z|\Big|,$$

where in the first inequality, we used the first part of Lemma 2.4 (restated). Applying this bound, and using the fact that the test functions J and $\overline{J}$ have compact support, we find that

$$\int_0^T \mathbb{E}_N|G_z(2)|dt \leq C\Big|\log|z|\Big|\big|\log\epsilon\big|^{-2}\mathbb{E}_N\int_0^T\sum_{k,l\in I_q}\alpha(m_k,m_l)V_\epsilon(x_k-x_l)dt$$

whose right-hand-side is bounded above by $C\big|\log|z|\big|\big|\log\epsilon\big|^{-1}$, according to Lemma 2.1. That is,

$$\int_0^T |G_z(2)|dt \leq C\big|\log|z|\big|\big|\log\epsilon\big|^{-1}.$$

2.3.5 The case of $\mathbb{E}_N|X_z - X_0|$

We now turn to $\mathbb{E}_N|X_z - X_0|$. Assume that $|z| \geq R_0\varepsilon$. Using the second part of Lemma 2.4, we have that

$$\mathbb{E}_N|X_z - X_0(0)| \leq C|z| \iint_{L^2} h_{M_1}(x)h_{M_2}(y)|x-y|^{-1}dxdy,$$

where L is a bounded set that contains the support of J and $\overline{J}$. Using our second assumption on the initial data h_n we obtain the bound $C|z|$ for $\mathbb{E}_N|X_z - X_0|(0)$.

2.4 The martingale term

This section is devoted to proving the estimate (2.8). Note that

$$M_z(T) = X_z\big(q(T),T\big) - X_z\big(q(0),0\big) - \int_0^T \left(\frac{\partial}{\partial t} + \mathbb{L}\right) X_z\big(q(t),t\big)dt$$

is a martingale which satisfies

$$\mathbb{E}_N\Big[M_z(T)^2\Big] = \sum_{i=1}^{3} \mathbb{E}_N \int_0^T A_i\big(q(t),t\big)dt,$$

where $A_1(q,t)$ and $A_2(q,t)$ are respectively set equal to

$$2\big|\log\epsilon\big|^{-4} \sum_{i\in I_q, m_i=M_1} d(M_1) \left|\nabla_{x_i} \sum_{j\in I_q, m_j=M_2} u^\epsilon(x_i - x_j + z)J(x_i,M_1,t)\overline{J}(x_j,M_2,t)\right|^2,$$

and

$$2\big|\log\epsilon\big|^{-4} \sum_{i\in I_q, m_i=M_2} d(M_2) \left|\nabla_{x_i} \sum_{j\in I_q, m_j=M_1} u^\epsilon(x_j - x_i + z)J(x_j,M_1,t)\overline{J}(x_i,M_2,t)\right|^2,$$

while A_3 is given by

$$
\begin{aligned}
&\frac{1}{2}\big|\log\epsilon\big|^{-4}\sum_{i,j\in I_q}\alpha(m_i,m_j)V_\epsilon(x_i-x_j) \qquad (2.41)\\
&\quad\Big\{\sum_{k\in I_q}\Big[\frac{m_i}{m_i+m_j}u^\epsilon(x_i-x_k+z)J(x_i,m_i+m_j)\overline{J}(x_k,m_k)\\
&\qquad\qquad+\frac{m_i}{m_i+m_j}u^\epsilon(x_k-x_i+z)J(x_k,m_k)\overline{J}(x_i,m_i+m_j)\\
&\qquad\qquad+\frac{m_j}{m_i+m_j}u^\epsilon(x_j-x_k+z)J(x_j,m_i+m_j)\overline{J}(x_k,m_k)\\
&\qquad\qquad+\frac{m_j}{m_i+m_j}u^\epsilon(x_k-x_j+z)J(x_k,m_k)\overline{J}(x_j,m_i+m_j)\\
&\qquad\qquad-u^\epsilon(x_i-x_k+z)J(x_i,m_i)\overline{J}(x_k,m_k)-u^\epsilon(x_k-x_i+z)J(x_k,m_k)\overline{J}(x_i,m_i)\\
&\qquad\qquad-u^\epsilon(x_j-x_k+z)J(x_j,m_j)\overline{J}(x_k,m_k)-u^\epsilon(x_k-x_j+z)J(x_k,m_k)\overline{J}(x_j,m_j)\Big]\\
&\qquad\qquad\quad-u^\epsilon(x_i-x_j+z)J(x_i,m_i)\overline{J}(x_j,m_j)\Big\}^2.
\end{aligned}
$$

Recall that, by our convention, we do not display the dependence of J and $\overline{J}$ on the t-variable. To bound these terms, we require two variants of Lemma 2.3 :

Lemma 2.5 *There exists a collection of constants $C:\mathbb{N}^2\to(0,\infty)$ such that, for any continuous functions $t,v,a_1,a_2,a_3:\mathbb{R}^2\to[0,\infty)$ and any $z\in\mathbb{R}^2$,*

$$
\begin{aligned}
\mathbb{E}_N\int_0^T dt\sum_{i,j,k\in I_q(t)}&\gamma(m_i,m_j)t(x_i-x_j+z)v(x_i-x_k+z)\\
&a_1(x_i)a_2(x_j)a_3(x_k)\mathbb{1}\big\{m_i=n_1,m_k=n_3\big\}\\
&\le C_{n_1,n_3}|\log\varepsilon|^3\sum_{n_2}\mathbb{E}_N\sum_{i,j,k\in I_{q(0)}}A^\epsilon_{n_1,n_2,n_3}(x_i,x_j,x_k),
\end{aligned}
$$

where $A^\epsilon_{n_1,n_2,n_3}:\mathbb{R}^6\to[0,\infty)$ is given by

$$
\begin{aligned}
A^\epsilon_{n_1,n_2,n_3}(x_1,x_2,x_3) &= c_0(6)\gamma(n_1,n_2)\int_{\mathbb{R}^6}\left(\frac{|x_1-z'|^2}{d(n_1)}+\frac{|x_2-y|^2}{d(n_2)}+\frac{|x_3-y'|^2}{d(n_3)}\right)^{-2}\\
&\qquad t(z'-y+z)v(z'-y'+z)a_1(z')a_2(y)a_3(y')dz'dydy,
\end{aligned}
$$

with γ as in Lemma 2.3.

