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The continuum limit of critical random graphs

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Abstract We consider the Erdős–Rényi random graph $G(n, p)$ inside the critical window, that is when $p = 1/n + \lambda n^{-4/3}$, for some fixed $\lambda \in \mathbb{R}$. We prove that the sequence of connected components of $G(n, p)$, considered as metric spaces using the graph distance rescaled by $n^{-1/3}$, converges towards a sequence of continuous compact metric spaces. The result relies on a bijection between graphs and certain marked random walks, and the theory of continuum random trees. Our result gives access to the answers to a great many questions about distances in critical random graphs. In particular, we deduce that the diameter of $G(n, p)$ rescaled by $n^{-1/3}$ converges in distribution to an absolutely continuous random variable with finite mean.

Keywords Random graphs · Gromov–Hausdorff distance · scaling limits · continuum random tree · diameter

Mathematics Subject Classification (2000) 05C80 · 60C05

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1 Introduction

Random graphs and the phase transition. Since its introduction by Erdős and Rényi [22], the model $G(n,p)$ of random graphs has received an enormous amount of attention [11, 30]. In this model, a graph on n labeled vertices $\{1,2,\ldots,n\}$ is chosen randomly by joining any two vertices by an edge with probability p, independently for different pairs of vertices. This model exhibits a radical change in structure (or *phase transition*) for large n when $p = p(n) \sim 1/n$. For $p \sim c/n$ with $c < 1$, the largest connected component has size (number of vertices) $O(\log n)$. On the other hand, when $c > 1$, there is a connected component containing a positive proportion of the vertices (the *giant component*). The cases $c < 1$ and $c > 1$ are called *subcritical* and *supercritical* respectively. This phase transition was discovered by Erdős and Rényi in their seminal paper [22]; indeed, they further observed that in the *critical* case, when $p = 1/n$, the largest components of $G(n,p)$ have sizes of order $n^{2/3}$. For this reason, the phase transition in random graphs is sometimes dubbed the *double jump*.

Understanding the critical random graph (when $p = p(n) \sim 1/n$) requires a different and finer scaling: the natural parameterization turns out to be of the form $p = p(n) = 1/n + \lambda n^{-4/3}$, for $\lambda = o(n^{1/3})$ [12, 37, 40]. In this paper, we will restrict our attention to $\lambda \in \mathbb{R}$; this parameter range is then usually called the *critical window*. One of the most significant results about random graphs in the critical regime was proved by Aldous [7]. He observed that one could encode various aspects of the structure of the random graph (specifically, the sizes and surpluses of the components) using stochastic processes. His insight was that standard limit theory for such processes could then be used to get at the relevant limiting quantities which could, moreover, be analyzed using powerful stochastic-process tools. Fix $\lambda \in \mathbb{R}$, set $p = 1/n + \lambda n^{-4/3}$ and write Z_i^n and S_i^n for the size and surplus (that is, the number of edges which would need to be removed in order to obtain a tree) of $\mathcal{C}_i^n$, the i-th largest component of $G(n,p)$. Set $\mathbf{Z}^n = (Z_1^n, Z_2^n, \ldots)$ and $\mathbf{S}^n = (S_1^n, S_2^n, \ldots)$.

Theorem 1 (Aldous [7]) *As $n \to \infty$.*

$$(n^{-2/3}\mathbf{Z}^n, \mathbf{S}^n) \xrightarrow{d} (\mathbf{Z}, \mathbf{S}).$$

Here, the convergence of the first co-ordinate takes place in $\ell^2_{\searrow}$, the set of infinite sequences $(x_1, x_2, \ldots)$ with $x_1 \geq x_2 \geq \cdots \geq 0$ and $\sum_{i \geq 1} x_i^2 < \infty$. (See also [29, 40].) The limit $(\mathbf{Z}, \mathbf{S})$ is described in terms of a Brownian motion with parabolic drift, $(W^\lambda(t), t \geq 0)$, where

$$W^\lambda(t) := W(t) + t\lambda - \frac{t^2}{2}$$

and $(W(t), t \geq 0)$ is a standard Brownian motion. The limit $\mathbf{Z}$ has the distribution of the ordered sequence of lengths of excursions of the reflected process $W^\lambda(t) - \min_{0 \leq s \leq t} W^\lambda(s)$ above 0, while $\mathbf{S}$ is the sequence of numbers of points of a Poisson point process with rate one in $\mathbb{R}^+ \times \mathbb{R}^+$ lying under the corresponding excursions. Aldous's limiting picture has since been extended to "immigration" models of random graphs [8], hypergraphs [27] and, most recently, to random regular graphs with fixed degree [44].

The purpose of this paper is to give a precise description of the limit of the sequence of *components* $\mathcal{C}^n = (\mathcal{C}_1^n, \mathcal{C}_2^n, \ldots)$. Here, we view $\mathcal{C}_1^n, \mathcal{C}_2^n, \ldots$ as metric spaces $M_1^n, M_2^n, \ldots$, where the metric is the usual graph distance, which we rescale by $n^{-1/3}$. The limit object is then a sequence of compact metric spaces $\mathbf{M} = (M_1, M_2, \ldots)$. The appropriate topology for our convergence result is that generated by the Gromov–Hausdorff distance on the set of compact metric spaces, which we now define. Firstly, for a metric space (M, δ), write d_H for the Hausdorff distance between two compact subsets K, K' of M, that is

$$d_H(K, K') = \inf\{\epsilon > 0 : K \subseteq F_\epsilon(K') \text{ and } K' \subseteq F_\epsilon(K)\},$$

where $F_\epsilon(K) := \{x \in M : \delta(x, K) \leq \epsilon\}$ is the ϵ-fattening of the set K. Suppose now that X and X' are two compact metric spaces, each "rooted" at a distinguished point, called ρ and ρ' respectively. Then we define the *Gromov–Hausdorff* distance between X and X' to be

$$d_{GH}(X, X') = \inf\{d_H(\phi(X), \phi'(X')) \vee \delta(\phi(\rho), \phi(\rho'))\}$$

where the infimum is taken over all choices of metric space (M, δ) and all isometric embeddings $\phi : X \to M$ and $\phi' : X' \to M$. (Throughout the paper, when viewing a connected labeled graph G as a metric space, we will consider G to be rooted at its vertex of smallest label.) The main result of the paper is the following theorem.

Theorem 2 *As $n \to \infty$,*

$$(n^{-2/3}\mathbf{Z}^n, n^{-1/3}\mathbf{M}^n) \xrightarrow{d} (\mathbf{Z}, \mathbf{M}),$$

for an appropriate limiting sequence of metric spaces $\mathbf{M} = (M_1, M_2, \ldots)$. Convergence in the second co-ordinate here is in the metric specified by

$$d(\mathbf{A}, \mathbf{B}) = \left(\sum_{i=1}^{\infty} d_{\mathrm{GH}}(A_i, B_i)^4 \right)^{1/4} \tag{1}$$

for any sequences of metric spaces $\mathbf{A} = (A_1, A_2, \ldots)$ and $\mathbf{B} = (B_1, B_2, \ldots)$.

We will eventually state and prove a more precise version of this theorem, Theorem 24, once we have introduced the appropriate limiting sequence. For the moment, we will remark only that convergence in the distance defined above implies convergence of distances in the graph, and so our results can be used to answer many questions about the asymptotic distribution of distances between vertices of $G(n, p)$ inside the critical window. We highlight one of the most important such consequences below in Theorem 5, and treat several others in a companion paper [3].

Before we can give an intuitive description of our limit object, we need to introduce one of its fundamental building blocks: the continuum random tree.

Fig. 1 Left: a tree on [9]. Right: the same tree but labeled in depth-first order.

THE CONTINUUM RANDOM TREE. In recent years, a huge literature has grown up around the notion of *random real trees*. Here we will concentrate on the most famous random example of such trees, Aldous' Brownian continuum random tree (see [4–6]), and encourage the interested reader to look at [20, 23, 35] and the references therein for more general cases.

The fundamental idea is to encode tree structures using functions. We will begin our discussion by considering a rooted combinatorial tree on n vertices labeled by $[n] := \{1, 2 \ldots, n\}$. There are two (somewhat different) encodings of such a tree which will be useful to us. We will introduce one of them here and explain the second, which plays a more technical role, in the main body of the paper (see Section 2). We need to introduce the notion of the *depth-first ordering* of the vertices. For each vertex v, there is a unique path from v to the root, ρ. Call the vertices along this path the *ancestors* of v. Relabel each vertex by the string which consists of the concatenation of all the labels of its ancestors and its own label, so that if the path from ρ to v is $\rho, a_1, a_2, \ldots, a_m, v$, relabel v by the string $\rho a_1 a_2 \ldots a_m v$. The depth-first ordering of the vertices is then precisely the lexicographical ordering on these strings. More intuitively, we look first at the root, then at its lowest-labeled child, then at the lowest-labeled child of that vertex, and so on until we hit a leaf. Then we backtrack one generation and look at the next lowest-labeled child and its descendants as before. See Figure 1.

The first encoding is in terms of the *height function* (or, when the tree is random, the *height process*). For $0 \leq i \leq n-1$, let $H(i)$ be the graph distance from the root of the $(i+1)$-st vertex visited in depth-first order (so that $H(0) = 0$, since we always start from the root). Then the height function of the tree is the discrete function $(H(i), 0 \leq i \leq n-1)$. (See Figure 2.) (It is, perhaps, unfortunate that we talk about a *depth*-first ordering and vertices having *heights*. The reason for this is that the two pieces of terminology originated in different communities. However, since both are now standard, we have chosen to keep them and hope that the reader will forgive the ensuing clumsiness.)

Note that the topology of the tree, but not the labels, can be recovered from the height function.

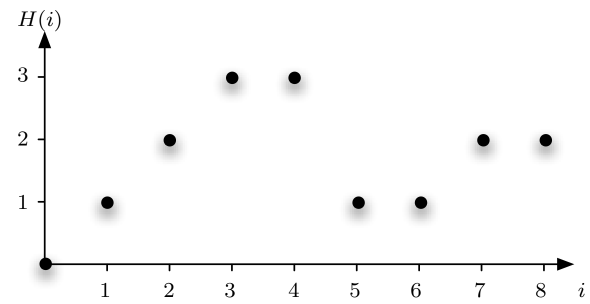

Fig. 2 The height process associated with the tree in Figure 1.

Suppose now that we take a uniform random tree on $[n]$, rooted at 1. Let $(H^n(i), 0 \leq i \leq n-1)$ be its height process. For convenience, set $H^n(n) = 0$.)

Theorem 3 (Aldous [6]) *Let $(e(t), 0 \leq t \leq 1)$ be a standard Brownian excursion. Then, as $n \to \infty$,*

$$\frac{1}{\sqrt{n}}(H^n(\lfloor nt \rfloor), 0 \leq t \leq 1) \xrightarrow{d} 2(e(t), 0 \leq t \leq 1).$$

Here, convergence is in the space $\mathbb{D}([0,1], \mathbb{R}^+)$ of non-negative càdlàg (right-continuous with left limits) functions, equipped with the Skorohod topology (see, for example, Billingsley [10]).

In fact, this convergence turns out to imply that the *tree itself* converges, in a sense which we will now make precise. We follow the exposition of Le Gall [35]. Take a uniform random tree on $[n]$ and view it as a path metric space T_n by taking the union of the line segments joining the vertices, each assumed to have length 1. (Note that the original tree-labels no longer play any role, except that we will think of the metric space T_n as being rooted at the point corresponding to the old label 1.) Then the distance between two elements x and y of T_n is simply the length of the shortest path between them; we will write $d_{T_n}(x, y)$ for this distance. We will abuse notation somewhat and write $n^{-1/2}T_n$ for the same metric space with all distances rescaled by $n^{-1/2}$.

In order to state the convergence result, we need to specify the limit object. We will start with some general definitions.

A compact metric space $(\mathcal{T}, d)$ is a *real tree* if for all $x, y \in \mathcal{T}$

- there exists a unique geodesic from x to y i.e. there exists a unique isometry $f_{x,y} : [0, d(x,y)] \to \mathcal{T}$ such that $f_{x,y}(0) = x$ and $f_{x,y}(d(x,y)) = y$. The image of $f_{x,y}$ is called $[\![x, y]\!]$;
- the only non-self-intersecting path from x to y is $[\![x, y]\!]$ i.e. if $q : [0,1] \to \mathcal{T}$ is continuous and injective and such that $q(0) = x$ and $q(1) = y$ then $q([0,1]) = [\![x, y]\!]$.

An element $x \in \mathcal{T}$ is called a *vertex*. A *rooted real tree* is a real tree $(\mathcal{T}, d)$ with a distinguished vertex ρ called the *root*. The *height* of a vertex v is $d(\rho, v)$.

By a *leaf*, we mean a vertex v which does not belong to $[\![\rho, w]\!]$ for any $w \neq v$. Write $\mathcal{L}(\mathcal{T})$ for the set of leaves of $\mathcal{T}$. Finally, write $[\![x, y[\![$ for $f_{x,y}([0, d(x, y)))$.

Suppose now that $h : [0, \infty) \to [0, \infty)$ is a continuous function of compact support such that $h(0) = 0$. Use it to define a pseudo-metric d by

$$d(x, y) = h(x) + h(y) - 2 \inf_{x \wedge y \leq t \leq x \vee y} h(t), \qquad x, y \in [0, \infty).$$

Let $x \sim y$ if $d(x, y) = 0$, so that $\sim$ is an equivalence relation. Let $\mathcal{T} = [0, \infty)/\sim$ and denote by $\tau : [0, \infty) \to \mathcal{T}$ the canonical projection. If σ is the supremum of the support of h then note that $\tau(s) = 0$ for all $s \geq \sigma$. This entails that $\mathcal{T} = \tau([0, \sigma])$ is compact. The metric space $(\mathcal{T}, d)$ can then be shown to be a real tree. Set $\rho = \tau(0)$ and take ρ to be the root.

Now take

$$h(t) = \begin{cases} 2e(t) & 0 \leq t \leq 1, \\ 0 & t > 1, \end{cases}$$

where, as in Theorem 3, and for the rest of the paper, $(e(t), 0 \leq t \leq 1)$ is a standard Brownian excursion. Then the resulting tree is the *Brownian continuum random tree* (or *CRT*, when this is unambiguous). We will always think of the CRT as rooted. We then have the following.

Theorem 4 (Aldous [6], Le Gall [36]) *Let T_n be the metric space corresponding to a uniform random tree on $[n]$ and let $\mathcal{T}$ be the CRT. Then*

$$n^{-1/2} T_n \xrightarrow{d} \mathcal{T},$$

as $n \to \infty$, where convergence is in the Gromov–Hausdorff sense.

It is perhaps useful to note here that the limit tree $\mathcal{T}$ comes equipped with a *mass measure* μ, which is simply the probability measure induced on $\mathcal{T}$ from Lebesgue measure on $[0, 1]$. Unsurprisingly, μ is the limit of the empirical measure on the uniform tree on $[n]$ which puts mass $1/n$ on each vertex. Later on, we will use the fact that $\mu(\mathcal{L}(\mathcal{T})) = 1$, i.e., μ is concentrated on the leaves of the CRT [5, p. 60].

THE LIMIT OF A CRITICAL RANDOM GRAPH. We now give a non-technical description of the limiting object in Theorem 2. Conditional on their size and surplus, components of $G(n, p)$ are uniform connected graphs with that size and surplus. Moreover, as we have discussed, in the critical window, where $p = n^{-1} + \lambda n^{-4/3}$ for some $\lambda \in \mathbb{R}$, the largest components have size of order $n^{2/3}$ and surplus of constant order. In order to understand better the structure of these components, we look at uniform connected graphs with "small" surplus. For definiteness, we will consider a uniform connected graph on m vertices with surplus s.

