The elementary renewal theorem: supplement for lecture 12

The proof of this theorem is non-examinable.

Please send comments or corrections to <hammond@stats.ox.ac.uk>.

**Theorem.** (Elementary renewal theorem.) Let \( X = (X_t)_{t \geq 0} \) be a renewal process with mean inter-arrival time \( \mu \) and renewal function \( m(t) = \mathbb{E}[X_t] \). Then

\[
\frac{m(t)}{t} \to \frac{1}{\mu} \quad \text{as} \quad t \to \infty.
\]

Note that this does not follow directly from the Strong Law of Renewal Theory because almost sure convergence does not imply convergence of means (see Question 4 on Problem Sheet 5).

We need a lemma.

**Lemma.** For a renewal process \( X \) with arrival times \((T_n)_{n \geq 1}\), we have

\[
\mathbb{E}[T_{X_t+1}] = \mu(m(t) + 1),
\]

where \( m(t) = \mathbb{E}[X_t] \) and \( \mu = \mathbb{E}[T_1] \).

This ought to be true, because \( T_{X_t+1} \) is the sum of \( X_t + 1 \) interarrival times, each with mean \( \mu \). Taking expectations, we should get \( m(t) + 1 \) times \( \mu \). However, if we condition on \( X_t \) we have to know the distribution of the residual inter-arrival time after \( t \) but, without the memoryless property, it’s not clear how to do this.

**Proof.** We do a one-step analysis of \( g(t) = \mathbb{E}[T_{X_t+1}] \):

\[
g(t) = \int_0^\infty \mathbb{E}[T_{X_t+1}|T_1 = s] f(s)ds = \int_0^t (s + \mathbb{E}[T_{X_{t-s} + 1}]) f(s)ds + \int_t^\infty s f(s)ds
\]

\[
= \mu + (g \ast f)(t).
\]

This is almost the renewal equation. In fact, \( h(t) = g(t)/\mu - 1 \) satisfies the renewal equation:

\[
h(t) = \frac{1}{\mu} \int_0^t g(t - s)f(s)ds = \int_0^t (h(t - s) + 1)f(s)ds = F(t) + (h \ast f)(t).
\]

Since we know that \( m(t) \) is the unique solution to the renewal equation which is bounded on finite intervals, \( h(t) = m(t) \), i.e. \( g(t) = \mu(1 + m(t)) \), as required. \( \square \)
Proof of the elementary renewal theorem. We certainly have $t < T_{X_{t+1}}$ and so $t < \mathbb{E}[T_{X_{t+1}}] = \mu(m(t) + 1)$ gives the lower bound

$$\liminf_{t \to \infty} \frac{m(t)}{t} \geq \frac{1}{\mu}.$$ 

For the upper bound we use a truncation argument and introduce

$$\tilde{Z}_j = Z_j \wedge a = \begin{cases} Z_j & \text{if } Z_j < a \\ a & \text{if } Z_j \geq a, \end{cases}$$

with associated renewal process $\tilde{X}$. $\tilde{Z}_j \leq Z_j$ for all $j \geq 0$ implies $\tilde{X}_t \geq X_t$ for all $t \geq 0$ and so $\tilde{m}(t) \geq m(t)$. We can apply the lemma again to obtain

$$t \geq \mathbb{E}\left[T_{\tilde{X}_t}\right] = \mathbb{E}\left[T_{\tilde{X}_{t+1}}\right] - \mathbb{E}\left[\tilde{Z}_{\tilde{X}_{t+1}}\right] = \tilde{\mu}(\tilde{m}(t) + 1) - \mathbb{E}\left[\tilde{Z}_{\tilde{X}_{t+1}}\right] \geq \tilde{\mu}(m(t) + 1) - a.$$ 

Therefore,

$$\frac{m(t)}{t} \leq \frac{1}{\tilde{\mu}} + \frac{a - \tilde{\mu}}{\tilde{\mu}t}$$

so that

$$\limsup_{t \to \infty} \frac{m(t)}{t} \leq \frac{1}{\mu}.$$ 

Now $\tilde{\mu} = \mathbb{E}\left[\tilde{Z}_1\right] = \mathbb{E}[Z_1 \wedge a] \to \mathbb{E}[Z_1] = \mu$ as $a \to \infty$ (by monotone convergence). Therefore,

$$\limsup_{t \to \infty} \frac{m(t)}{t} \leq \frac{1}{\mu},$$

which completes the proof. \qed

Note that truncation was necessary to get $\mathbb{E}\left[\tilde{Z}_{\tilde{X}_{t+1}}\right] \leq a$. It would have been enough to have had $\mathbb{E}[Z_{X_{t+1}}] = \mathbb{E}[Z_1] = \mu$, but this is not true. Consider at the Poisson process as an example. We know that the residual lifetime already has mean $\mu = 1/\lambda$, but there is also the part of $Z_{X_{t+1}}$ before time $t$ to take care of.