Lemma 2.6 *There exists a collection of constants* $C : \mathbb{N}^3 \to [0,\infty)$ *such that, for any* $z \in \mathbb{R}^d$, *any continuous functions* $v, w : \mathbb{R}^2 \to [0,\infty)$ *and another* $(a_1,a_2,a_3) : \mathbb{R}^6 \to [0,\infty)$,

$$\begin{aligned}
&\mathbb{E}_N \int_0^T dt \sum_{k,l,i,j \in I_q} \gamma(n_i,n_j) V_\varepsilon(x_i - x_j) v(x_i - x_k + z) w(x_i - x_l + z) a_1(x_i) a_2(x_k) a_3(x_l) \\
&\qquad \mathbb{1}\big\{ m_i = n_1, m_k = n_3, m_l = n_3 \big\} \\
&\leq C_{n_1,n_3,n_3} \big| \log \epsilon \big|^4 \sum_{n_2} \mathbb{E}_N \sum_{i,j,k,l \in I_{q(0)}} B^\epsilon_{m_i,m_j,m_k,m_l}(x_i,x_j,x_k,x_l) \\
&\qquad \mathbb{1}\big\{ m_i \leq n_1, m_j \leq n_2, m_k \leq n_3, m_l \leq n_3 \big\},
\end{aligned}$$

where $B^\epsilon_{n_1,n_2,n_3,n_3} : \mathbb{R}^8 \to [0,\infty)$ *is given by*

$$\begin{aligned}
& B^\epsilon_{n_1,n_2,n_3,n_3}\big(x_1,x_2,x_3,x_3\big) \\
= \ & c_0(8) \int_{\mathbb{R}^8} \bigg(\frac{|x_1 - \hat{z}|^2}{d(n_1)} + \frac{|x_2 - z'|^2}{d(n_2)} + \frac{|x_3 - y|^2}{d(n_3)} + \frac{|x_3 - y'|^2}{d(n_3)} \bigg)^{-3} \\
& \gamma(n_1,n_2) V_\varepsilon(\hat{z} - z') v(\hat{z} - y + z) w(z' - y' + z) a_1(\hat{z}) a_2(y) a_3(y') d\hat{z} dz' dy dy',
\end{aligned}$$

with the function $\gamma : \mathbb{N}^2 \to (0,\infty)$ *satisfying*

$$n_2 \gamma\big(n_1, n_2 + n_3\big) \max\bigg\{ 1, \Big[\frac{d(n_2+n_3)}{d(n_2)} \Big]^3 \bigg\} \leq \big(n_2 + n_3\big) \gamma(n_1,n_2).$$

The proof of Lemma 2.5 is identical to that of Lemma 2.3. The proof of Lemma 2.6 is very similar to the proof of Lemma 2.3 and is omitted.

We now bound the three terms. Of the first two, we treat only A_1, the other being bounded by an identical argument. By multiplying out the brackets appearing in the definition of A_1, we obtain that this quantity is bounded above by

$$\begin{aligned}
& C \big| \log \epsilon \big|^{-4} \sum_{i,j,k \in I_q} \big| \nabla u^\epsilon \big| (x_i - x_j + z) \big| \nabla u^\epsilon \big| (x_i - x_k + z) J^2(x_i, m_i) \\
& \qquad \big| \overline{J}(x_j,m_j) \big| \big| \overline{J}(x_k,m_k) \big| \mathbb{1}\big\{ m_i = M_1, m_j = m_k = M_2 \big\} \qquad (2.42) \\
+ \ & C \big| \log \epsilon \big|^{-4} \sum_{i,j,k \in I_q} |u^\epsilon(x_i - x_j + z)| \, |u^\epsilon(x_i - x_k + z)| \, |\nabla J(x_i,m_i)|^2 \\
& \qquad \big| \overline{J}(x_j,m_j) \big| \big| \overline{J}(x_k,m_k) \big| \mathbb{1}\big\{ m_i = M_1, m_j = m_k = M_2 \big\}. \qquad (2.43)
\end{aligned}$$

Let us assume that $z = 0$ because this will not affect our arguments. We are required to bound the quantity appearing in the statement of Lemma 2.5, for each of the following cases:

$$\big(t, v, a_1, a_2, a_3\big) \in \Big\{ \big(|\nabla u^\epsilon|, |\nabla u^\epsilon|, J^2, \big|\overline{J}\big|, \big|\overline{J}\big| \big) \ , \ \big(u^\epsilon, u^\epsilon, |\nabla J|^2, \big|\overline{J}\big|, \big|\overline{J}\big| \big) \Big\}. \qquad (2.44)$$