Such connected graphs always possess spanning subtrees. A particular one of these, which we will refer to as the *depth-first tree*, will be very useful to us. As its name suggests, this tree is constructed via a depth-first search procedure (which we will not detail until later). The depth-first tree of a uniform random connected graph with s surplus edges is *not* a uniform random tree, but has a "tilted" distribution which is biased (in a way depending on s) in favor of trees with large *area* (for "typical" trees, this is essentially the sum of the heights of the vertices of the tree). We will define this tilting precisely later, and will then spend much of the paper studying it.

The limit of a uniform random tree on m vertices, thought of as a metric space with graph distances rescaled by $m^{-1/2}$, is the continuum random tree. It turns out that the limit of the depth-first tree associated with a connected component with surplus s, with the same rescaling, is a continuum random tree coded by a Brownian excursion whose distribution is biased in favor of excursions having large area (where area now has its habitual meaning; once again, the bias depends on s).

The difference between the depth-first tree and the connected graph is precisely the s surplus edges. The depth-first tree is convenient because not only can we describe its continuum limit but, given the tree, it is straightforward to describe where surplus edges may go (we call such locations *permitted edges*). Indeed, the surplus edges are equally likely to be any of the possible s-sets of permitted edges. A careful analysis of the locations of the surplus edges in the finite graph leads to the following surprisingly simple limit description for a uniform random connected graph with surplus s. Take a continuum random tree with tilted distribution and independently select s of its leaves with density proportional to their height. For each of these leaves there is a unique path to the root of the tree. Independently for each of the s leaves, pick a point uniformly along the path and identify the leaf and the selected point. (Note that we identify the points because edge-lengths have shrunk to 0 in the limit.)

Having thus described the limit of a *single* component of a critical random graph, it remains to describe the limit of the *collection* of components. The key is Aldous' description of the limiting sizes and surpluses of the components in terms of the excursions above 0 of the reflected Brownian motion with parabolic drift. The excursion lengths give the limiting component sizes. The auxilliary Poisson process of points with unit intensity under the graph of the reflected process gives the limit of the numbers of surplus edges. In fact, more is true. The excursions themselves can be viewed as coding the sequence of limits of depth-first trees of the components; the locations of the Poisson points under the excursions can be seen to correspond in a natural way to the locations of the surplus edges. Intuitively, the successive excursions are selected according to an inhomogeneous excursion measure associated with the process. Under this measure, the length and area of an excursion are related in precisely the correct "tilted" manner, so that, conditional on an excursion having length σ and s Poisson points, the metric space it codes has precisely the distribution of the limit of a uniform random connected graph on $\sim \sigma n^{2/3}$ vertices with s surplus edges, whose edge-lengths have been rescaled by $n^{-1/3}$.

The diameter of random graphs. The *diameter* of a connected graph is the largest distance between any pair of vertices of the graph. For a general graph G, we define the diameter of G to be the greatest distance between any pair of vertices lying in the same connected component.

The behavior of the diameter of $G(n,p)$ for $p = O(1/n)$ is a pernicious problem for which few detailed results were known until extremely recently [48]. (For references on distances in dense graphs $G(n,p)$ with p fixed, see [11].) In the subcritical phase, when $p = p(n) = 1/n + \lambda(n)n^{-4/3}$ and $\lambda \to -\infty$, Łuczak [39] showed that the diameter of $G(n,p)$ is within one of the largest diameter of a tree component with probability tending to one. Chung and Lu [15] focused on the early supercritical phase, when $np > 1$ and $np \leq c \log n$. (Problem 4 in their paper asks about the diameter of $G(n,p)$ inside the critical window.) More recently, Riordan and Wormald [48] have addressed the problem for the range $p = c/n$, $c > 1$ fixed, proving essentially best possible bounds on the behavior of the diameter for such p. They also have optimal results for the diameter in the parameter range $p = 1/n + \lambda n^{-4/3}$, with $\lambda = o(n^{1/3})$ and $\lambda \to \infty$ as $n \to \infty$. (Ding et al. [19] were the first to obtain the first-order asymptotics of the diameter in this entire range, but the precise chronology of results is a bit complicated; see Remark 1 in [48] for details.) When λ is fixed, $G(n,p)$ contains several complex (i.e., with multiple cycles) components of comparable size, and any one of them has a non-vanishing probability of accounting for the diameter. Nachmias and Peres [42] have shown that the greatest diameter of any connected component of $G(n,p)$ is with high probability $\Theta(n^{1/3})$ for p in this range; this result also follows trivially from work of Addario-Berry, Broutin, and Reed [1, 2] on the diameter of the minimum spanning tree of a complete graph in which each edge e has an independent uniform $[0,1]$ edge weight.

In this paper, we demonstrate how Theorem 2 allows us straightforwardly to derive precise results on the diameter of $G(n,p)$ for $p = 1/n + \lambda n^{-4/3}$ with λ fixed.

Theorem 5 *Suppose that $p = 1/n + \lambda n^{-4/3}$ for $\lambda \in \mathbb{R}$. Let D_i^n denote the diameter of the i-th largest connected component of $G(n,p)$. Let $D^n = \sup_{i \geq 1} D_i^n$ denote the diameter of $G(n,p)$ itself. For each $i \geq 1$ there is a random variable $D_i \geq 0$ with $\mathbb{E}[D_i] < \infty$ such that*

$$(n^{-1/3} D_i^n, i \geq 1) \xrightarrow{d} (D_i, i \geq 1).$$

Furthermore, there is a random variable $D \geq 0$ with an absolutely continuous distribution and $\mathbb{E}[D] < \infty$ such that $n^{-1/3} D^n \xrightarrow{d} D$.

Plan of the paper. The depth-first procedure is presented in Section 2. This procedure allows us to associate a "canonical" spanning tree $\tilde{T}_m^p$ to a connected component of $G(n,p)$ of size m. The distribution of $\tilde{T}_m^p$ is studied in Section 3. In Section 4, we describe the graphs obtained by adding random surplus edges to the trees $\tilde{T}_m^p$ and introduce the continuous limit of connected components conditional on their size. We then prove that a single component of $G(n,p)$ conditioned to have size m converges to its continuous counterpart when appropriately rescaled. Finally, very much as $G(n,p)$ may be obtained by taking a sequence of connected components which are independent given their sizes, the continuum limit of $G(n,p)$ can be constructed by first setting the sizes of the components to have the correct distribution, and then generating components independently. This is described in Section 5.

2 Depth-first search and random graphs

Let $G=(V,E)$ be an undirected graph with $V=[n]=\{1,\ldots,n\}$. The *ordered depth-first search forest* for G is the spanning forest of G obtained by running depth-first search (DFS) on G, using the rule that whenever there is a choice of which vertex to explore, the smallest-labeled vertex is always explored first. For clarity, we explain more precisely what we mean by this description, and introduce some relevant notation. For $i \geq 0$, we define the ordered set (or *stack* [see 16]) $\mathcal{O}_i$ of open vertices at time i, and the set $\mathcal{A}_i$ of the vertices that have already been explored at time i. We say that a vertex u has been *seen* at time i if $u \in \mathcal{O}_i \cup \mathcal{A}_i$. Let c_i be a counter which keeps track of how many components have been discovered up to time i.

oDFS(G)
INITIALIZATION Set $\mathcal{O}_0=(1)$, $\mathcal{A}_0=\emptyset$, $c_0=1$.
STEP i $(0 \leq i \leq n-1)$: Let v_i be the first vertex of $\mathcal{O}_i$ and let $\mathcal{N}_i$ be the set of neighbors of v_i in $[n] \setminus (\mathcal{A}_i \cup \mathcal{O}_i)$. Set $\mathcal{A}_{i+1} = \mathcal{A}_i \cup \{v_i\}$. Construct $\mathcal{O}_{i+1}$ from $\mathcal{O}_i$ by removing v_i from the start of $\mathcal{O}_i$ and affixing the elements of $\mathcal{N}_i$ in increasing order to the start of $\mathcal{O}_i \setminus \{v_i\}$. If now $\mathcal{O}_{i+1}=\emptyset$, add to it the lowest-labeled element of $[n] \setminus \mathcal{A}_{i+1}$ and set $c_{i+1}=c_i+1$. Otherwise, set $c_{i+1}=c_i$.

After step $n-1$, we have $\mathcal{O}_n=\emptyset$. We remark that this procedure defines a reordering $\{v_0,\ldots,v_{n-1}\}$ of $[n]$ and, for any G, **oDFS**(G) always sets $v_0=1$. We refer to DFS run according to this rule as *ordered DFS*. (The terminology *lexicographic-DFS* may seem natural; however, this has been given a slightly different definition by Corneil and Krueger [17].) We note also that we increment the counter c_i precisely when v_i is the last vertex explored in a component, so that $(c_i, 0 \leq i < n)$ really does count components. (We observe that if, in STEP i, we affix the elements of $\mathcal{N}_i$ to the *end* of $\mathcal{O}_i \setminus \{v_i\}$ instead of the start, we obtain the *breadth-first ordering* exploited by Aldous [7]; we will discuss this further in Section 5.

The forest corresponding to **oDFS**(G) consists of all edges xy such that for some $i \in \{0,\ldots,n-1\}$, x is the vertex explored at step i (so $x=v_i$) and $y \in \mathcal{N}_i$. We refer to this as the *ordered depth-first search forest* for G. (Note that, because of the way the nodes are marked open, the ordered depth-first search forest considered here is not the one usually defined by depth-first search in graphs as in [16] or [53].)

For $i=0,1,\ldots,n-1$, let $X(i)=|\mathcal{O}_i \setminus \{v_i\}|-(c_i-1)$. The process $(X(i), 0 \leq i < n)$ is called the *depth-first walk* of the graph G. (The terminology "walk"

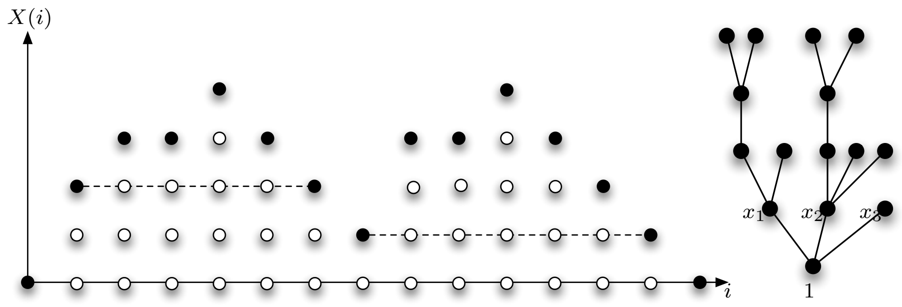

Fig. 3 The depth-first walk $(X(i), 0 \le i < 16)$ (black dots) of the tree T displayed on the right is shown. We have emphasized the integral points that contribute to the area $a(T)$ (white dots). The portions of the walk above the dashed lines correspond to the $\mathbf{oDFS}(T_1)$ and $\mathbf{oDFS}(T_2)$ processes (started from x_1 and x_2 respectively).

may seem odd here, but in the random context which is the focus of this paper, $(X(i), 0 \le i < n)$ turns out to be something like a random walk.)

We will particularly make use of these ideas in the case where G is connected. In that situation, we have $c_i = 1$ for all $0 \le i < n$ and so the algorithm is simpler. (In particular, the set of open vertices only becomes empty at the end of the procedure.) Furthermore, since G is connected, the ordered depth-first search forest is now a tree, which we will refer to as the *depth-first tree* and write $T(G)$. The depth-first walk $(X(i), 0 \le i < n)$ now has the simpler representation $X(i) = |\mathcal{O}_i \setminus \{v_i\}| = |\mathcal{O}_i| - 1$ for $0 \le i < n$ and can be interpreted as the number of vertices seen but not yet fully explored at step i of the $\mathbf{oDFS}(G)$ procedure. The following observation will be important later: the vertices in $\mathcal{O}_i \setminus \{v_i\}$ all lie at distance 1 from the path from the root to v_i. Put differently, the vertices of $\mathcal{O}_i \setminus \{v_i\}$ are all younger siblings of ancestors of v_i.

We next consider running **oDFS** on a tree $T = ([n], E)$. Of course, now the depth-first tree will be T itself. We define the *area* of a tree T to be

$$a(T) = \sum_{i=1}^{n-1} X(i).$$

This corresponds to the number of integral points in $\{(i,j) : 0 \le i < n, 0 \le j < X(i)\}$ (see Figure 3). Say that an edge $uv \notin E$ is *permitted by* $\mathbf{oDFS}(T)$ if, in the $\mathbf{oDFS}(T)$ procedure run on T, at some stage of the process, i and j are both seen but neither is fully explored: there exists $i \in \{0, \ldots, n-1\}$ such that $u, v \in \mathcal{O}_i$. The following lemma is then straightforward.

Lemma 6 *The number of edges permitted by* $\mathbf{oDFS}(T)$ *is precisely* $a(T)$.

Proof We proceed by induction on n. For $n = 1, 2$ the claim is clear. For $n \ge 3$, let $x_1, \ldots, x_i$ be the neighbors of 1 in T, listed in increasing order, and let $T_1, \dots, T_i$ be the trees containing $x_1, \dots, x_i$, respectively, when vertex 1 is removed from T.

By its definition, the procedure **oDFS**(T) simply uncovers $x_1, \dots, x_i$, then runs **oDFS**(T_j), for each $j = 1, \dots, i$, in this this order, but started (exceptionally) from x_j in each case. In particular, for each $j = 1, \dots, i$, each edge from x_j to $x \in T_k$, $k \leq j$ is permitted by **oDFS**(T). Thus, the total number of edges with one endpoint in $\{x_1, \dots, x_i\}$ permitted by **oDFS**(T) is precisely

$$\sum_{j=1}^{i} (i-j)|T_j|.$$

Write X_T and X_{T_j}, $1 \leq j \leq i$ in order to distinguish the depth-first walks on T and on its subtrees. By induction, it thus follows that the number of edges permitted by **oDFS**(T) is

$$\begin{aligned}\sum_{j=1}^{i}((i-j)|T_j| + a(T_j)) &= \sum_{j=1}^{i}\sum_{k=1}^{|T_j|}\big((i-j) + X_{T_j}(k)\big) \\ &= \sum_{j=1}^{i}\sum_{\ell=|T_1|+\ldots+|T_{j-1}|+1}^{|T_1|+\ldots+|T_j|} X_T(\ell) = a(T),\end{aligned}$$

since, for $0 \leq k < |T_j|$ and $\ell = |T_1| + \cdots + |T_{j-1}| + k$, the ℓ-th step of the **oDFS**(T) process explores a vertex v_ℓ of the tree T_j and $X_T(\ell) = i-j+X_{T_j}(k)$ (see Figure 3). □

The next lemma characterizes the connected graphs G which have a given depth-first tree. This lemma essentially appears in [26], though that paper uses slightly different terminology and a different canonical vertex ordering for **oDFS**. The correspondence of Spencer [51] is a precise analog of our lemma when the tree extracted from the connected graph is constructed by breadth-first search rather than depth-first search. (Spencer used this correspondence to show that the so-called "Wright constants" [54] are essentially factorial weightings of the moments of the area of a standard Brownian excursion.)

Lemma 7 *Given any tree T and connected graph G on $[n]$, $T(G) = T$ if and only if G can be obtained from T by adding some subset of the edges permitted by* **oDFS**(T)*.*

Proof First, if $T(G) = T$ then T is certainly a subgraph of G. Next, suppose that G can be obtained from T by adding a subset of the edges permitted by **oDFS**(T). We proceed by induction on k, the number of edges of G not in T. The case $k = 0$ is clear, so suppose $k \geq 1$ and let $v_0, \dots, v_{n-1}$ be the ordering of $[n]$ obtained by running **oDFS**(T). Let $v_i v_j$ be the lexicographically least edge of G not in T (written so that $i < j$).