Recall that each of the test functions J, $\overline{J}$, and their gradients, is assumed to be uniformly bounded with compact support. To each of the two cases, Lemma 2.5 applies. For either of them, the right-hand-side of the inequality in Lemma 2.5 may be written as a finite sum of the expectations appearing there, with the sum being taken over triples of given masses n_1, n_2 and n_3. Such an expectation is bounded above by

$$
\begin{aligned}
& C\big|\log\epsilon\big|^{-4}\sum_{n_2\in\mathbb{N}}\int\int_{K^3}\bigg(\frac{|x_1-z'|^2}{d(n_1)}+\frac{|x_2-y|^2}{d(n_2)}+\frac{|x_3-y'|^2}{d(n_3)}\bigg)^{-2} \\
& \qquad t(z'-y)v(z'-y')h_{n_1}(x_1)h_{n_2}(x_2)h_{n_3}(x_3)dz'dydy'dx_1dx_2dx_3,
\end{aligned}
\tag{2.45}
$$

where $K=\{x:|x|\leq\ell\}\subseteq\mathbb{R}^2$ is chosen to contain the support of J and $\overline{J}$. As in Section 3.4 of [2], we can use our bounds in the first two parts of Lemma 2.4 and repeat the proof of the eighth part of Lemma 2.4 to obtain

$$
\mathbb{E}_N\int_0^T\big[A_1\big(q(t)\big)+A_2\big(q(t)\big)\big]\,dt\leq C\big|\log\epsilon\big|^{-1}. \tag{2.46}
$$

We must treat the third term, A_3. An application of the inequality

$$
\Big(a_1+\ldots+a_n\Big)^2\leq n\Big(a_1^2+\ldots+a_n^2\Big)
$$

to the bound on A_3 provided in (2.41) implies that

$$
A_3(q)\leq\frac{9}{2}\big|\log\epsilon\big|^{-4}\sum_{i,j\in I_q}\alpha(m_i,m_j)V_\epsilon(x_i-x_j)\bigg[\sum_{n=1}^{8}\Big(\sum_{k\in I_q}Y_n\Big)^2+Y_9^2\bigg], \tag{2.47}
$$

where Y_1 is given by

$$
\frac{m_i}{m_i+m_j}u^\epsilon(x_i-x_k+z)J(x_i,m_i+m_j)\overline{J}(x_k,m_k),
$$

and where $\big\{Y_i:i\in\{2,\ldots,8\}\big\}$ denote the other seven expressions in (2.41) that appear in a sum over $k\in I_q$, while Y_9 denotes the last term in (2.41) that does not appear in this sum. There are nine cases to consider. The first eight are practically identical, and we treat only the fifth. Note that

$$
\begin{aligned}
& \big|\log\epsilon\big|^{-4}\sum_{i,j\in I_q}\alpha(m_i,m_j)V_\varepsilon(x_i-x_j)\Big(\sum_{k\in I_q}Y_5\Big)^2 \\
= \;& C\big|\log\epsilon\big|^{-4}\sum_{i,j\in I_q}\alpha(m_i,m_j)V_\varepsilon(x_i-x_j) \\
& \quad\Big[\sum_{k,l\in I_q}u^\epsilon(x_i-x_k+z)u^\epsilon(x_i-x_l+z)J^2(x_i,m_i)\overline{J}(x_k,m_k)\overline{J}(x_l,m_l)\Big].
\end{aligned}
\tag{2.48}
$$

In the sum with indices involving $k, l \in I_q$, we permit the possibility that these two may be equal, though they must be distinct from each of i and j (which of course must themselves be distinct by the overall convention).

Note that the expression (2.48) appears in the statement of Lemma 2.6, provided that the choice

$$\Big(v, w, a_1, a_2, a_3\Big) = \Big(\big|u^\epsilon\big|, \big|u^\epsilon\big|, J^2, \big|\overline{J}\big|, \big|\overline{J}\big|\Big)$$

is made. Again we set $z = 0$ because this does not affect the estimates. Given that the support of each of the functions $a_1, a_2, a_3 : \mathbb{R}^6 \to [0, \infty)$ is bounded, we must bound

$$\sum_{n_2 \in \mathbb{N}} \int \int_{L^4} \Big(\frac{|x_1 - \hat{z}|^2}{d(n_1)} + \frac{|x_2 - z'|^2}{d(n_2)} + \frac{|x_3 - y|^2}{d(n_3)} + \frac{|x_3 - y'|^2}{d(n_3)} \Big)^{-3} V\Big(\frac{\hat{z} - z'}{\epsilon} \Big)$$
$$\big|u^\epsilon(\hat{z} - y)\big|\big|u^\epsilon(z' - y')\big| h_{n_1}(x_1) h_{n_2}(x_2) h_{n_3}(x_3) h_{n_3}(x_3) d\hat{z} dz' dy dy' dx_1 dx_2 dx_3 dx_3,$$

for a compact set L. This expression is bounded above by

$$\int_{L^3} V\Big(\frac{\hat{z} - z'}{\epsilon} \Big) u^\epsilon(\hat{z} - y) u^\epsilon(z' - y') d\hat{z} dz' dy dy'$$
$$\sum_{n_2 \in \mathbb{N}} \int_{K^3} \left(\frac{|x_1 - \hat{z}|^2}{d(n_1)} + \frac{|x_2 - z'|^2}{d(n_2)} + \frac{|x_3 - y|^2}{d(n_3)} + \frac{|x_3 - y'|^2}{d(n_3)} \right)^{-3}$$
$$h_{n_1}(x_1) h_{n_2}(x_2) h_{n_3}(x_3) h_{n_3}(x_3) dx_1 dx_2 dx_3 dx_3,$$

which is less than

$$C \int_{L^3} V\Big(\frac{\hat{z} - z'}{\epsilon} \Big) u^\epsilon(\hat{z} - y) u^\epsilon(z' - y') d\hat{z} dz' dy dy'.$$

The proof of this follows the proof of the eighth part of Lemma 2.4; we use the elementary inequality $abcd \leq (a^2 + b^2 + c^2 + d^2)^2$ and the fact that the function

$$\hat{k}(x) = \sum_n d(n)^{3/4} \int h_n(y) |x - y|^{-3/2} dy$$

is locally bounded. Noting that the bound $|u^\epsilon(x)| \leq \big|\log|x|\big|$ implies that

$$\int_L u^\varepsilon(\hat{z} - y) dy$$

is bounded above by a constant, we find that

$$\int_{L^3} V\Big(\frac{\hat{z} - z'}{\epsilon} \Big) u^\epsilon(\hat{z} - y) u^\epsilon(z' - y') d\hat{z} dz' dy dy' \leq C \int_{L^2} V\Big(\frac{\hat{z} - z'}{\epsilon} \Big) d\hat{z} dz'.$$

This is at most $C\varepsilon^2$. Applying Lemma 2.6, we find that the contribution to

$$\mathbb{E}_N \int_0^T A_3\big(q(t)\big)dt$$

arising from the fifth term in (2.47) is at most

$$C\epsilon^{-2}\big|^{-1}\epsilon^2 = C\big|\log\epsilon\big|^{-1}.$$

We now treat the ninth term, as they are classified in (2.47). It takes the form

$$\big|\log\epsilon\big|^{-4}\sum_{i,j\in I_q}\alpha(m_i,m_j)V_\varepsilon\Big(x_i-x_j\Big)$$
$$u^\epsilon\big(x_i-x_j+z\big)^2 J\big(x_i,m_i\big)^2\overline{J}\big(x_j,m_j\big)^2.$$

This is bounded above by

$$C\big|\log\epsilon\big|^{-3}\sum_{i,j\in I_q}\alpha(m_i,m_j)V_\varepsilon\Big(x_i-x_j\Big),$$

because $u^\varepsilon \le C\big|\log\epsilon\big|$ in the support of V_ε by the first part of Lemma 2.4. The expected value of the integral on the interval of time $[0,T]$ of this last expression is bounded above by

$$C\big|\log\epsilon\big|^{-3}\mathbb{E}_N\int_0^T dt\sum_{i,j\in I_q}\alpha(m_i,m_j)V_\varepsilon(x_i-x_j) \le C\big|\log\epsilon\big|^{-1},$$

where we used Lemma 2.1 for the last inequality. This completes the proof of (2.8).

2.5 Using the estimates

The inequalities (2.6), (2.7) and (2.8) imply that, for large T,

$$\lim_{|z|\to 0}\limsup_{\epsilon\downarrow 0}\mathbb{E}_N\bigg|\int_0^T H_{11}\big(t\big)dt + \int_0^T H_{13}\big(t\big)dt\bigg|dt = 0.$$

That is,

$$\lim_{|z|\to 0}\limsup_{\epsilon\downarrow 0}\big|\log\epsilon\big|^{-2}\mathbb{E}_N\bigg|\int_0^T dt\sum_{i,j\in I_{q(t)}}\alpha(m_i,m_j)J(x_i,m_i)\overline{J}(x_j,m_j)$$
$$\Big[V^\epsilon(x_i-x_j+z)-V^\epsilon(x_i-x_j)+V_\epsilon(x_i-x_j+z)u^\epsilon(x_i-x_j+z)\Big]\ \bigg| = 0.$$

(Recall that we simply write $J(x_i, m_i)$ and $\overline{J}(x_i, m_i)$ for $J(x_i, m_i, t)$ and $\overline{J}(x_i, m_i, t)$.) This implies that

$$\begin{aligned}
& \big|\log \epsilon\big|^{-2} \int_0^T \sum_{i,j \in I_{q(t)}} \alpha(m_i, m_j) V^\epsilon(x_i - x_j) J(x_i, m_i) \overline{J}(x_j, m_j) dt \\
= \; & \big|\log \epsilon\big|^{-2} \int_0^T \sum_{i,j \in I_{q(t)}} \alpha(m_i, m_j) V^\epsilon(x_i - x_j + z) J(x_i, m_i) \overline{J}(x_j, m_j) \\
& \qquad \Big[1 + \big|\log \epsilon\big|^{-1} u^\epsilon(x_i - x_j + z)\Big] dt \; + \; Err_1(\epsilon, z), \qquad (2.49)
\end{aligned}$$

where Err_1 satisfies

$$\lim_{|z| \to 0} \limsup_{\epsilon \downarrow 0} \mathbb{E}_N \big|Err_1\big(\epsilon, z\big)\big| = 0. \qquad (2.50)$$