Now, vertex v_i is explored at step i of **oDFS**(T). By our choice of $v_i v_j$, prior to step i the behavior of **oDFS**(T) and **oDFS**(G) is identical, so in particular $\mathcal{O}_i(T) = \mathcal{O}_i(G)$. Furthermore, since $v_i v_j$ is permitted by **oDFS**(T), and v_j is explored after v_i, we must have $v_j \in \mathcal{O}_i(T) = \mathcal{O}_i(G)$. Thus, $v_iv_j \notin T(G)$, and so $T(G) = T(G \setminus \{v_iv_j\})$. The "if" part of the lemma follows by induction.

Finally, suppose that G contains an edge not permitted by **oDFS**(T), and let v_iv_j be the lexicographically least such edge (in the order given by **oDFS**(T)). Then as before, the behavior of **oDFS**(T) and **oDFS**(G) is necessarily identical prior to step i, so in particular $v_j \notin A_i(T) = A_i(G)$. Furthermore, since v_iv_j is not permitted by **oDFS**(T), $v_j \notin \mathcal{O}_i(T) = \mathcal{O}_i(G)$. But $v_iv_j \in E(G)$, so we will have $v_j \in \mathcal{N}_i(G)$ and thus $v_iv_j \in T(G)$. Hence, $T(G) \neq T$, which proves the "only if" part of the lemma. □

Let $\mathbb{T}_{[n]}$ denote the set of trees on $[n]$ and write $\mathbb{G}_T$ for the set of connected graphs G with $T(G) = T$. Then it follows from Lemmas 6 and 7 that

$$\{\mathbb{G}_T : \ T \in \mathbb{T}_{[n]}\}$$

is a partition of the connected graphs on $[n]$, and that the cardinality of $\mathbb{G}_T$ is $2^{a(T)}$. Recall that the *surplus* of a connected graph G is the minimum number of edges that must be removed in order to obtain a tree, and call it $s(G)$. Then, for any $k \in \mathbb{Z}^+$, the number of graphs in $\mathbb{G}_T$ with surplus k is precisely $\binom{a(T)}{k}$. (We interpret $\binom{a}{k}$ as 0 if $k > a$ throughout the paper.)

We use these ideas to give a method of generating a connected component on a fixed number m of vertices. More precisely, write G_m^p for a graph with the same distribution as $G(m,p)$ but conditioned to be connected. Suppose that the connected components of $G(n,p)$ induce a partition $A_1, A_2, \ldots, A_r$ of $[n]$ with $|A_i| = m_i$ for $1 \leq i \leq r$. Then it is straightforward to see that, conditional on this partition, the components are independent. Moreover, for $1 \leq i \leq r$, the component on A_i has the same distribution as a copy of $G_{m_i}^p$ in which the vertices have been relabeled by A_i in an exchangeable manner. We will thus focus our attention on generating G_m^p.

Now fix $p \in (0,1)$. First pick a labeled tree $\tilde{T}_m^p$ on $[m]$ in such a way that $\mathbb{P}(\tilde{T}_m^p = T) \propto (1-p)^{-a(T)}$. Now add to $\tilde{T}_m^p$ each of the $a(\tilde{T}_m^p)$ edges permitted by **oDFS**$(\tilde{T}_m^p)$ independently with probability p, so that, given $\tilde{T}_m$, we add a Binomial$(a(\tilde{T}_m), p)$ number of surplus edges. Call the graph thus generated $\tilde{G}_m^p$.

Proposition 8 *For any $p \in (0,1)$ and $m \geq 1$, $\tilde{G}_m^p$ has the same distribution as G_m^p.*

Proof For a graph G on $[m]$, we write $s(G) = |E(G)| - (m-1)$, so if G is connected then $s(G)$ is its surplus. It is then immediate from the definition of G_m^p that, for a connected graph G on $[m]$,

$$\begin{aligned}\mathbb{P}\left(G_m^p = G\right) \propto \mathbb{P}\left(G(m,p) = G\right) &= p^{m-1+s(G)}(1-p)^{\binom{m}{2}-m+1-s(G)} \\ &\propto p^{s(G)}(1-p)^{-s(G)}.\end{aligned}$$

Also, by its definition,

$$\begin{aligned}\mathbb{P}(\tilde{G}_m^p = G) &\propto (1-p)^{-a(T)}\mathbb{P}(\tilde{G}_m^p = G \mid T(G) = T) \\ &= (1-p)^{-a(T)}p^{s(G)}(1-p)^{a(T)-s(G)},\end{aligned}$$

which completes the proof. □

Corollary 9 *Conditional on $s(\tilde{G}^p_m) = s \geq 0$, $\tilde{G}^p_m$ is a uniformly chosen connected graph on $[m]$ with $m+s-1$ edges, irrespective of the value of $p \in (0,1)$.*

3 Tilted trees and tilted excursions

In the introduction, we observed that twice the standard Brownian excursion appears as the limit of the height process $(H^n(i), 0 \leq i \leq n-1)$ of a uniform random tree on $[n]$ (see, e.g., [6, 35]). Let $(X^n(i), 0 \leq i \leq n-1)$ be the corresponding depth-first walk (and, for convenience, take $X^n(n) = 0$). Then

$$\frac{1}{\sqrt{n}}(X^n(\lfloor nt \rfloor), 0 \leq t \leq 1) \xrightarrow{d} (e(t), 0 \leq t \leq 1),$$

as $n \to \infty$, with convergence in $\mathbb{D}([0,1], \mathbb{R}^+)$ equipped with the Skorohod topology (see Marckert and Mokkadem [41]). (Note that the results in [41] are stated in the more general situation of an ordered Galton–Watson tree with an arbitrary offspring distribution having exponential moments, conditioned to have n vertices. If the offspring distribution is taken to be Poisson mean 1 then the conditioned tree has precisely the metric structure of a uniform labeled tree.) It is no coincidence that the limits of these two processes should be the same: they are not only the same in distribution, but are actually the *same* excursion.

Theorem 10 (Marckert and Mokkadem [41]) *As $n \to \infty$,*

$$\left(\frac{1}{\sqrt{n}}(X^n(\lfloor nt \rfloor), 0 \leq t \leq 1), \frac{1}{\sqrt{n}}(H^n(\lfloor nt \rfloor), 0 \leq t \leq 1)\right) \xrightarrow{d} (e, 2e).$$

We will make considerable use of this fact. An essential tool in what follows will be the following estimate on the distance between the depth-first walk and the height process (Theorem 3 of [41]).

Theorem 11 (Marckert and Mokkadem [41]) *For any $\nu > 0$, there exist n_ν and $\gamma > 0$ such that, for all $n \geq n_\nu$,*

$$\mathbb{P}\left(\sup_{0 \leq i < n}\left|X^n(i) - \frac{H^n(i)}{2}\right| \geq n^{1/4+\nu}\right) \leq e^{-\gamma n^\nu}.$$

In this section, we focus on understanding the distribution of the *tilted trees* $\tilde{T}^p_m$. Note that in the case of critical $G(n,p)$ the largest components have size m of order $n^{2/3}$ and $p \sim 1/n$, so that we shall take $p = p(m)$ of order $m^{-3/2}$. We write $\tilde{X}^m = (\tilde{X}^m(i), 0 \leq i < m)$ and $\tilde{H}^m = (\tilde{H}^m(i), 0 \leq i < m)$ for the depth-first walk and height process of $\tilde{T}^p_m$ (in **oDFS** order). Although it is usually impossible to reconstruct the labelling from $\tilde{X}^m$ or $\tilde{H}^m$, the structure of the trees (as unlabeled rooted ordered trees) can be recovered from either one. We start with a description of the scaling limit of these discrete excursions, which is closely related to the scaling limit of the corresponding processes, X^m and H^m, for uniform trees.

Write $\mathcal{E}$ for the space of excursions; that is,

$$\mathcal{E} = \{f \in C(\mathbb{R}^+, \mathbb{R}^+) : f(0) = 0, \exists\, \sigma < \infty \text{ s.t. } f(x) > 0\ \forall\, x \in (0, \sigma), \\ f(x) = 0\ \forall\, x \geq \sigma\}. \quad (2)$$

Given a function $f \in C([0, \sigma], \mathbb{R}^+)$ with $f(0) = f(\sigma) = 0$ and $f(x) > 0\ \forall\, x \in (0, \sigma)$, we will abuse notation by identifying f with the function $g \in \mathcal{E}$ which has $g(x) = f(x)$, $0 \leq x < \sigma$ and $g(x) = 0$, $x \geq \sigma$. (We will apply the same principle for discrete coding functions such as $\tilde{X}^m$ and $\tilde{H}^m$ and set $\tilde{X}^m(i) = \tilde{H}^m(i) = 0$ for $i \geq m$.) The distance of interest for us on $\mathcal{E}$ is given by the supremum norm: for a function $f \in C(\mathbb{R}^+, \mathbb{R})$, we write $\|f\| = \sup_{s \geq 0} |f(s)|$. Let $e^{(\sigma)} = (e^{(\sigma)}(t), 0 \leq t \leq \sigma)$ be a Brownian excursion of length $\sigma > 0$. We omit the superscript in the case of a standard Brownian excursion $(e(t), 0 \leq t \leq 1)$. Note that, by Brownian scaling, we have

$$(e^{(\sigma)}(t), 0 \leq t \leq \sigma) \stackrel{d}{=} (\sqrt{\sigma} \cdot e(t/\sigma), 0 \leq t \leq \sigma).$$

Then the *area under the excursion* $e^{(\sigma)}$ is

$$A(\sigma) := \int_0^\sigma e^{(\sigma)}(t) dt \stackrel{d}{=} \sigma^{3/2} \int_0^1 e(t) dt.$$

The random variable $A(2)$ has the so-called *Airy distribution*. This distribution has a rather complicated form but, for our purposes, it will suffice to note that its Laplace transform $\phi : \mathbb{C} \to \mathbb{C}$, given by $\phi(z) = \mathbb{E}[\exp(-z \int_0^1 e(t) dt)]$, is an entire function (see Janson [28] for details); in particular, it is finite for $z = -1$. For $\sigma > 0$, we define the *tilted excursion* of length σ, $\tilde{e}^{(\sigma)} = (\tilde{e}^{(\sigma)}(t), 0 \leq t \leq \sigma) \in \mathcal{E}$, to be an excursion whose distribution is characterized by

$$\mathbb{P}\left(\tilde{e}^{(\sigma)} \in \mathcal{B}\right) = \frac{\mathbb{E}\left[\mathbb{1}_{\{e^{(\sigma)} \in \mathcal{B}\}} \exp\left(\int_0^\sigma e^{(\sigma)}(t) dt\right)\right]}{\mathbb{E}\left[\exp\left(\int_0^\sigma e^{(\sigma)}(t) dt\right)\right]}, \quad (3)$$

for $\mathcal{B} \subseteq \mathcal{E}$ a Borel set. Here, the Borel sigma-algebra on $\mathcal{E}$ is that generated by open sets in the supremum norm $\|\cdot\|$. (Equation (3) gives a well-defined distribution since $\phi(-1) < \infty$.) As with the standard Brownian excursion, we will omit the superscript whenever the length of the tilted excursion is 1. As previously, let $\mathbb{D}([0, \sigma], \mathbb{R}^+)$ be the space of non-negative càdlàg paths on $[0, \sigma]$, equipped with the Skorohod topology.

Theorem 12 *Suppose that $p = p(m)$ is such that $mp^{2/3} \to \sigma$ as $m \to \infty$. Then, as $m \to \infty$,*

$$((m/\sigma)^{-1/2} \tilde{X}^m(\lfloor (m/\sigma) t \rfloor), 0 \leq t \leq \sigma) \xrightarrow{d} (\tilde{e}^{(\sigma)}(t), 0 \leq t \leq \sigma),$$

in $\mathbb{D}([0, \sigma], \mathbb{R}^+)$.

The proof consists in transferring known limits for uniform random trees over to tilted trees. We must first ensure that the change of measure defined by $\mathbb{P}(\tilde{T}_m^p = T) \propto (1-p)^{-a(T)}$ is well-behaved when $p = O(m^{-3/2})$. To do so, we shall in fact first derive tail bounds on the maximum of the depth-first walk. (Khorunzhiy and Marckert [33] have proved similar bounds for the maxima of Dyck paths, which are essentially the contour processes of Catalan trees.) Let T_m be a uniformly random tree on $[m]$, and let X^m be the associated depth-first walk.

Lemma 13 *There exist constants $C \geq 0$ and $\alpha > 0$ such that for all $m \in \mathbb{Z}^+$ and all $x \geq 0$,*

$$\mathbb{P}\left(\|X^m\| \geq x\sqrt{m}\right) \leq Ce^{-\alpha x^2}.$$

Proof To prove the lemma we use a connection with a queueing process which is essentially due to Borel [13]. Consider a queue with Poisson rate 1 arrivals and constant service time, started at time zero with a single customer in the queue. We may form a rooted tree (rooted at the first customer) associated with the queue process, run until the first time there are no customers in the queue, in the following manner: if a new customer joins the queue at time t, he is joined to the customer being served at time t. We denote the resulting rooted tree by $\mathscr{T}$. Then $\mathscr{T}$ is distributed as a Poisson(1) Galton–Watson tree and, hence, conditional on its size being m, as T_m (viewed as an unlabeled tree) [13]. Viewing the arrivals as given by a Poisson process $\mathcal{Q}$ of intensity 1 on $\mathbb{R}^+$, let $S_t = |\mathcal{Q} \cap [0,t)| - t$, for $t \in \mathbb{R}^+$, be the compensated process. Then $|\mathscr{T}|$ is precisely the first time t that $S_t = -1$, i.e., that $|\mathcal{Q} \cap [0,t)| = t-1$. Furthermore, $\{S_t, t = 1, 2, \ldots, |\mathscr{T}|-1\}$ is distributed precisely as the depth-first walk of $\mathscr{T}$. (It is not *equal* to the depth-first walk, but to the breadth-first walk discussed in Section 5.)

Using the above facts, we may thus generate $\mathscr{T}$ conditional upon $|\mathscr{T}| = m$ as follows. First let $U_1, \ldots, U_{m-1}$ be independent and uniformly distributed on $[0,m]$, and let $\mathcal{U} = \{U_1, \ldots, U_{m-1}\}$, so that $\mathcal{U}$ is distributed as $\mathcal{Q} \cap [0,m)$ conditional on $|\mathcal{Q} \cap [0,m)| = m-1$. Next, let $\mu \in \{1, \ldots, m\}$ minimize $|\mathcal{U} \cap [0,\mu]| - \mu$, and apply a *cyclic rotation by* $-\mu$ to all the points in $\mathcal{U}$ to obtain $\mathcal{U}'$. In other words, let

$$\mathcal{U}' = \{(U_1 - \mu) \bmod m, \ldots, (U_{m-1} - \mu) \bmod m\}.$$

Then m is precisely the first time t that $|\mathcal{U}' \cap [0,t)| = t-1$, and $\mathcal{U}'$ is distributed precisely as $|\mathcal{Q} \cap [0,m)|$ conditional on $|\mathscr{T}| = m$ (see [21]; this type of "rotation argument" was introduced in [9]).