By Theorem 3.2, the expression $1 + |\log \varepsilon|^{-1} u^\varepsilon_{m_1, m_2}\big(a\big)$ is uniformly close to

$$\left(1 - \frac{\tau(m_1, m_2)}{2\pi + \tau(m_1, m_2)}\right),$$

for a satisfying $V^\varepsilon(a) \neq 0$ and $\tau(m_1, m_2) = \alpha(m_1, m_2)/(d(m_1) + d(m_2))$. Recalling from (1.9) that

$$Q = \big|\log \epsilon\big|^{-1} \sum_{(i,j) \in I_q} \alpha(m_i, m_j) V_\epsilon(x_i - x_j) J(x_i, m_i) \overline{J}(x_j, m_j)$$

and writing

$$\overline{Q}(z) = \big|\log \epsilon\big|^{-1} \sum_{i,j \in I_q} \beta(m_i, m_j) V^\varepsilon(x_i - x_j + z) J(x_i, m_i) \overline{J}(x_j, m_j), \qquad (2.51)$$

it follows from (2.49) and Theorem 3.2 that

$$\int_0^T Q(t) dt = \int_0^T \overline{Q}(z)(t) dt + Err_2\big(\epsilon, z\big), \qquad (2.52)$$

where Err_2 satisfies

$$\lim_{|z| \to 0} \limsup_{\epsilon \downarrow 0} \mathbb{E}_N \big|Err_2\big(\epsilon, z\big)\big| = 0.$$

From this, it is not hard to deduce that

$$\begin{aligned}
\overline{Q}(z_2 - z_1) \;\; = \;\; & |\log \varepsilon|^{-2} \sum_{i,j \in I_q} \beta(m_i, m_j) V^\varepsilon(x_i - x_j + z_2 - z_1) \qquad (2.53) \\
& \qquad J(x_i - z_1, m_i) \overline{J}(x_j - z_2, m_j) \; + \; Err(\epsilon, z_1, z_2),
\end{aligned}$$

where

$$\mathbb{E}_N\big|Err(\epsilon, z_1, z_2)\big| \le C\big(|z_1| + |z_2|\big).$$

(See Section 3.5 of [2].) By (2.52) and (2.53),

$$\begin{aligned}
&\int_0^T Q(t)dt \\
= \;& |\log\varepsilon|^{-2}\int_0^T dt \sum_{i,j\in I_{q(t)}} \beta(m_i,m_j)J(x_i - z_1, m_i)\overline{J}(x_j - z_2, m_j) \\
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} V^\varepsilon((x_i - z_1) - (x_j - z_2))\delta^{-2}\eta\Big(\frac{z_1}{\delta}\Big)\delta^{-2}\eta\Big(\frac{z_2}{\delta}\Big)dz_1dz_2 \,+\, Err_3\big(\epsilon,\delta\big) \\
= \;& |\log\varepsilon|^{-2}\int_0^T dt\int_{\mathbb{R}^{2d}} d\omega_1 d\omega_2 \sum_{i,j\in I_{q(t)}} V^\varepsilon(\omega_1 - \omega_2)\beta(m_i,m_j)J(\omega_1,m_i)\overline{J}(\omega_2,m_j) \\
&\delta^{-2}\eta\Big(\frac{x_i-\omega_1}{\delta}\Big)\delta^{-2}\eta\Big(\frac{x_j-\omega_1}{\delta}\Big) \,+\, Err_3\big(\epsilon,\delta\big) \\
= \;& \int_0^T dt\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} d\omega_1 d\omega_2 v^\varepsilon(\omega_1-\omega_2)\beta(M_1,M_2)J(\omega_1,M_1)\overline{J}(\omega_2,M_2) \\
&\Bigg[\big|\log\epsilon\big|^{-1}\sum_{i\in I_q:m_i=M_1}\delta^{-2}\eta\Big(\frac{x_i-\omega_1}{\delta}\Big)\Bigg] \\
&\Bigg[\big|\log\epsilon\big|^{-1}\sum_{j\in I_q;m_j=M_2}\delta^{-2}\eta\Big(\frac{x_j-\omega_2}{\delta}\Big)\Bigg] \,+\, Err_3\big(\epsilon,\delta\big),
\end{aligned}$$

where Err_3 satisfies

$$\lim_{\delta\downarrow 0}\limsup_{\epsilon_\downarrow 0}\mathbb{E}_N\big|Err_3\big(\epsilon,\delta\big)\big| = 0,$$

and where in the last equality, we made use of the fact that the test functions J and $\overline{J}$ take non-zero values only on particles of a given mass, respectively M_1 and M_2. Thus,

$$\begin{aligned}
&\int_0^T Q(t)dt \\
= \;& \int_0^T dt\int_{\mathbb{R}^2} d\omega\beta(M_1,M_2)J(\omega,M_1)\overline{J}(\omega,M_2)\Bigg[\big|\log\epsilon\big|^{-1}\sum_{i\in I_q:m_i=M_1}\delta^{-2}\eta\Big(\frac{x_i-\omega}{\delta}\Big)\Bigg] \\
&\Bigg[\big|\log\epsilon\big|^{-1}\sum_{j\in I_q;m_j=M_2}\delta^{-2}\eta\Big(\frac{x_j-\omega}{\delta}\Big)\Bigg] + Err\big(\epsilon,\delta\big)
\end{aligned}$$

where $Err = Err_5 + err$ with the function $err = O\big(\varepsilon\delta^{-5}\big)$ also satisfies

$$\lim_{\delta\downarrow 0}\limsup_{\epsilon_\downarrow 0}\mathbb{E}_N\big|err\big(\epsilon,\delta\big)\big| = 0.$$

This completes the proof of Proposition 1.