Now let $X_i = |\mathcal{U}' \cap [0,i)| - i$ for $i = 1, \ldots, m$. By the above, we have that $\{X_1^m, \ldots, X_{m-1}^m, -1\}$ and $\{X_1, \ldots, X_m\}$ are identically distributed. In particular,

$$\begin{aligned}\|X^m\| &\stackrel{d}{=} \max_{0\le i\le m} X_i \\ &= \max_{0\le i\le m} (|\mathcal{U}\cap[0,i)| - i) - \min_{0\le i\le m} (|\mathcal{U}\cap[0,i)| - i) - 1 \\ &= \max_{0\le i\le m} (|\mathcal{U}\cap[0,i)| - i) + \max_{0\le i\le m} (|\mathcal{U}\cap[i,m)| - (m-i)) \\ &\le \sup_{0\le t\le m} (|\mathcal{U}\cap[0,t)| - t) + \sup_{0\le t\le m} (|\mathcal{U}\cap[t,m)| - (m-t)).\end{aligned}$$

The two suprema in the last line are identically distributed, and so

$$\mathbb{P}\left(\|X^m\| \ge 2x\right) \le 2\mathbb{P}\left(\sup_{0\le t\le m} (|\mathcal{U}\cap[0,t)| - t) \ge x\right). \tag{4}$$

For any fixed $t \in [0,m]$, let P_t be the event that there is a point of $\mathcal{U}$ at t. For fixed $x \ge 0$ and $t \in [0,m]$, $E_{t,x}$ be the event that $|\mathcal{U}\cap[0,t]| = \lceil t+x\rceil$ but that $|\mathcal{U}\cap[0,s)| < s+x$ for all $0 \le s \le t$ (so in particular, there is a point at t). We then have

$$\begin{aligned}&\mathbb{P}(E_{t,x} \mid |\mathcal{U}\cap[0,t]| = \lceil t+x\rceil, P_t) \\ &= \mathbb{P}(|\mathcal{U}\cap[0,s)| < s+x \;\forall\; 0\le s<t \mid |\mathcal{U}\cap[0,t)| = \lceil t+x\rceil - 1).\end{aligned}$$

Applying the ballot theorem for stochastic processes to this probability (see, e.g., [52] or p. 218 of [31]), we obtain the bound

$$\mathbb{P}(E_{t,x} \mid |\mathcal{U}\cap[0,t]| = \lceil t+x\rceil, P_t) \le 1 - \frac{\lceil t+x\rceil - 1}{t+x} < \frac{1}{t}. \tag{5}$$

Furthermore, in order for $\{\sup_{0\le t\le m}(|\mathcal{U}\cap[0,t)| - t) \ge x\}$ to occur, $E_{t,x}$ must occur for some $0 \le t \le m$. Since an infinitesimal interval $[t, t+dt]$ contains a point of $\mathcal{U}$ with probability $dt(m-1)/m$, and for each t, $|\mathcal{U}\cap[0,t)|$ is distributed as $\mathrm{Bin}(m-1, t/m)$, it follows from (4) and (5) that

$$\begin{aligned}\mathbb{P}\left(\|X^m\| \ge 2x\right) &\le 2\int_0^m \mathbb{P}\left(E_{t,x}\right) dt \\ &< 2\int_0^m \frac{1}{t}\mathbb{P}\left(|\mathcal{U}\cap[0,t)| = \lceil t+x\rceil\right)\frac{(m-1)}{m} dt \\ &\le 2\int_0^m \frac{1}{t} e^{-x^2/(2(t+x/3))} dt,\end{aligned}$$

where the last inequality follows from Chernoff's bound for Binomial random variables (see, for example, [30]). The conclusion follows easily for $x \le m/2$, and thus for all x (since we always have $\|X^m\| < m$). □

Lemma 14 *There exist universal constants $K, \kappa > 0$ such that the following holds. Fix $c > 0$ and $\xi > 0$, and suppose that $p \in (0,1)$ and $m \in \mathbb{Z}^+$ are such that $p \le cm^{-3/2}$. Let T_m be a uniform random tree on $[m]$. Then*

$$\mathbb{E}[(1-p)^{-\xi a(T_m)}] < Ke^{\kappa c^2 \xi^2}.$$

Proof Fix c and ξ as above, and let $\lambda = 2c\xi$. We may clearly restrict our attention to m sufficiently large that $p \le cm^{-3/2} \le 1/2$. For all such m we have

$$(1-p)^{-\xi a(T_m)} \le e^{2\xi p a(T_m)} \le e^{\lambda m^{-3/2} a(T_m)},$$

so it suffices to prove that $\sup_{m\ge 1} \mathbb{E}[e^{\lambda m^{-3/2} a(T_m)}] < Ke^{\kappa\lambda^2/4}$ for some universal constants K and κ. But $a(T_m) \le m\|X^m\|$, and so

$$\mathbb{P}(m^{-3/2} a(T_m) \ge x) \le \mathbb{P}(m^{-1/2}\|X^m\| \ge x).$$

Since $\|X^m\| \le m$, it follows by Lemma 13 that

$$\begin{aligned} \mathbb{E}\left[e^{\lambda m^{-3/2} a(T_m)}\right] &\le \int_0^{m^{1/2}} e^{\lambda x} \mathbb{P}(m^{-1/2}\|X^m\| \ge x) dx \\ &\le \int_0^{m^{1/2}} e^{\lambda x} \cdot Ce^{-\alpha x^2} dx. \end{aligned} \tag{6}$$

Completing the square in the last integrand so as to express the right-hand side as a Gaussian integral yields the claim with $\kappa = \alpha^{-2}$ and $K = C\sqrt{\pi/\alpha}$.
□

Proof of Theorem 12 We assume $\sigma = 1$ for notational simplicity; the general result follows by Brownian scaling. Again, let $(X^m(i), 0 \le i \le m)$ be the depth-first walk associated with a uniformly-chosen labeled tree. Its area is $a(T_m) = \sum_{i=0}^{m-1} X^m(i)$. We know from Theorem 10 that

$$(m^{-1/2} X^m(\lfloor mt \rfloor), 0 \le t \le 1) \xrightarrow{d} (e(t), 0 \le t \le 1). \tag{7}$$

We will henceforth want to think of X^m as a function in $\mathbb{D}([0,1], \mathbb{R}^+)$ and will write $\bar{X}^m(t)$ instead of $m^{-1/2} X^m(\lfloor mt \rfloor)$. For $h \in \mathbb{D}([0,1], \mathbb{R}^+)$, let $I(h) = \int_0^1 h(t)dt$; I is a continuous functional of the path h. It then follows from (7) that

$$m^{-3/2} a(T_m) = \frac{1}{m^{3/2}} \sum_{i=0}^{m-1} X^m(i) = \int_0^1 \bar{X}^m(t) dt \xrightarrow{d} \int_0^1 e(t) dt,$$

jointly with the convergence in distribution of the depth-first walk.

Now suppose that $f : \mathbb{D}([0,1], \mathbb{R}^+) \to \mathbb{R}^+$ is any bounded continuous function. Then

$$\mathbb{E}\left[f\left(m^{-1/2} \tilde{X}^m(\lfloor mt \rfloor), 0 \le t \le 1\right)\right] = \frac{\mathbb{E}\left[f(\bar{X}^m)(1-p)^{-m^{3/2}\int_0^1 \bar{X}^m(t)dt}\right]}{\mathbb{E}\left[(1-p)^{-m^{3/2}\int_0^1 \bar{X}^m(t)dt}\right]}$$

and

$$\mathbb{E}\left[f(\tilde{e})\right] = \frac{\mathbb{E}\left[f(e)\exp\left(\int_0^1 e(t)dt\right)\right]}{\mathbb{E}\left[\exp\left(\int_0^1 e(t)dt\right)\right]}.$$

Since $\int_0^1 \bar{X}^m(t)dt \xrightarrow{d} \int_0^1 e(t)dt$ and $p \sim m^{-3/2}$, we also have

$$(1-p)^{-m^{3/2}\int_0^1 \bar{X}^m(t)dt} \xrightarrow{d} \exp\left(\int_0^1 e(t)dt\right).$$

By Lemma 14, the above sequence is uniformly integrable, and so we can deduce that

$$\mathbb{E}[f(m^{-1/2}\tilde{X}^m(\lfloor mt \rfloor), 0 \le t \le 1)] \to \mathbb{E}[f(\tilde{e})],$$

which implies that $(m^{-1/2}\tilde{X}^m(\lfloor mt \rfloor), 0 \le t < 1) \xrightarrow{d} \tilde{e}$. □

Unsurprisingly, as in the case of uniform trees, we also have convergence of the height processes of a sequence of tilted trees towards a tilted excursion.

Theorem 15 *Suppose that $p = p(m)$ is such that $mp^{2/3} \to \sigma$ as $m \to \infty$. Then, as $m \to \infty$,*

$$((m/\sigma)^{-1/2}\tilde{H}^m(\lfloor (m/\sigma)t \rfloor), 0 \le t \le \sigma) \xrightarrow{d} (2\tilde{e}^{(\sigma)}(t), 0 \le t \le \sigma)$$

in $\mathbb{D}([0,\sigma], \mathbb{R}^+)$.

Theorem 15 follows straightforwardly from the following lemma and Lemma 14, much as Theorem 12 folllowed from Lemma 14; its proof is omitted.

Lemma 16 *Suppose that $p = p(m)$ is such that $mp^{2/3} \to \sigma$ as $m \to \infty$. For $m \ge 1$ let $\tilde{T}_m^p$ be a tree on $[m]$ sampled according to the distribution $\mathbb{P}(\tilde{T}_m^p = T) \propto (1-p)^{-a(T)}$. Let $\tilde{X}^m$ and $\tilde{H}^m$ be the associated depth-first walk and height process. Then, there are constants K and $m_0 \ge 0$, such that for all $m \ge m_0$.*

$$\mathbb{P}(\|\tilde{X}^m - \tilde{H}^m/2\| \ge m^{3/8}) \le Km^{-1/16}.$$

Proof Let T_m be a tree on $[m]$ chosen uniformly at random, and write X^m and H^m for its depth-first walk and height process. Then, by definition,

$$\mathbb{P}(\|\tilde{X}^m - \tilde{H}^m/2\| \ge m^{3/8}) = \frac{\mathbb{E}[\mathbb{1}_{\{\|X^m - H^m/2\| \ge m^{3/8}\}}(1-p)^{-a(T_m)}]}{\mathbb{E}[(1-p)^{-a(T_m)}]}.$$

Distinguishing between the trees with a "large" area, $a(T_m) \ge m^{25/16}$, and the others, we obtain

$$\begin{aligned}
&\mathbb{P}(\|\tilde{X}^m - \tilde{H}^m/2\| \ge m^{3/8}) \\
&\le \frac{\mathbb{E}[\mathbb{1}_{\{a(T_m) \ge m^{25/16}\}}(1-p)^{-a(T_m)}]}{\mathbb{E}[(1-p)^{-a(T_m)}]} + \frac{\mathbb{P}(\|X^m - H^m/2\| \ge m^{3/8})(1-p)^{-m^{25/16}}}{\mathbb{E}[(1-p)^{-a(T_m)}]} \\
&\le \mathbb{P}(a(\tilde{T}_m) \ge m^{25/16}) + \frac{e^{-\gamma m^{1/8}} e^{m^{1/16}}}{\mathbb{E}[(1-p)^{-a(T_m)}]},
\end{aligned}$$

where the second inequality follows from the bound $\mathbb{P}(\|X^m - H^m/2\| \geq m^{3/8}) \leq e^{-\gamma m^{1/8}}$, for some $\gamma > 0$ and all m large enough, which is obtained from Theorem 11. By Markov's inequality,

$$\mathbb{P}(\|\tilde{X}^m - \tilde{H}^m/2\| \geq m^{3/8}) \leq \mathbb{E}[m^{-3/2}a(\tilde{T}_m)] \cdot m^{-1/16} + \frac{e^{-\gamma(1+o(1))m^{1/8}}}{\mathbb{E}[(1-p)^{-a(T_m)}]}.$$

Finally, by Theorem 12 and Theorem 10 together with Lemma 14 we have

$$\mathbb{E}\left[m^{-3/2}a(\tilde{T}_m)\right] \to \mathbb{E}\left[\int_0^\sigma \tilde{e}^{(\sigma)}(t)dt\right] < \infty$$

and

$$\mathbb{E}\left[(1-p)^{-a(T_m)}\right] \to \mathbb{E}\left[\exp\left(\int_0^\sigma e^{(\sigma)}(t)dt\right)\right] > 0,$$

as $m \to \infty$. □

To conclude this section, we observe that Theorem 15 yields the convergence of tilted trees in the Gromov–Hausdorff distance (an analog of Theorem 4 for tilted trees): we have $m^{-1/2}\tilde{T}_m^p \to \tilde{\mathcal{T}}$ in distribution, where $\tilde{\mathcal{T}}$ is the real tree encoded by $2\tilde{e}^{(\sigma)}$. Although we do not require this convergence in the sequel, we find it instructive to give a brief version of the argument because we will prove our main result along the same lines. By Brownian scaling, it is sufficient to consider the case $\sigma = 1$.

First, let us give an alternative formulation of the Gromov–Hausdorff distance. Let (X, d_X) and (Y, d_Y) be two metric spaces. Define a *correspondence* between X and Y to be a set $\mathcal{R} \subseteq X \times Y$ such that for every $x \in X$ there exists at least one y such that $(x, y) \in \mathcal{R}$ and for every $y \in Y$ there exists at least one x such that $(x, y) \in \mathcal{R}$. Let $\mathscr{C}(X, Y)$ be the set of correspondences between X and Y. Define the *distortion* of a correspondence $\mathcal{R}$ to be

$$\mathrm{dis}(\mathcal{R}) = \sup\{|d_X(x, x') - d_Y(y, y')| : (x, y), (x', y') \in \mathcal{R}\}.$$

Then it is standard [14] that

$$d_{\mathrm{GH}}(X, Y) = \frac{1}{2} \inf_{\mathcal{R} \in \mathscr{C}(X,Y)} \mathrm{dis}(\mathcal{R}).$$

We use the height processes of $\tilde{T}_m^p$ and $\tilde{\mathcal{T}}$ to give us a correspondence $\mathcal{R}_m$ between them. (We learnt this approach from Grégory Miermont.) Suppose that $m^{-1/2}\tilde{T}_m^p$ has vertices labeled $v_0, v_1, \ldots, v_{m-1}$ in depth-first order. Let τ denote the canonical projection from $[0, 1]$ onto $\tilde{\mathcal{T}}$. Declare that $(v_i, \tau(t)) \in \mathcal{R}_m$ if $i = \lfloor mt \rfloor$, $0 \leq t < 1$. In order to find the distortion of $\mathcal{R}_m$, we need to consider

$$\left| m^{-1/2} d_{\tilde{T}_m^p}(v_i, v_j) - d_{\tilde{\mathcal{T}}}(\tau(t), \tau(u)) \right|$$

for $(v_i, \tau(t)), (v_j, \tau(u)) \in \mathcal{R}_m$. We note a technical lemma whose proof is straightforward and thus omitted.

Lemma 17 *Suppose that T is a discrete tree with vertices $v_0, v_1, \ldots, v_{n-1}$ labeled in depth-first order. Write d_T for the graph distance on T and consider the height process H, i.e., $H(i) = d_T(v_0, v_i)$ for $0 \le i \le n-1$. Write $u \wedge v$ for the the common ancestor of vertices u and v furthest from v_0. Then for $0 \le i \le j \le n-1$,*

$$\left| d_T(v_0, v_i \wedge v_j) - \min_{i \le k \le j} H(k) \right| \le 1.$$

By Skorohod's representation theorem, there exists a probability space where the convergence

$$(m^{-1/2} \tilde{H}^m(\lfloor mt \rfloor), 0 \le t \le 1) \xrightarrow{d} 2\tilde{e} \tag{8}$$

of Theorem 15 occurs almost surely. Without loss of generality, we may work on this probability space. By Lemma 17, we have for any $t, u \in (0,1)$,

$$\begin{aligned} &\left| d_{\tilde{T}^p_m}(\lfloor mt \rfloor, \lfloor mu \rfloor) - d_{\tilde{\mathcal{T}}}(t,u) \right| \\ &\le \left| m^{-1/2} \tilde{H}^m(\lfloor mt \rfloor) + m^{-1/2} \tilde{H}^m(\lfloor mu \rfloor) - 2 \min_{t \wedge u \le r \le t \vee u} \tilde{H}^m(\lfloor mr \rfloor) \right. \\ &\qquad \left. - \left(2\tilde{e}(t) + 2\tilde{e}(u) - 4 \inf_{t \wedge u \le r \le t \vee u} \tilde{e}(r) \right) \right| + 2m^{-1/2}. \end{aligned}$$

which clearly goes to 0 uniformly in t and u by (8). It follows that $\mathrm{dis}(\mathcal{R}_m) \to 0$ as $m \to \infty$. As a consequence, $m^{-1/2} \tilde{T}^p_m \to \tilde{\mathcal{T}}$ in distribution in the Gromov–Hausdorff distance.

4 The limit of connected components

Generating connected components of the random graph. Consider a connected labeled graph G on m vertices, and recall that running the **oDFS** process on G produces the depth-first tree $T(G)$. Recall that the edges permitted by **oDFS** are those between vertices which both lie in the stack $\mathcal{O}_i$ for some i. By Lemma 7, G can be recovered from $T = T(G)$ by adding some specific subset of the permitted edges.

By Lemma 6, we may consider permitted edges to be in bijective correspondence with the integral points (i,j) lying above or on the x-axis and strictly under the depth-first walk $X = (X(i), 1 \le i < m)$ encoding T. Place a mark at the point (i,j) if there is an edge in G between v_i and $v_{k(i,j)}$ where $k(i,j) = \inf\{k \ge i : X(k) = j\}$. Call the resulting object a *marked depth-first walk*. Then clearly this gives a bijection between marked depth-first walks and connected graphs G.

For a tree T with depth-first walk X and a pointset $\mathcal{Q} \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$, let $G^X = G^X(T, \mathcal{Q})$ be the graph obtained by adding to T the edges corresponding to the points in $\mathcal{Q} \cap X$ where, for convenience, we define

$$\mathcal{S} \cap f := \{(x,y) \in \mathcal{S} : 0 \le y < f(x)\} \text{ for all } \mathcal{S} \subseteq \mathbb{R}^+ \times \mathbb{R}^+ \text{ and } f : \mathbb{R}^+ \to \mathbb{R}^+. \tag{9}$$

A *Binomial pointset of intensity* p is random subset of $\mathbb{Z}^+ \times \mathbb{Z}^+$ in which each point is present independently with probability p. The following lemma follows straightforwardly from Proposition 8.

Lemma 18 *Let $p = p \in (0,1)$. Let $\tilde{T}^p_m$ be a tree on $[m]$ sampled in such a way that $\mathbb{P}(\tilde{T}^p_m = T) \propto (1-p)^{-a(T)}$. Let $(\tilde{X}^m(i), 0 \leq i < m)$ be the associated depth-first walk. Let $\mathcal{Q}^p \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$ be a Binomial pointset with intensity p, independent of $\tilde{T}^p_m$. Then, the graph $G^X(\tilde{T}^p_m, \mathcal{Q}^p)$ has the same distribution as G^p_m.*

We now write $(m/\sigma)^{-1/2}\tilde{X}^m(\lfloor (m/\sigma)\cdot\rfloor)$ as shorthand for the process

$$((m/\sigma)^{-1/2}\tilde{X}^m(\lfloor (m/\sigma)t\rfloor), 0 \leq t \leq \sigma).$$

Lemma 19 *Let $p = p(m)$ be such that $mp^{2/3} \to \sigma$ as $m \to \infty$. Pick a labeled tree $\tilde{T}^p_m$ on $[m]$ in such a way that $\mathbb{P}(\tilde{T}^p_m = T) \propto (1-p)^{-a(T)}$ and let $\tilde{X}^m$ be the associated depth-first walk. Let $\mathcal{Q}^p \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$ be a Binomial pointset of intensity p. Let $\mathcal{P}_m = \{((m/\sigma)^{-1}i, (m/\sigma)^{-1/2}j) : (i,j) \in \mathcal{Q}^p\}$. Then*

$$\left((m/\sigma)^{-1/2}\tilde{X}^m(\lfloor (m/\sigma)\cdot\rfloor), \mathcal{P}_m \cap ((m/\sigma)^{-1/2}\tilde{X}^m(\lfloor (m/\sigma)\cdot\rfloor)\right) \xrightarrow{d} \left(\tilde{e}^{(\sigma)}, \mathcal{P} \cap \tilde{e}^{(\sigma)}\right)$$

as $n \to \infty$, where $\mathcal{P}$ is a homogeneous Poisson point process with intensity measure the Lebesgue measure $\mathscr{L}$ on $\mathbb{R}^+ \times \mathbb{R}^+$, and $\mathcal{P}$ is independent of $\tilde{e}^{(\sigma)}$. Convergence in the first co-ordinate is in $\mathbb{D}([0,\sigma], \mathbb{R}^+)$, and in the second co-ordinate is in the sense of the Hausdorff distance.

Proof We assume for notational simplicity that $\sigma = 1$; the result for general σ follows by Brownian scaling. (Note that the point process is rescaled independently of σ.) Let $k \geq 1$ and let $A_1, A_2, \ldots, A_k \subseteq [0,1] \times \mathbb{R}^+$ be disjoint measurable sets. Then for any $n \geq 1$ and any $1 \leq i \leq k$, the discrete counting function

$$N_m(A_i) = \#\{(\lfloor mx\rfloor, \lfloor m^{1/2}y\rfloor) \in \mathcal{Q}^p : (x,y) \in A_i\}$$

is a Binomial random variable with parameters $\eta_m(A_i) = \#\{(\lfloor mx\rfloor, \lfloor m^{1/2}y\rfloor) : (x,y) \in A_i, 0 < \lfloor m^{1/2}y\rfloor \leq \lfloor mx\rfloor\}$ and p. Since

$$m^{-3/2}\eta_m(A_i) \to \mathscr{L}(A_i)$$

and $m^{3/2}p \to 1$ as $m \to \infty$, $N_m(A_i) \xrightarrow{d} \text{Poisson}(\mathscr{L}(A_i))$ for $1 \leq i \leq k$. Moreover, the random variables $N_m(A_1), N_m(A_2), \ldots, N_m(A_k)$ are independent, since they count the points of $\mathcal{Q}^p$ in disjoint sets. Thus $\mathcal{P}_m \to \mathcal{P}$ in distribution [18, 34].

Suppose now that for each $m \geq 1$, $f_m : [0,1] \to \mathbb{R}^+$ is a continuous function, and that f_m converges uniformly to some function $f : [0,1] \to \mathbb{R}^+$. Then for any open set $A \subseteq [0,1] \times \mathbb{R}^+$, $\{(x,y) \in A : 0 < y < f_m(x)\} \to \{(x,y) \in A : 0 < y < f(x)\}$ (in the sense of the Hausdorff distance). It follows that $\mathcal{P}_m \cap f_m \to \mathcal{P} \cap f$ in distribution, since the Poisson process almost surely puts no points in the set $\{(x,y) \in A : y = 0\}$.

Now suppose that $g_m : \{0,1,\dots,m\} \to \mathbb{Z}^+$ and $(m^{-1/2} g_m(\lfloor mt \rfloor), 0 \le t \le 1) \to (g(t), 0 \le t \le 1)$ in $\mathbb{D}([0,1], \mathbb{R}^+)$, where g is continuous. Then letting $\tilde{g}_m : [0,1] \to \mathbb{R}^+$ be the continuous interpolation of $(m^{-1/2} g_m(mt) : t = 0, m^{-1}, 2m^{-1}, \dots, 1)$, we also have $\tilde{g}_m \to g$ uniformly. Moreover,

$$\mathcal{P}_m \cap (m^{-1/2} g_m(\lfloor mt \rfloor), 0 \le t \le 1) = \mathcal{P}_m \cap \tilde{g}_m$$

since the functions agree at lattice points. So we obtain that

$$\mathcal{P}_m \cap (m^{-1/2} g_m(\lfloor mt \rfloor), 0 \le t \le 1) \xrightarrow{d} \mathcal{P} \cap g.$$

Finally, since $\mathcal{P}_m$ and $\tilde{X}^m$ are independent, and $(m^{-1/2} \tilde{X}^m(\lfloor mt \rfloor), 0 \le t \le 1) \to (\tilde{e}(t), 0 \le t \le 1)$ in distribution by Theorem 15, it follows easily that

$$\mathcal{P}_m \cap (m^{-1/2} \tilde{X}^m(\lfloor mt \rfloor), 0 \le t \le 1) \xrightarrow{d} \mathcal{P} \cap \tilde{e},$$

jointly with the convergence of the depth-first walk. □

THE LIMIT OBJECT. For $p = p(n) = 1/n + \lambda n^{-4/3}$, $\lambda \in \mathbb{R}$, any single one of the largest components of the random graph $G(n,p)$ has (random) size $m = m(n) \sim \sigma n^{2/3}$ [7, 11, 30]. Conditioned on having size m, such a connected component has the same distribution as $G^X(\tilde{T}_m^p, \mathcal{Q}^p)$. Lemma 19 suggests that $G^X(\tilde{T}_m^p, \mathcal{Q}^p)$ should have a non-trivial limit in distribution when distances are rescaled by $n^{-1/3}$, which should, thus, also be the limit in distribution of G_m^p similarly rescaled. We will now define this limit object, $\mathcal{M}^{(\sigma)}$, by analogy with $G^X(\tilde{T}_m^p, \mathcal{Q}^p)$.

Let $\mathcal{T}$ be the real tree encoded by a height process h, as described in the introduction. Recall that we think of $\mathcal{T}$ as a metric space, rooted at a vertex we call ρ. Recall that τ denotes the canonical projection $\tau : [0,\sigma] \to \mathcal{T}$. Let $\mathcal{Q}$ be a pointset in $\mathbb{R}^+ \times \mathbb{R}^+$ such that there are only finitely many points in any compact set. To each point $\xi = (\xi^x, \xi^y) \in \mathcal{Q} \cap (h/2)$, there corresponds a (unique) vertex $\tau(\xi^x) \in \mathcal{T}$ of height $h(\xi^x)$. Now let $\hat{\tau}(\xi) = \hat{\tau}(\xi^x, \xi^y)$ be the vertex at distance $2\xi^y$ from ρ on the path $[\![\rho, \tau(\xi^x)]\!]$. In fact, $\hat{\tau}(\xi) = \tau(u(\xi))$ where $u(\xi) = \inf\{u \ge \xi^x : h(u) = 2\xi^y\}$. Define a new "glued" metric space $g(h, \mathcal{Q})$ by identifying the vertices $\tau(\xi^x)$ and $\hat{\tau}(\xi^x, \xi^y)$ in $\mathcal{T}$, for each point $\xi \in \mathcal{Q} \cap (h/2)$, and taking the obvious induced metric.

Now let $\mathcal{P}$ be a Poisson point process with intensity measure $\mathscr{L}$ on $\mathbb{R}^+ \times \mathbb{R}^+$, independent of $\tilde{e}^{(\sigma)}$, a tilted excursion of length σ. Note that $\mathcal{P}$ almost surely only has finitely many points in any compact set. Then we define the random metric space $\mathcal{M}^{(\sigma)} = g(2\tilde{e}^{(\sigma)}, \mathcal{P})$ and write $\mathcal{M} = \mathcal{M}^{(1)}$.

In Theorem 22 below, we will see that $\mathcal{M}^{(\sigma)}$ is indeed the scaling limit of a connected component of $G(n,p)$ conditioned to have size $m \sim \sigma n^{2/3}$. In order to do this, we show that the metric space corresponding to $G^X(\tilde{T}_m^p, \mathcal{Q}^p)$ and $\mathcal{M}^{(\sigma)}$ are close in the Gromov–Hausdorff distance for large $m \sim \sigma n^{2/3}$. The scaling limit of the whole of $G(n,p)$ is then a collection of such continuous random components with random sizes; the components of size $o(n^{2/3})$ rescale to trivial continuous limits. The proof of this is dealt with in Section 5.

It is possible to give a more intuitively appealing description of $\mathcal{M}^{(\sigma)}$. Take a continuum random tree with tilted distribution and independently select a Poisson number of its leaves, each picked with density proportional to its height. For each of these leaves there is a unique path to the root of the tree. Independently for each of the selected leaves, pick a point uniformly along the path and identify the leaf and the selected point. Before we move on, we justify this description.

Proposition 20 *Condition on the excursion $\tilde{e}^{(\sigma)}$ and let $\tilde{\mathcal{T}}$ be the tree encoded by $2\tilde{e}^{(\sigma)}$ with metric $d_{\tilde{\mathcal{T}}}$. Then using the same notation as above, the following statements hold.*

1. *The number $|\mathcal{P} \cap \tilde{e}^{(\sigma)}|$ of vertex identifications in $\mathcal{M}^{(\sigma)}$ has a Poisson distribution with mean $\int_0^\sigma \tilde{e}^{(\sigma)}(u)du$.*
2. *Given $|\mathcal{P} \cap \tilde{e}^{(\sigma)}| = s$, the points of $\mathcal{P} \cap \tilde{e}^{(\sigma)}$ have co-ordinates distributed as s independent copies of the random pair (ξ^x, ξ^y), where ξ^x has density*
$$\frac{\tilde{e}^{(\sigma)}(u)}{\int_0^\sigma \tilde{e}^{(\sigma)}(t)dt}$$
on $[0, \sigma]$ and, given ξ^x, ξ^y is uniformly distributed on $[0, \tilde{e}^{(\sigma)}(\xi^x)]$.
3. *For (ξ^x, ξ^y) having the distribution specified in 2., the vertex $\tau(\xi^x)$ is almost surely a leaf of $\tilde{\mathcal{T}}$. Moreover, $d_{\tilde{\mathcal{T}}}(\rho, \hat{\tau}(\xi^x, \xi^y)) \stackrel{d}{=} U d_{\tilde{\mathcal{T}}}(\rho, \tau(\xi^x))$, where U is a uniform random variable on $[0, 1]$, independent of ξ^x and $\tilde{e}^{(\sigma)}$.*

Proof It is useful to keep in mind that there are two (independent) sources of randomness: one which gives rise to the excursion $\tilde{e}^{(\sigma)}$ (and hence the tree $\tilde{\mathcal{T}}$), and another which gives rise to the point process $\mathcal{P}$.

The first statement is immediate from the construction. It is standard that, given that $|\mathcal{P} \cap \tilde{e}^{(\sigma)}| = s$, the points of $\mathcal{P} \cap \tilde{e}^{(\sigma)}$ are independently and uniformly distributed in the area under $\tilde{e}^{(\sigma)}$. It is then straightforward to see that ξ^x has claimed density on $[0, \sigma]$ and that, conditional on ξ^x, ξ^y must be uniform in the set of values it may take i.e. $[0, \tilde{e}^{(\sigma)}(\xi^x)]$.