3 Potential theory

The purpose of this section is twofold. Firstly, we show the existence of the function u^ε that satisfies (3.1). Secondly we evaluate the limit of $u^\varepsilon |\log\varepsilon|^{-1}$ in the support of V^ε, as $\varepsilon \to 0$. This limit was used in the evaluation of β in Section 2.5. We start with the statements of the main results of this section. Let $V : \mathbb{R}^d \to \mathbb{R}$ be a continuous function of compact support with $V \geq 0$ and $\int_{\mathbb{R}^2} V(x)dx = 1$. We also write K_0 for the topological closure of U_0 where

$$U_0 = \{x : V(x) \neq 0\}. \tag{3.1}$$

Given a measure μ, let us define

$$\mathcal{G}\mu(x) = \int \log|x-y|\mu(dy).$$

When the measure μ is absolutely continuous with respect to the Lebesgue measure with a density g, we simply write $\mathcal{G}g$ for $\mathcal{G}\mu$.

Theorem 3.1 *There exists a number $\gamma_0 > 0$ such that for every $\gamma \in (0,\gamma_0)$ and $a \in \mathbb{R}$, there exists a unique function $u \in C^2(\mathbb{R}^2)$ such that $u(x) = O\big(\big|\log|x|\big|\big)$ as $|x| \to \infty$ and*

$$u = \gamma\mathcal{G}\big((u+a)V\big). \tag{3.2}$$

Moreover $Z = \gamma\int(u+a)V\,dx \neq 0$ and $(u+a)Z^{-1} \geq 0$.

Recall that we are searching for a function u^ε such that

$$\Delta u^\varepsilon = \tau(n,m)\left[V_\varepsilon u^\varepsilon + V^\varepsilon\right],$$

where $\tau = \tau(n,m) = \alpha(n,m)/(d(n)+d(m))$. For this it suffices to have

$$u^\varepsilon = \mathcal{G}\mu^\varepsilon, \tag{3.3}$$

where $\mu^\varepsilon(dx) = \frac{1}{2\pi}\tau(V_\varepsilon u^\varepsilon + V^\varepsilon)dx$. This can be rewritten as

$$u^\varepsilon = \frac{1}{2\pi}\tau^\varepsilon\mathcal{G}\left(V^\varepsilon\left[u^\varepsilon + |\log\varepsilon|\right]\right),$$

where $\tau^\varepsilon = \tau|\log\varepsilon|^{-1}$. Evidently we can apply Theorem 3.1 to deduce the existence of the function u^ε for sufficiently small ε.

Our next theorem was used in the previous section for the evaluation of β.

Theorem 3.2 *For every positive* k*,*

$$\lim_{\varepsilon\to 0}\sup_{|x|\le k}\left|u^\varepsilon(\varepsilon x)|\log\varepsilon|^{-1}+\frac{\tau}{2\pi+\tau}\right|=0.$$

Proof of Theorem 3.1

Step 1. Let J be a bounded continuous function with $J>0$ and

$$\int_{|x|\ge 1} J(x)\left(\log|x|\right)^2 dx<\infty.$$

Define

$$\mathcal{H}=\left\{u:\ u \text{ is measurable and } \int_{\mathbb{R}^2} u^2(x)J(x)dx<\infty\right\}. \tag{3.4}$$

We then define $\mathcal{F}:\mathcal{H}\to\mathcal{H}$ by $\mathcal{F}(u)=\mathcal{G}(uV)$. Observe that $\mathcal{H}$ is a Hilbert space with respect to the inner product

$$\langle u,v\rangle=\int_{\mathbb{R}^2} u(x)v(x)J(x)dx.$$

Let us verify that $\mathcal{F}$ is a bounded operator. To see this, write

$$\Gamma(x)=\int_{\mathbb{R}^2}\big|\log|x-y|\big|V(y)dy. \tag{3.5}$$

When $|x|$ is sufficiently large, we have that $\Gamma(x)\le\log(2|x|)$ because $0<\log|x-y|\le\log(2|x|)$ whenever $V(y)\ne 0$. Otherwise we have

$$\Gamma(x)\le c_0\int_{|x-y|\le c_1}\big|\log|x-y|\big|dy\le c_2,$$

for constants c_0, c_1 and c_2. As a result

$$\Gamma(x)\le c+\log^+|x| \tag{3.6}$$

for a constant c. Also, we may use Hölder's inequality to assert

$$\begin{aligned}(\mathcal{F}(u)(x))^2 &\le \left[\Gamma(x)\int_{\mathbb{R}^2}\big|\log|x-y|\big|V(y)|u(y)|\frac{dy}{\Gamma(x)}\right]^2 \qquad (3.7)\\ &\le \Gamma(x)\int_{\mathbb{R}^2}\big|\log|x-y|\big|V(y)u^2(y)dy.\end{aligned}$$

From this we deduce

$$\int_{\mathbb{R}^2} (\mathcal{F}(u)(x))^2 J(x)dx \le \int_{\mathbb{R}^2} V(y)u^2(y) \left[\int_{\mathbb{R}^2} \Gamma(x) \big| \log|x-y| \big| J(x)dx \right] dy.$$

If $V(y) \neq 0$ then $|y| \le R_0$ for a suitable R_0. Define

$$I(y) = \int_{\mathbb{R}^2} \Gamma(x) \big| \log|x-y| \big| J(x)dx,$$

Note

$$\begin{aligned} I(y) &= \int_{|x|\le 2R_0} + \int_{|x|>2R_0} \Gamma(x) \big| \log|x-y| \big| J(x)dx \\ &\le c_1 \int_{|x-y|\le 3R_0} \big| \log|x-y| \big| dx + c_1 \int_{|x|>2R_0} \big| \log|x| \big|^2 J(x)dx, \end{aligned}$$

where for the second line we have used (3.6). From this and our assumption on J we deduce that $\sup_{|y|\le R_0} I(y) < \infty$. As a result,

$$\int_{\mathbb{R}^2} (\mathcal{F}(u)(x))^2 J(x)dx \le c_1 \int_{\mathbb{R}^2} V(y)u^2(y)dy \le c_2 \int_{\mathbb{R}^2} u^2(y)J(y)dy$$

because V is of compact support and $J > 0$. This shows the boundedness of the operator $\mathcal{F} : \mathcal{H} \to \mathcal{H}$.