Hence,

$$\mathbb{P}(\tau(\xi^x) \in \mathcal{L}(\tilde{\mathcal{T}}) \mid \tilde{e}^{(\sigma)}) = \int_0^\sigma \mathbb{1}_{\{\tau(u) \in \mathcal{L}(\tilde{\mathcal{T}})\}} \frac{\tilde{e}^{(\sigma)}(u)}{\int_0^x \tilde{e}^{(\sigma)}(t)dt} du.$$

Let $\tilde{\mu}$ be the natural measure induced on $\tilde{\mathcal{T}}$ from Lebesgue measure on $[0, \sigma]$. Since the distribution of $\tilde{e}^{(\sigma)}$ is absolutely continuous with respect to that of $e^{(\sigma)}$, a Brownian excursion of length σ, and $\mu(\mathcal{L}(\mathcal{T})) = \sigma$ [5, p. 60], where μ is the corresponding measure on $\mathcal{T}$, we must also have $\tilde{\mu}(\mathcal{L}(\tilde{\mathcal{T}})) = \sigma$. It follows that

$$\mathbb{P}(\tau(\xi^x) \in \mathcal{L}(\tilde{\mathcal{T}}) \mid \tilde{e}^{(\sigma)}) = 1$$

for almost all $\tilde{e}^{(\sigma)}$. Hence, $\tau(\xi^x)$ is almost surely a leaf. Finally, by construction we have $d_{\tilde{\mathcal{T}}}(\rho, \hat{\tau}(\xi^x, \xi^y)) = 2\xi^y$ and $d_{\tilde{\mathcal{T}}}(\rho, \tau(\xi^x)) = 2\tilde{e}^{(\sigma)}(\xi^x)$. Since $2\xi^y$ is uniform on $[0, 2\tilde{e}^{(\sigma)}(\xi^x)]$, the last statement follows. □

Gromov–Hausdorff convergence of connected components. We begin by discussing the identification of points in metric spaces. Suppose that (X, d_X) is a metric space and that $(x_i^0, x_i^1), 1 \leq i \leq k$ are k pairs of points in X which we are going to identify. Write d'_X for the quasi-metric on X which is defined by setting $d'_X(x, x')$ equal to

$$\min\left(d_X(x,x'), \inf_{\substack{i_1,\ldots,i_r,\\ \epsilon_1,\ldots,\epsilon_r}}\left(d_X(x, x_{i_1}^{\epsilon_1}) + \sum_{j=1}^{r-1} d_X(x_{i_j}^{\epsilon_j+1}, x_{i_{j+1}}^{\epsilon_{j+1}}) + d_X(x_{i_r}^{\epsilon_r+1}, x')\right)\right), \tag{10}$$

where the infimum is taken over integers r, indices $i_1, i_2, \ldots, i_r$ (which may be chosen to be distinct) and $\epsilon_1, \epsilon_2, \ldots, \epsilon_r \in \{0, 1\}$. (For such an ϵ, addition $\epsilon + 1$ is taken modulo 2.) The quasi-metric d'_X gives the shortest distance between x and y when we identify x_i^0 and x_i^1 for $1 \leq i \leq k$; it is not a true metric since $d'_X(x_i^0, x_i^1) = 0$, but it clearly induces a metric on the quotient space $X/\sim$ where $x \sim x'$ if and only if $d'_X(x, x') = 0$. Indeed, this makes explicit how the vertex identification referred to in the previous subsection should be carried out rigorously.

Lemma 21 *Let (X, d_X) and (Y, d_Y) be metric spaces and let $\{(x_i^0, x_i^1), 1 \leq i \leq k\}$ be points in X and $\{(y_i^0, y_i^1), 1 \leq i \leq k\}$ be points in Y. Suppose that $\mathcal{R}$ is a correspondence between X and Y such that $(x_i^\epsilon, y_i^\epsilon) \in \mathcal{R}$ for $\epsilon \in \{0, 1\}$ and all $1 \leq i \leq k$. Let d'_X and d'_Y be the induced metrics when we identify x_i^0 with x_i^1 and y_i^0 with y_i^1 for $1 \leq i \leq k$. Then*

$$d_{\mathrm{GH}}((X, d'_X), (Y, d'_Y)) \leq \frac{k+1}{2}\mathrm{dis}(\mathcal{R}).$$

Proof Let $(x, y), (x', y') \in \mathcal{R}$. Suppose that $i_1, i_2, \ldots, i_r$ and $\epsilon_1, \epsilon_2, \ldots, \epsilon_r$ are such that

$$d'_Y(y, y') = d_Y(y, y_{i_1}^{\epsilon_1}) + \sum_{j=1}^{r-1} d_Y(y_{i_j}^{\epsilon_j+1}, y_{i_{j+1}}^{\epsilon_{j+1}}) + d_Y(y_{i_r}^{\epsilon_r+1}, y').$$

Then by definition, for this particular choice of indices,

$$d'_X(x, x') \leq d_X(x, x_{i_1}^{\epsilon_1}) + \sum_{j=1}^{r-1} d_X(x_{i_j}^{\epsilon_j+1}, x_{i_{j+1}}^{\epsilon_{j+1}}) + d_X(x_{i_r}^{\epsilon_r+1}, x').$$

Hence, since $(x_i^\epsilon, y_i^\epsilon) \in \mathcal{R}$ for every i and ϵ,

$$\begin{aligned}
&d'_X(x, x') - d'_Y(y, y') \\
&\leq \left(d_X(x, x_{i_1}^{\epsilon_1}) - d_Y(y, y_{i_1}^{\epsilon_1})\right) + \sum_{j=1}^{r-1}\left(d_X(x_{i_j}^{\epsilon_j+1}, x_{i_{j+1}}^{\epsilon_{j+1}}) - d_Y(y_{i_j}^{\epsilon_j+1}, y_{i_{j+1}}^{\epsilon_{j+1}})\right) \\
&\quad + \left(d_X(x_{i_r}^{\epsilon_r+1}, x') - d_Y(y_{i_r}^{\epsilon_r+1}, y')\right) \\
&\leq (k+1)\mathrm{dis}(\mathcal{R}),
\end{aligned}$$

since we can always take $r \le k$. A symmetric argument gives the same expression with the roles of X and Y reversed and the result follows. □

We are now ready to prove the convergence of G_m^p (informally, a connected component of $G(n,p)$ conditioned to have size m) to a continuum random graph.

Theorem 22 *Suppose that $\sigma > 0$. Let $m = m(n) \in \mathbb{Z}^+$ be a sequence of integers such that $n^{-2/3}m \to \sigma$ as $n \to \infty$, and let $p = p(n) \in (0,1)$ be such that $pn \to 1$. Then, as $n \to \infty$,*

$$n^{-1/3} G_m^p \xrightarrow{d} \mathcal{M}^{(\sigma)},$$

in the Gromov–Hausdorff distance.

Proof As argued earlier, by Brownian scaling we may reduce to the case $\sigma = 1$. Furthermore, by Skorohod's representation theorem, there exists a probability space where the convergence

$$(m^{-1/2}\tilde{X}^m(\lfloor m\cdot\rfloor), \mathcal{P}_m \cap m^{-1/2}\tilde{X}^m(\lfloor m\cdot\rfloor)) \xrightarrow{d} (\tilde{e}, \mathcal{P} \cap \tilde{e}) \qquad (11)$$

of Lemma 19 occurs almost surely. Without loss of generality, we may work on this probability space. Since the random variables involved are integer-valued, there exists an almost surely finite random variable M such that for $m \ge M$,

$$|\mathcal{P}_m \cap (m^{-1/2}\tilde{X}^m(\lfloor m\cdot\rfloor))| = |\mathcal{P} \cap \tilde{e}| = s,$$

for some $s \ge 0$. We will suppose henceforth that $s \ge 1$ since in the $s = 0$ case there is nothing to prove. We can label the points of $\mathcal{P}_m \cap (m^{-1/2}\tilde{X}^m(\lfloor m\cdot\rfloor))$ as $(i_\ell^m, j_\ell^m), 1 \le \ell \le s$ and the points of $\mathcal{P} \cap \tilde{e}$ as $(\xi_\ell^x, \xi_\ell^y), 1 \le \ell \le s$ in such a way that $\max_{1\le\ell\le s}\{|i_\ell^m - \xi_\ell^x| + |j_\ell^m - \xi_\ell^y|\} \to 0$ as $m \to \infty$.

Recall that in the present context, for integers i, j, we have $k(i,j) = \inf\{k \ge i : \tilde{X}^m(i) = j\}$ and define $u(\xi) = u(\xi^x, \xi^y) = \inf\{u \ge \xi^x : \tilde{e}(u) = \xi^y\}$. Now let $k_\ell^m = m^{-1}k(mi_\ell^m, m^{1/2}j_\ell^m)$ (the indices mi_ℓ^m and $m^{1/2}j_\ell^m$ are integers by the definition of $\mathcal{P}_m$). Then $\max_{1\le\ell\le s}|k_\ell^m - u(\xi_\ell)| \to 0$ as $m \to \infty$. This follows from the convergence (11) and the fact that almost surely for every $\epsilon > 0$ small enough we have

$$\inf_{0\le t\le\epsilon} \tilde{e}(u(\xi_\ell) + t) < \tilde{e}(u(\xi_\ell)) < \sup_{0\le t\le\epsilon} \tilde{e}(u(\xi_\ell) - t).$$

This can be deduced from the corresponding result for the Brownian excursion e and from the absolute continuity of the law of $\tilde{e}$ with respect to that of e.

We now construct a correspondence $\mathcal{R}_m$ between $\tilde{T}_m^p$ and $\tilde{\mathcal{T}}$, the real tree encoded by $2\tilde{e}$. Let

$$\epsilon_m = \max\left\{\max_{1\le\ell\le s}|i_\ell^m - \xi_\ell^x|, \max_{1\le\ell\le s}|k_\ell^m - u(\xi_\ell)|\right\}$$

and note that we have already argued that $\epsilon_m \to 0$ almost surely. Let τ denote the canonical projection from $[0,1]$ onto $\tilde{\mathcal{T}}$. Then let $(v_i, \tau(t)) \in \mathcal{R}_m$

for $0 \le i \le m-1$ and $0 \le t \le 1$ if and only if $|m^{-1}i - t| \le \epsilon_m$. It follows in particular that $(v_{mi_\ell^m}, \tau(\xi_\ell^x)) \in \mathcal{R}_m$ and $(v_{mk_\ell^m}, \tau(u(\xi_\ell^x))) \in \mathcal{R}_m$ for all $1 \le \ell \le s$.

Now recall that $\mathcal{M}$ is the real tree $\tilde{\mathcal{T}}$ in which $\tau(\xi_\ell^x)$ has been identified with $\tau(u(\xi_\ell))$ for $1 \le \ell \le s$, and that G_m^p may be seen as the metric space $\tilde{T}_m^p$ in which an edge (of length 1) has been added between $v_{mi_\ell^m}$ and $v_{mk_\ell^m}$ for every $1 \le \ell \le s$. This last space is clearly at Gromov–Hausdorff distance one from the space G'_m obtained by instead identifying $v_{mi_\ell^m}$ and $v_{mk_\ell^m}$ for every $1 \le \ell \le s$. By Lemma 21,

$$d_{\mathrm{GH}}(n^{-1/3}G'_m, \mathcal{M}) \le \frac{s+1}{2}\mathrm{dis}(\mathcal{R}_m),$$

where the distortion $\mathrm{dis}(\mathcal{R}_m)$ is with respect to the metrics $n^{-1/3}d_{\tilde{T}_m^p}$ and $d_{\tilde{\mathcal{T}}}$ on the trees $\tilde{T}_m^p$ and $\tilde{\mathcal{T}}$ respectively.

It remains to prove that $\mathrm{dis}(\mathcal{R}_m) \to 0$ as $n \to \infty$. Let $i \le j$ and $t \le u$ be such that $(v_i, \tau(t)), (v_j, \tau(u)) \in \mathcal{R}_m$. Recall that we assumed that we are working on a probability space such that the convergence in (11) is almost sure. On this space, by Lemma 16, we also have

$$(m^{-1/2}\tilde{H}(\lfloor mt \rfloor), 0 \le t \le 1) \to 2\tilde{e}$$

almost surely. Then by Lemma 17,

$$\begin{aligned}
&|n^{-1/3}d_{\tilde{T}_m^p}(v_i, v_j) - d_{\tilde{\mathcal{T}}}(\tau(t), \tau(u))| \\
&\le \Bigg| n^{-1/3}m^{1/2}\left(m^{-1/2}\tilde{H}^m(i) + m^{-1/2}\tilde{H}^m(j) - 2\min_{r \in [i,j]\cap\mathbb{Z}} m^{-1/2}\tilde{H}^m(r)\right) \\
&\quad - \left(2\tilde{e}(t) + 2\tilde{e}(u) - 4\inf_{v \in [t,u]}\tilde{e}(v)\right)\Bigg| + 2n^{-1/3}.
\end{aligned}$$

By Theorem 15, the right-hand side converges to 0 uniformly. This entails that $d_{\mathrm{GH}}(n^{-1/3}G'_m, \mathcal{M}) \to 0$ and, since $d_{\mathrm{GH}}(n^{-1/3}G'_m, n^{-1/3}G_m^p) \le n^{-1/3}$, the result follows. □

We finish this section by stating an easy corollary on the number of surplus edges of G_m^p.

Corollary 23 *Suppose that $m = m(n)$ is such that $n^{-2/3}m \to \sigma$ as $n \to \infty$ and let $p = p(n)$ be such that $pn \to 1$. Then as $n \to \infty$,*

$$s(G_m^p) \xrightarrow{d} \mathrm{Poisson}\left(\int_0^\sigma \tilde{e}^{(\sigma)}(u)du\right).$$

Proof This follows immediately from the observation that

$$s(G_m^p) \stackrel{d}{=} |\mathcal{Q}^p \cap \tilde{X}^m| = |\mathcal{P}_m \cap ((m/\sigma)^{-1/2}\tilde{X}^m(\lfloor (m/\sigma)t \rfloor), 0 \le t \le \sigma)|,$$

and from Lemma 19 and Proposition 20. □

5 The limit of the critical random graph

Recall that we are interested in $G(n,p)$ with $p = n^{-1} + \lambda n^{-4/3}$, for $\lambda \in \mathbb{R}$. We will begin by recalling some more details of Aldous' limit result from [7]. His principal tool is the so-called *breadth-first walk* on $G(n,p)$. This is very similar to our depth-first walk, except that the vertices are considered in a different order. The *breadth-first ordering* $v_0, v_1, \ldots, v_{n-1}$ on the vertices of the graph is obtained as follows. (We deliberately use the same notation as in our definition of the depth-first ordering.) For $i \geq 0$, we define the ordered set $\mathcal{O}_i$ of open vertices at time i, and the set $\mathcal{A}_i$ of the vertices that have already been explored at time i. We say that a vertex u has been *seen* at time i if $u \in \mathcal{O}_i \cup \mathcal{A}_i$. Let c_i be a counter which keeps track of how many components we have looked at so far.

INITIALIZATION Set $\mathcal{O}_0 = (1)$, $\mathcal{A}_0 = \emptyset$, $c_0 = 1$.

STEP i $(0 \leq i \leq n-1)$: Let v_i be the first vertex of $\mathcal{O}_i$ and let $\mathcal{N}_i$ be the set of neighbors of v_i in $[n] \setminus (\mathcal{A}_i \cup \mathcal{O}_i)$. Set $\mathcal{A}_{i+1} = \mathcal{A}_i \cup \{v_i\}$. Construct $\mathcal{O}_{i+1}$ from $\mathcal{O}_i$ by removing v_i from the start of $\mathcal{O}_i$ and affixing the elements of $\mathcal{N}_i$ in increasing order to the *end* of $\mathcal{O}_i \setminus \{v_i\}$. If now $\mathcal{O}_{i+1} = \emptyset$, add to it the lowest-labeled element of $[n] \setminus \mathcal{A}_{i+1}$ and set $c_{i+1} = c_i + 1$. Otherwise, set $c_{i+1} = c_i$.