Step 2. Since the operator $\mathcal{F}$ is bounded, the equation

$$(id - \gamma\mathcal{F})(u) = g$$

has a solution, where $g(x) = -\gamma a \Gamma(x)$ with Γ as in (3.5) and id denotes the identity transformation. Note that our assumption on Γ implies that $\Gamma \in \mathcal{H}$ because of (3.6). So far we have shown the existence of a unique solution $u \in \mathcal{H}$ of $u - \tau\mathcal{F}(u) = g$. From this and the Hölder continuity of V we can readily show that in fact $u \in C^2$ and that u is a classical solution of

$$\Delta u = 2\pi\gamma(u+a)V. \tag{3.8}$$

(See for example Section 4.2 of [1].)

Step 3. In this step we verify $Z \neq 0$. Observe that $u = \mathcal{G}\mu$ for a measure μ with a bounded support. From this we can readily deduce

$$u(x) = \mu\left(\mathbb{R}^2\right) \log|x| + O\left(|x|^{-1}\right), \tag{3.9}$$

$$\nabla u(x) = \mu\left(\mathbb{R}^2\right)\frac{x}{|x|^2} + O\left(|x|^{-2}\right). \tag{3.10}$$

We now choose $R > R_0$ and use $\Delta u = 2\pi\gamma(u+a)V$ to write

$$\int_{|x|\leq R}(u+a)\Delta u dx = 2\pi\gamma\int_{|x|\leq R}V(u+a)^2dx.$$

After an integration by parts we obtain

$$-\int_{|x|\leq R}|\nabla u|^2dx + \int_{|x|=R}(u+a)\nabla u\cdot n dS = 2\pi\gamma\int_{|x|\leq R}V(u+a)^2dx,$$

where $n = \dfrac{x}{|x|}$ is the normal vector and dS is the Lebesgue measure on $|x| = R$. Now if $Z = \mu\big(\mathbb{R}^2\big) = 0$, then we can use (3.9)–(3.10) to deduce that

$$\int_{|x|=R}(u+a)\nabla u\cdot n\ dS = O(R^{-1}).$$

As a result,

$$-\int_{\mathbb{R}^2}|\nabla u|^2dx = 2\pi\gamma\int_{\mathbb{R}^2}V(u+a)^2dx.$$

From this we deduce that $\int_{\mathbb{R}^2}|\nabla u|^2dx = \int_{\mathbb{R}^2}(u+a)^2Vdx = 0$. This in turn implies that $u \equiv 0$. But this contradicts $u = \mathcal{G}(V(u+a))$. Hence we can not have $Z = 0$.

Step 4. It remains to show that $(u+a)Z^{-1} \geq 0$. We only establish this when $Z > 0$. The case $Z < 0$ can be treated likewise. First take a smooth function $\varphi_\delta : \mathbb{R} \to (-\infty, 0]$ such that $\varphi_\delta' \geq 0$ and

$$\varphi_\delta(r) = \begin{cases} 0 & r > -a, \\ a+r & r < -a-\delta. \end{cases}$$

We then have

$$\int_{|x|=R}\varphi_\delta(u)\nabla u.ndS - \int_{|x|\leq R}\varphi_\delta'(u)|\nabla u|^2dx = 2\pi\gamma\int_{|x|\leq R}\varphi_\delta(u)(u+a)Vdx \tag{3.11}$$

by an integration by parts. (Here $n = x/|x|$.) If $Z > 0$, then we can use (3.9) to assert that $u(x) > 0$ and $\varphi_\delta(u(x)) = 0$ whenever $|x| = R$ and R is sufficiently large. Since the left-hand side of (3.11) is negative for such large R and $(u+a)\varphi_\delta(u) \geq 0$ we deduce

$$\int_{\mathbb{R}^2}\varphi_\delta'(u)|\nabla u|^2dx = \int_{\mathbb{R}^2}V(u+a)\varphi_\delta(u)dx = 0.$$

We now send $\delta \to 0$ to deduce

$$0 = \int_{\mathbb{R}^2} |\nabla u|^2 1\!\!1(u + a \le 0)dx = \int_{\mathbb{R}^2} V(u+a)^2 1\!\!1(u + a \le 0)dx.$$

As a result, on the set $A = \{x : a + u(x) < 0\}$ we have $\nabla u = 0$. Hence u is constant on each component B of A. But this constant can only be $-a$ because on the boundary of A we have $u + a = 0$. This is impossible unless A is empty and we deduce that $u \ge -a$ everywhere. □

We now turn to the proof of Theorem 3.2. We first state and prove a lemma. Let us write Λ_ε for $\mu^\varepsilon(\mathbb{R}^2)$ where μ^ε was defined right after (3.3).