The only difference between this procedure and the one introduced in Section 2 is that the word "start" has been changed to "end" (italicized above). Now define $Y_n(i) = |\mathcal{O}_i \setminus \{v_i\}| - (c_i - 1)$. Then $(Y_n(i), 0 \leq i < n)$ is called the breadth-first walk on the graph. It is straightforward to see that $(Y_n(i), 0 \leq i < n)$ attains a new minimum every time that v_i is the root of a new component. This enables us to interpret component sizes as excursions above past minima of the breadth-first walk. Aldous proved that

$$n^{-1/3}(Y_n(\lfloor n^{2/3} t \rfloor), t \geq 0) \xrightarrow{d} (W^\lambda(t), t \geq 0) \tag{12}$$

as $n \to \infty$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^+)$, with convergence (as is usual) uniform on compact time-intervals. Here,

$$W^\lambda(t) = W(t) + \lambda t - \frac{t^2}{2},$$

where W is a standard Brownian motion. It is not hard to see that the breadth-first and depth-first walks are interchangeable here, and that the identical result holds for the depth-first walk. Since we do not actually need this result, we will not go further into details here.

Now define

$$B^\lambda(t) = W^\lambda(t) - \min_{0 \leq s \leq t} W^\lambda(s),$$

the reflecting process. The excursions of this process correspond to "components" of the limiting graph. As stated by Aldous, there is an inhomogeneous excursion measure associated with this B^λ, in the same way as Itô's excursion measure is associated to a reflecting Brownian motion. Recall from (2) that $\mathcal{E}$ is the space of continuous excursions of finite length.

Denote Itô's measure by $\mathbb{N}$ and the excursion measure associated to B^λ by $\mathbb{N}_t^\lambda$ (i.e. for excursions starting at time t). Then, as argued by Aldous in the proof of his Lemma 26, we can calculate the density of $\mathbb{N}_t^\lambda$ with respect to $\mathbb{N}$. For clarity, we will repeat his argument here. Firstly note that $\mathbb{N}_t^\lambda = \mathbb{N}_0^{\lambda-t}$ and so it suffices to find $\mathbb{N}_0^\lambda$ for all $\lambda \in \mathbb{R}$.

Write $W = (W(t), 0 \le t \le \sigma)$ for the canonical process under $\mathbb{N}$. Then by the Cameron–Martin–Girsanov formula [47, 49], applied under $\mathbb{N}$,

$$\frac{d\mathbb{N}_0^\lambda}{d\mathbb{N}} = \exp\left(\int_0^\sigma (\lambda - s) dW(s) - \frac{1}{2}\int_0^\sigma (\lambda - s)^2 ds\right).$$

By integration by parts, we have

$$\int_0^\sigma (\lambda - s) dW(s) = \int_0^\sigma W(s) ds,$$

the area under the excursion W. So we can re-write

$$\frac{d\mathbb{N}_0^\lambda}{d\mathbb{N}} = \exp\left(\int_0^\sigma W(s) ds - \frac{1}{6}((\sigma - \lambda)^3 + \lambda^3)\right).$$

We know that we can define a normalized excursion measure $\mathbb{N}(\,\cdot\,|\sigma = x)$ for each $x > 0$, which is, in fact, a probability measure. There is a corresponding probability measure $\mathbb{N}_0^\lambda(\,\cdot\,|\sigma = x)$ which, for $\mathcal{B} \subseteq \mathcal{E}$ a Borel set, is determined by

$$\mathbb{N}_0^\lambda[\mathbb{1}_{\mathcal{B}}|\sigma = x] = \frac{\mathbb{N}\left[\exp\left(\int_0^x W(s)ds\right)\mathbb{1}_{\mathcal{B}}\middle|\sigma = x\right]}{\mathbb{N}\left[\exp\left(\int_0^x W(s)ds\right)\middle|\sigma = x\right]}.$$

Note that this quantity is independent of λ. By Brownian scaling,

$$\begin{aligned}\mathbb{N}\left[\exp\left(\int_0^x W(s)ds\right)\middle|\sigma = x\right] &= \mathbb{N}\left[\exp\left(x^{3/2}\int_0^1 W(s)ds\right)\middle|\sigma = 1\right] \\ &= \mathbb{E}\left[\exp\left(x^{3/2}\int_0^1 e(s)ds\right)\right]\end{aligned}$$

where $(e(s), 0 \le s \le 1)$ is a standard Brownian excursion under $\mathbb{E}$. Similarly, for any suitable test function f of the excursion,

$$\begin{aligned}&\mathbb{N}_0^\lambda[f(W(s), 0 \le s \le x)|\sigma = x] \\ &= \frac{\mathbb{E}\left[f(\sqrt{x}e(s/x), 0 \le s \le 1)\exp\left(x^{3/2}\int_0^1 e(s)ds\right)\right]}{\mathbb{E}\left[\exp\left(x^{3/2}\int_0^1 e(s)ds\right)\right]}.\end{aligned}$$

Putting all of this together, we see that the inhomogenity of the excursion measure lies entirely in the selection of the length of the excursion. So to give a complete description, we just need to determine $\mathbb{N}_0^\lambda(\sigma \in dx)$. We know that $\mathbb{N}(\sigma \in dx) = (2\pi)^{-1/2} x^{-3/2} dx$ and so

$$\begin{aligned}&\mathbb{N}_0^\lambda(\sigma \in dx)\\&= \frac{x^{-3/2}}{\sqrt{2\pi}} \exp\left(-\frac{1}{6}((x-\lambda)^3+\lambda^3)\right) \mathbb{N}\left[\exp\left(\int_0^x W(s)ds\right) \middle| \sigma = x\right] dx.\end{aligned}$$

To recapitulate: the excursion measure at time t picks an excursion length according to $\mathbb{N}_0^{\lambda-t}(\sigma \in dx)$. Then, given $\sigma = x$, it picks a tilted Brownian excursion of that length. This is the crucial fact that allows us to use Theorem 22 and the results of Section 4 about the limit of connected components. It is not surprising that it holds, however, since the components of $G(n,p)$ likewise have the property that one can first sample the size and then, given the size, sample a connected component of that size.

Let $\mathcal{C}^n = (\mathcal{C}_1^n, \mathcal{C}_2^n, \ldots)$ be the components of the random graph $G(n,p)$ with $p = 1/n + \lambda n^{-4/3}$, in decreasing order of their sizes, $Z_1^n \geq Z_2^n \geq \ldots$ respectively. Let $\mathbf{Z}^n = (Z_1^n, Z_2^n, \ldots)$. As a consequence of (12), Aldous [7] proves that

$$n^{-2/3}\mathbf{Z}^n \xrightarrow{d} \mathbf{Z}, \tag{13}$$

where $\mathbf{Z}$ is the ordered sequence of excursion lengths of B^λ and convergence is in $\ell^2_\searrow$. Let $\mathbf{M}^n = (M_1^n, M_2^n, \ldots)$ be the sequence of metric spaces corresponding to these components. Recall the definition of $\mathcal{M}^{(\sigma)}$ from Section 4. We next state a more precise version of Theorem 2.

Theorem 24 *As $n \to \infty$,*

$$(n^{-2/3}\mathbf{Z}^n, n^{-1/3}\mathbf{M}^n) \xrightarrow{d} (\mathbf{Z}, \mathbf{M}),$$

where $\mathbf{M} = (M_1, M_2, \ldots)$ is a sequence of metric spaces such that, conditional on $\mathbf{Z}$, $M_1, M_2, \ldots$ are independent and $M_i \stackrel{d}{=} \mathcal{M}^{(Z_i)}$. Convergence in the second co-ordinate here is in the metric specified by (1).

In proving Theorem 24, we need one additional result, on the expected height of the tilted trees $\tilde{T}_m^p$ introduced in Section 2. This lemma is essentially what allows us to use the distance (1), rather than product convergence.

Lemma 25 *Let $p = 1/n + \lambda n^{-4/3}$. There exists a universal constant $M > 0$ such that for all n large enough that $1/(2n) < p < 2/n$ and $p < 1/2$, and all $1 \leq m \leq n$,*

$$\mathbb{E}[\|\tilde{T}_m^p\|^4] \leq M \cdot \max(m^6 n^{-4}, 1) \cdot m^2.$$

Before we proceed with the proof, note that the bound in Lemma 25 tells us that tilted trees of size of order $n^{2/3}$ behave more or less like uniform trees. (See the moments of the height $\|T^m\|$ of uniform trees T^m in [24, 25, 46].) Trees of size much larger than $n^{2/3}$ are much more influenced by the tilting (as witnessed by the factor $m^6 n^{-4}$).

Proof We assume throughout that $m \geq 2$. For any $x > 0$ and $\alpha > 0$, we have

$$\begin{aligned}&\mathbb{P}(\|\tilde{T}_m^p\| > xm^{1/2})\\ &\leq \mathbb{P}(\|\tilde{T}_m^p\| > xm^{1/2} \mid a(\tilde{T}_m^p) \leq \alpha x^2 m^{3/2}) + \mathbb{P}(a(\tilde{T}_m^p) > \alpha x^2 m^{3/2}). \qquad (14)\end{aligned}$$

We will bound each of the terms on the right-hand side of (14) then integrate over x to obtain the desired bound on $\mathbb{E}[\|\tilde{T}_m^p\|^4]$. (We will optimize our choice of α later in the proof.) The intuition is that when $a(\tilde{T}_m^p)$ is not too large, the distribution of $\tilde{T}_m^p$ is not too different from that of the uniformly random labeled rooted tree T_m, and so we should be able to use pre-existing bounds on the tails of $\|T_m\|$. On the other hand, we have already proved (c.f. Lemma 14) bounds that will allow us to control the probability that $a(\tilde{T}_m^p)$ is large. We now turn to the details.

Let $q = \max(m^{-3/2}, p)$. By Markov's inequality and the definition of $\tilde{T}_m^p$ we have

$$\begin{aligned}\mathbb{P}(a(\tilde{T}_m^p) > \alpha x^2 m^{3/2}) &\leq \frac{\mathbb{E}[(1-q)^{-a(\tilde{T}_m^p)}]}{(1-q)^{-\alpha x^2 m^{3/2}}}\\ &\leq \frac{\mathbb{E}\left[((1-p)(1-q))^{-a(T_m)}\right]}{(1-q)^{-\alpha x^2 m^{3/2}}}\\ &\leq \frac{\mathbb{E}\left[(1-q)^{-2a(T_m)}\right]}{(1-q)^{-\alpha x^2 m^{3/2}}}. \qquad (15)\end{aligned}$$

Let $c = 2m^{3/2}/n$, so that $cm^{-3/2}/4 < p < cm^{-3/2}$, and observe that $q \leq \delta m^{-3/2}$ for $\delta := \max(c, 1)$. By Lemma 14, there exist absolute constants $K, \kappa > 0$ such that

$$\sup_{m \geq 1} \mathbb{E}[(1-q)^{-2a(T_m)}] \leq K e^{4\kappa\delta^2}.$$

Furthermore, since $qm^{3/2} \geq \delta/4$, (15) yields

$$\mathbb{P}(a(\tilde{T}_m^p) > \alpha x^2 m^{3/2}) \leq K e^{4\kappa\delta^2 - \alpha x^2 \delta/4} \leq K e^{-\alpha x^2 \delta/8}, \qquad (16)$$

for all x such that $x^2 \geq 32\kappa\delta/\alpha$. For $x \geq \sqrt{8\ln(2K)/(\alpha\delta)}$, so that $e^{-\alpha x^2\delta/8} \leq 1/2$, since $p < cm^{-3/2}$ and $(1-p)^{-1/p} < e^2$, we also have

$$\begin{aligned}\mathbb{P}(\|\tilde{T}_m^p\| \geq xm^{1/2} \mid a(\tilde{T}_m^p) \leq \alpha x^2 m^{3/2}) &\leq \frac{(1-p)^{-\alpha x^2 m^{3/2}} \cdot \mathbb{P}(\|T_m\| \geq xm^{1/2})}{\mathbb{P}(a(\tilde{T}_m^p) \leq \alpha x^2 m^{3/2})}\\ &\leq 2e^{2c\alpha x^2} \cdot \mathbb{P}(\|T_m\| \geq xm^{1/2}). \qquad (17)\end{aligned}$$

We can now use tail bounds on the height of uniform labeled trees. Łuczak [38, Corollary 1] provides a uniform tail bound on $\|T_m\|$: for some universal constant K', and all integers $m \geq 1$,

$$\mathbb{P}(\|T_m\| \geq xm^{1/2}) \leq K' x^3 e^{-x^2/2},$$

and so taking $\alpha^{-1} = 8\delta$, (17) yields

$$\mathbb{P}(\|\tilde{T}^p_m\| \geq xm^{1/2} \mid a(\tilde{T}^p_m) \leq \alpha x^2 m^{3/2}) \leq 2K' x^3 e^{(2c\alpha - 1/2)x^2} \leq 2K' x^3 e^{-x^2/4}. \tag{18}$$

Notice that our requirements that $x^2 \geq 32\kappa\delta/\alpha$ and $x^2 \geq 8\ln(2K)/(\alpha\delta)$ now reduce to $x \geq 16\kappa^{1/2}\delta$ and $x \geq 8\sqrt{\ln(2K)}$. So, in particular, setting $L = 16\kappa^{1/2} + 8\sqrt{\ln(2K)}$, and recalling that $\delta = \max(c, 1)$, it suffices to require that $x \geq L\delta$.

For all such x, combining (16) and (18) with (14) and substituting in the value of α then yields

$$\mathbb{P}(\|\tilde{T}^p_m\| > xm^{1/2}) \leq e^{-x^2/64} + 2K' x^3 e^{-x^2/4} \leq \max(4K' x^3, 2) e^{-x^2/64}.$$

Writing $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X \geq t)\, dt$, we then have

$$\begin{aligned}\mathbb{E}[\|\tilde{T}^p_m\|^4] &\leq m^2 \int_0^{m^2} \mathbb{P}(\|\tilde{T}^p_m\|^4 > xm^2) dx \\ &\leq m^2 L^4 \delta^4 + m^2 \int_{L^4\delta^4}^{m^2} \max(4K' x^{3/4}, 2) e^{-x^{1/2}/64} dx.\end{aligned}$$

So, since $\delta = \max(c, 1)$ with $c = 2m^{3/2}/n$, we have

$$\mathbb{E}[\|\tilde{T}^p_m\|^4] \leq m^2 M (\max(m^{3/2}/n, 1))^4,$$

for some absolute constant $M > 0$, as required. □

Proof of Theorem 24 In the random graph $G(n, p)$, conditional on $\mathbf{Z}^n$, the components $M^n_1, M^n_2, \ldots$ are independent and (up to an unimportant relabeling),

$$M^n_i \stackrel{d}{=} G^p_{Z^n_i}$$

where as above, $p = n^{-1} + \lambda n^{-4/3}$. Note that $np \to 1$ as $n \to \infty$. By (13) and Skorohod's representation theorem, there exists a probability space and random variables $\tilde{\mathbf{Z}}^n, \tilde{\mathbf{M}}^n, n \geq 1$ and $\tilde{\mathbf{Z}}, \tilde{\mathbf{M}}$ defined on that space such that $(\tilde{\mathbf{Z}}^n, \tilde{\mathbf{M}}^n) \stackrel{d}{=} (\mathbf{Z}^n, \mathbf{M}^n)$, $n \geq 1$, and $(\tilde{\mathbf{Z}}, \tilde{\mathbf{M}}) \stackrel{d}{=} (\mathbf{Z}, \mathbf{M})$ with $n^{-2/3}\tilde{\mathbf{Z}}^n \to \tilde{\mathbf{Z}}$ a.s. But then the convergence $(n^{-2/3}\mathbf{Z}^n, n^{-1/3}\mathbf{M}^n) \stackrel{d}{\to} (\mathbf{Z}, \mathbf{M})$ in the product topology follows immediately from Theorem 22. We can, and will hereafter assume, again by applying Skorohod's theorem, that $(n^{-2/3}Z^n_i, n^{-1/3}M^n_i) \to (Z_i, M_i)$ almost surely for all i. It remains to prove convergence in distribution in the metric specified by (1). In doing so, we will need to use the **oDFS** procedure. For any n and i for which M^n_i is defined, we may view M^n_i as a finite connected graph; this graph is uniquely specified (up to isomorphism) by M^n_i. When we write **oDFS**(M^n_i) we mean the **oDFS** procedure run on a uniformly random labelling of the graph corresponding to M^n_i.