Lemma 3.1 *We have that $\Lambda_\varepsilon > 0$ for small ε. Moreover*

$$\limsup_{\varepsilon \to 0} \Lambda_\varepsilon \le 1.$$

Proof Let us write $\hat{u}^\varepsilon$ for $(u^\varepsilon + |\log \varepsilon|)\Lambda_\varepsilon^{-1}$ and $\hat{\mu}^\varepsilon$ for $\Lambda_\varepsilon^{-1}\mu^\varepsilon$. By Theorem 3.1 we have that $\hat{\mu}^\varepsilon$ is a probability measure and $u^\varepsilon \ge 0$. Note that the support of the probability measure $\hat{\mu}^\varepsilon$ is the set εK_0. Moreover u^ε is harmonic off εK_0 and

$$\hat{u}^\varepsilon = \mathcal{G}\hat{\mu}^\varepsilon + \frac{|\log \varepsilon|}{\Lambda_\varepsilon}. \tag{3.12}$$

By a well-known theorem in potential theory we have

$$\varepsilon Cap(K_0) = Cap(\varepsilon K_0) \ge \exp\left(-\frac{|\log \varepsilon|}{\Lambda_\varepsilon}\right),$$

where Cap denotes the logarithmic capacity. (See for example Theorem 9.8 of [4].) As a result

$$\frac{\log Cap(K_0)}{|\log \varepsilon|} - 1 \ge -\frac{1}{\Lambda_\varepsilon}.$$

From this, we can readily deduce the claims of the Lemma. □

Proof of Theorem 3.2 It suffices to show that for every positive k,

$$\lim_{\varepsilon \to 0} \sup_{|x| \le k} \left| u^\varepsilon(\varepsilon x)|\log \varepsilon|^{-1} + \Lambda_\varepsilon \right| = 0, \tag{3.13}$$

and

$$\lim_{\varepsilon \to 0} \Lambda_\varepsilon = \frac{\tau}{2\pi + \tau}. \tag{3.14}$$

Recall that by Theorem 3.1 and Lemma 3.1 the expression $u^\varepsilon(\varepsilon y)|\log\varepsilon|^{-1}+1$ is nonnegative Also recall that R_0 is defined so that the ball $B_{R_0}(0)$ contains the support of V. Let us write ℓ_ε for the maximum of $u^\varepsilon(\varepsilon x)$ over the ball $B_{R_0}(0)$. We then have

$$\begin{aligned} u^\varepsilon(\varepsilon x) &= \frac{\tau}{2\pi}\int \log|\varepsilon x-\varepsilon y|\left(u^\varepsilon(\varepsilon y)|\log\varepsilon|^{-1}+1\right)V(y)dy \\ &= \Lambda_\varepsilon\log\varepsilon+\frac{\tau}{2\pi}\int \log|x-y|\left(u^\varepsilon(\varepsilon y)|\log\varepsilon|^{-1}+1\right)\ V(y)dy \\ &\le \Lambda_\varepsilon\log\varepsilon+\frac{\tau}{2\pi}\left(\ell_\varepsilon|\log\varepsilon|^{-1}+1\right)\int_{|x-y|>1}\log|x-y|\ V(y)dy. \end{aligned} \tag{3.15}$$

Hence for every x with $|x|\le k$ we have

$$u^\varepsilon(\varepsilon x)\le\Lambda_\varepsilon\log\varepsilon+c\left(\ell_\varepsilon|\log\varepsilon|^{-1}+1\right),$$

where c is a constant that depends on k only. By choosing $k=R_0$ we deduce that

$$\left(1-c|\log\varepsilon|^{-1}\right)\ell_\varepsilon\le\Lambda_\varepsilon\log\varepsilon+c.$$

Hence

$$\ell_\varepsilon=\max_{|x|\le R_0}u^\varepsilon(\varepsilon x)\le 2\Lambda_\varepsilon\log\varepsilon+2c \tag{3.16}$$

for sufficiently small ε. This in turn implies

$$\ell_\varepsilon|\log\varepsilon|^{-1}+1\le 2, \tag{3.17}$$

for small ε. Moreover, by the second equality in (3.15),

$$u^\varepsilon(\varepsilon x)|\log\varepsilon|^{-1}+\Lambda_\varepsilon=\frac{\tau}{2\pi|\log\varepsilon|}\int\log|x-y|\left(u^\varepsilon(\varepsilon y)|\log\varepsilon|^{-1}+1\right)V(y)dy=X_1+X_2,$$

where

$$\begin{aligned} X_1 &= \frac{\tau}{2\pi|\log\varepsilon|}\int_{|x-y|\le1}\log|x-y|\left(u^\varepsilon(\varepsilon y)|\log\varepsilon|^{-1}+1\right)V(y)dy, \\ X_2 &= \frac{\tau}{2\pi|\log\varepsilon|}\int_{|x-y|\ge1}\log|x-y|\left(u^\varepsilon(\varepsilon y)|\log\varepsilon|^{-1}+1\right)V(y)dy. \end{aligned}$$

Since the expression $u^\varepsilon(\varepsilon y)|\log\varepsilon|^{-1}+1$ is nonnegative and bounded above for y in the ball $B_{R_0}(0)$, we deduce that both X_1 and X_2 converge to 0 in low ε limit. This completes the proof of (3.15).

We now turn to the proof (3.16). By the definition of Λ_ε and (3.15),

$$\begin{aligned}\Lambda_\varepsilon &= \frac{\tau}{2\pi}\int \big(u^\varepsilon(\varepsilon y)|\log\varepsilon|^{-1}+1\big)\,V(y)dy\\ &= \frac{\tau}{2\pi}(1-\Lambda_\varepsilon)\int V(y)dy+o(1)=\frac{\tau}{2\pi}(1-\Lambda_\varepsilon)+o(1).\end{aligned}$$

This immediately implies (3.16). □

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