To prove convergence in the metric specified by (1), we first observe that for any sequences of metric spaces $\mathbf{A}, \mathbf{B}$ and any integer $N \geq 1$, we have

$$d(\mathbf{A},\mathbf{B}) \leq \left(\sum_{i=1}^{N-1} d_{\mathrm{GH}}(A_i,B_i)^4\right)^{1/4} + \left(\sum_{i=N}^{\infty} d_{\mathrm{GH}}(A_i,B_i)^4\right)^{1/4}.$$

Since we have already established convergence in the product topology, to complete the proof it thus suffices to show that for all $\epsilon > 0$,

$$\lim_{N\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(\sum_{i=N}^{\infty} d_{\mathrm{GH}}(n^{-1/3}M_i^n, M_i)^4 > \epsilon\right) = 0. \tag{19}$$

As earlier, we write $\|\cdot\|$ for the height of a rooted tree or the supremum of a finite excursion. For any i and n, we may bound use the bound

$$d_{\mathrm{GH}}(n^{-1/3}M_i^n, M_i)^4 \leq 16(n^{-4/3}\|\tilde{T}_i^n\|^4 + \|\tilde{e}^{(Z_i)}\|^4), \tag{20}$$

where $\tilde{T}_i^n$ is the depth-first tree corresponding to $\mathbf{oDFS}(M_i^n)$ started at its smallest vertex, and $\tilde{e}^{(Z_i)}$ is the excursion corresponding to M_i. Now let

$$\varXi_i^n = \|\tilde{T}_i^n\|^4 \cdot (Z_i^n)^{-2}.$$

By Brownian scaling, given the length Z_i, we have that $\|\tilde{e}^{(Z_i)}\|^4 = Z_i^2 \cdot \|\tilde{e}_i\|^4$, where $\{\tilde{e}_i, i \geq 1\}$ are independent and identically distributed copies of $\tilde{e}$, a tilted excursion of length one, and which are independent of $\{Z_i, i \geq 1\}$. Combining the preceding equalities with (20), we thus have

$$\sum_{i=N}^{\infty} d_{\mathrm{GH}}\left(n^{-1/3}M_i^n, M_i\right)^4 \leq 16\sum_{i=N}^{\infty}\left(\varXi_i^n\left(n^{-2/3}Z_i^n\right)^2 + Z_i^2\|\tilde{e}_i\|^4\right).$$

Next, given $\delta > 0$ write $N_\delta = N_\delta(\mathbf{Z})$ for the smallest N such that $Z_N < \delta$; N is almost surely finite since $\mathbf{Z}$ is almost surely an element of $\ell^2_\searrow$. For any $\delta > 0$ and all n, N, setting $\epsilon_1 = \epsilon/16$ we then have

$$\begin{aligned}&\mathbb{P}\left(\sum_{i=N}^{\infty} d_{\mathrm{GH}}(n^{-1/3}M_i^n, M_i)^4 > \epsilon\right)\\ &\leq \mathbb{P}\left(\sum_{i>N_\delta}\left(\varXi_i^n\left(n^{-2/3}Z_i^n\right)^2 + Z_i^2\|\tilde{e}_i\|^4\right) > \epsilon_1\right) + \mathbb{P}\left(N_\delta > N\right).\end{aligned}$$

Since $\lim_{N\to\infty}\mathbb{P}\left(N_\delta > N\right) = 0$, and the first probability on the right-hand side of the preceding inequality does not depend on N, we thus have

$$\begin{aligned}&\lim_{N\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\sum_{i=N}^{\infty} d_{\mathrm{GH}}\left(n^{-1/3}M_i^n, M_i\right)^4 > \epsilon\right)\\ &\leq \limsup_{n\to\infty}\mathbb{P}\left(\sum_{i>N_\delta}\left(\varXi_i^n n^{-4/3}(Z_i^n)^2 + Z_i^2\|\tilde{e}_i\|^4\right) > \epsilon_1\right).\end{aligned}$$

Since this holds for any $\delta > 0$ and the left-hand side does not depend on δ, we then obtain

$$\lim_{N\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(\sum_{i=N}^{\infty} d_{\mathrm{GH}}\left(n^{-1/3}M_i^n, M_i\right)^4 > \epsilon \right) \\ \leq \lim_{\delta\downarrow 0} \limsup_{n\to\infty} \mathbb{P}\left(\sum_{i>N_\delta} \left(\mathit{\Xi}_i^n \left(\frac{Z_i^n}{n^{2/3}}\right)^2 + Z_i^2 \|\tilde{e}_i\|^4 \right) > \epsilon_1 \right). \qquad (21)$$

But it follows from Corollary 2 and Lemma 14 (b) of [7] that for all $\gamma > 0$,

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \mathbb{P}\left(\sum_{i>N_\delta} \left(\frac{Z_i^n}{n^{2/3}}\right)^2 > \gamma \right) = 0 \text{ and } \lim_{\delta\downarrow 0} \mathbb{P}\left(\sum_{i>N_\delta} Z_i^2 > \gamma \right) = 0,$$

from which we may complete the proof straightforwardly. First recall that $\tilde{e}_i$, $i \geq 1$ are independent and identically distributed tilted excursions of length one. Moreover, $\mathbb{E}\left[\|\tilde{e}_i\|^4\right] < \infty$, using the change of measure in the definition of $\tilde{e}_i$ and the Gaussian tails for the maximum $\|e\|$ of a standard Brownian excursion e [32]. Let $\epsilon_2 = \epsilon_1/2$, and choose $\delta > 0$ small enough that

$$\mathbb{P}\left(\sum_{i>N_\delta} Z_i^2 \geq \frac{\epsilon_2^2}{2\mathbb{E}\left[\|\tilde{e}\|^4\right]} \right) \leq \frac{\epsilon_2}{2}.$$

Then, by Markov's inequality, we have

$$\mathbb{P}\left(\sum_{i>N_\delta} Z_i^2 \|\tilde{e}_i\|^4 \geq \epsilon_2 \right) \leq \frac{\epsilon_2}{2} + \mathbb{P}\Big(\sum_{i>N_\delta} Z_i^2 \|\tilde{e}_i\|^4 \geq \epsilon_2 \;\Big|\; \sum_{i>N_\delta} Z_i^2 < \frac{\epsilon_2^2}{2\mathbb{E}\left[\|\tilde{e}\|^4\right]} \Big) \\ \leq \frac{\epsilon_2}{2} + \frac{1}{\epsilon_2} \mathbb{E}\left[\sum_{i>N_\delta} Z_i^2 \|\tilde{e}_i\|^4 \;\middle|\; \sum_{i>N_\delta} Z_i^2 < \frac{\epsilon_2^2}{2\mathbb{E}\left[\|\tilde{e}\|^4\right]} \right] \\ \leq \epsilon_2,$$

since $\{\tilde{e}_i, i \geq 1\}$ is independent of the set of lengths $\{Z_i, i \geq 1\}$. Since $\epsilon = 32\epsilon_2$ was arbitrary, it follows that

$$\lim_{\delta\downarrow 0} \mathbb{P}\left(\sum_{i>N_\delta} \|\tilde{e}_i\|^4 Z_i^2 \geq \epsilon_2 \right) = 0.$$

We may apply an identical argument to bound the terms involving discrete random variables and show that

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \mathbb{P}\left(\sum_{i>N_\delta} \mathit{\Xi}_i^n \left(Z_i^n n^{-2/3}\right)^2 > \epsilon_2 \right) = 0,$$

and thereby complete the proof, as long as we can show that $\sup_{i,n} \mathbb{E}[\mathit{\Xi}_i^n] < \infty$. We now prove that this is true. Recall that, by definition, $\mathit{\Xi}_i^n = \|\tilde{T}_i^n\|^4/(Z_i^n)^2$.

Recall also that for a given n and i, conditional on $Z_i^n = m$, the tree $\tilde{T}_i^n$ is distributed as $\tilde{T}_m^p$ from Section 2. In other words, we have

$$\mathbb{P}(\tilde{T}_i^n = T) \propto (1-p)^{-a(T)},$$

for each tree T on $[m]$, where by $\tilde{T}_i^n = T$ we mean that the increasing map (i.e. respecting the increasing order of the labels) between vertices of $\tilde{T}_i^n$ and T induces an isomorphism.

To bound $\mathbb{E}[\Xi_i^n]$, we use Lemma 25, together with bounds on the size of the largest component of $G(n,p)$ for $p = 1/n + \lambda n^{-4/3}$. For our purposes, these bounds are most usefully stated by Nachmias and Peres [43] (see also [45, 50]). They proved that for all fixed $\lambda \in \mathbb{R}$, there exist $\gamma = \gamma(\lambda) > 0$ and $C = C(\lambda) > 1$ such that for all n and for all $x \geq C$,

$$\mathbb{P}(Z_1^n \geq x n^{2/3}) \leq e^{-\gamma x^3}.$$

(In fact, the bound in [43] is slightly stronger than this.) For all integer $i \geq 1$, we thus have

$$\begin{aligned}
\mathbb{E}[\Xi_i^n] \leq & \sup_{m \leq Cn^{2/3}} m^{-2}\mathbb{E}[\|\tilde{T}_i^n\|^4 \mid Z_i^n = m\,] \\
& \quad + \sum_{m=\lceil Cn^{2/3}\rceil}^{n} m^{-2}\mathbb{E}[\|\tilde{T}_i^n\|^4 \mid Z_i^n = m\,] \cdot \mathbb{P}(Z_i^n = m) \\
\leq & \sup_{m \leq Cn^{2/3}} m^{-2}\mathbb{E}[\|\tilde{T}_m^p\|^4] + \sum_{m=\lceil Cn^{2/3}\rceil}^{n} m^{-2}\mathbb{E}[\|\tilde{T}_m^p\|^4] \cdot \mathbb{P}(Z_1^n \geq m) \\
\leq & \sup_{m \leq Cn^{2/3}} m^{-2}\mathbb{E}[\|\tilde{T}_m^p\|^4] + \sum_{m=\lceil Cn^{2/3}\rceil}^{n} m^{-2}\mathbb{E}[\|\tilde{T}_m^p\|^4] \cdot e^{-\gamma m^3 n^{-2}}.
\end{aligned}$$

Applying Lemma 25 to the last expression, we immediately obtain

$$\mathbb{E}[\Xi_i^n] \leq MC^6 + \sum_{m=\lceil Cn^{2/3}\rceil}^{n} M \cdot m^6 n^{-4} \cdot e^{-\gamma m^3 n^{-2}}$$

which is uniformly bounded in both $i \geq 1$ and $n \geq 1$, as required. □

We finally turn to the diameter of the random graph $G(n,p)$. Recall from the introduction that for $i \geq 1$, $D_i^n = \mathrm{diam}(M_i^n)$ if $\mathbf{M}^n$ has at least i components, and $D_i^n = 0$ otherwise. Hence, $D^n = \max_{i \geq 1} D_i^n$ is the diameter of $G(n,p)$. We can now prove Theorem 5 in the introduction, which states that $(n^{-1/3}D_i^n, i \geq 1) \xrightarrow{d} (D_i, i \geq 1)$ and that $n^{-1/3}D^n \xrightarrow{d} D$, for some random variables $D_i, i \geq 1$ of finite mean and some random variable $D \geq 0$ of finite mean and absolutely continuous distribution. It is now clear what the limiting random variables should be and we make the appropriate definitions: for each $i \geq 1$, let $D_i = \mathrm{diam}(M_i)$, and let $D = \sup_{i \geq 1} D_i$. We remark that Aldous discusses the diameter of continuum metric spaces and of the Brownian CRT in particular [5, Section 3.4].

Proof of Theorem 5 Observe that for any two metric spaces M and M', $|\text{diam}(M) - \text{diam}(M')| \leq 2d_{\text{GH}}(M, M')$. For fixed i, the claimed convergence is immediate from Theorem 24 since

$$|n^{-1/3}D_i^n - D_i| \leq 2d_{\text{GH}}(n^{-1/3}M_i^n, M_i)$$

and $n^{-1/3}M_i^n \to M_i$ in distribution in the Gromov–Hausdorff distance. Also D_i is a non-negative random variable with finite mean since it is at most the diameter of the underlying continuum random tree: $D_i \leq 2\|\tilde{e}_i\|\sqrt{Z_i}$, where $\tilde{e}_i$ is a tilted excursion of length one. It follows immediately that for any fixed N,

$$n^{-1/3} \max_{1 \leq i \leq N} D_i^n \xrightarrow{d} \max_{1 \leq i \leq N} D_i. \tag{22}$$

Next, observe that for any $\epsilon > 0$, there exists $c > 0$ such that

$$\mathbb{P}\left(D < c\right) \leq \mathbb{P}\left(D_1 < c\right) < \epsilon/2. \tag{23}$$

Since $n^{-1/3}D_1^n \xrightarrow{d} D_1$, we must then have that, for all n large enough,

$$\mathbb{P}(n^{-1/3}D^n < c) \leq \mathbb{P}(n^{-1/3}D_1^n < c) < \epsilon/2. \tag{24}$$

In order to prove convergence of $n^{-1/3}\mathbf{M}^n$ in the metric (1), we in fact proved that for all $\epsilon > 0$,

$$\lim_{N\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(\sum_{i \geq N} n^{-4/3}\|\tilde{T}_i^n\|^4 \geq \epsilon\right) = 0$$

and

$$\lim_{N\to\infty} \mathbb{P}\left(\sum_{i \geq N} \|\tilde{e}^{(Z_i)}\|^4 \geq \epsilon\right) = 0.$$

Since $D_i^n \leq 2\|\tilde{T}_i^n\|$ and $D_i \leq 2\|\tilde{e}^{(Z_i)}\|$, we must thus have

$$\lim_{N\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(\sup_{i \geq N} n^{-4/3}(D_i^n)^4 > \epsilon\right) = 0$$

and

$$\lim_{N\to\infty} \mathbb{P}\left(\sup_{i \geq N} D_i^4 > \epsilon\right) = 0.$$

It follows that for any fixed $\epsilon > 0$, we can find N large enough that for all n sufficiently large,

$$\mathbb{P}\left(\sup_{i \geq N} n^{-1/3}D_i^n \geq c\right) \leq \epsilon/2 \quad \text{and} \quad \mathbb{P}\left(\sup_{i \geq N} D_i \geq c\right) \leq \epsilon/2. \tag{25}$$

Then for all n sufficiently large, by (23), (24) and (25),

$$\mathbb{P}\left(D^n \neq \max_{1\le i\le N} D_i^n\right) \le \epsilon \quad \text{and} \quad \mathbb{P}\left(D \neq \max_{1\le i\le N} D_i\right) \le \epsilon.$$

Since $\epsilon > 0$ was arbitrary, combining these inequalities with (22) yields that $n^{-1/3}D^n \xrightarrow{d} D$, as required. It follows straightforwardly from the behavior of the tail of the sequence $(D_i, i \ge 1)$ that D has an absolutely continuous distribution. Finally, the fact that $\mathbb{E}[D] < \infty$ is a direct consequence of [2], Theorem 1. □

6 Concluding remarks

In this paper we have proved that it is possible to define a scaling limit for critical random graphs using random continuum metric spaces. This gives us a systematic way to consider a great many questions about distances in critical random graphs. In particular, it allows us to prove that critical random graphs have a diameter of order $n^{1/3}$ which is not concentrated around its mean. Focussing just on the diameter, there are now several questions which might deserve another look: what is the probability that the largest component achieves the diameter? What is the distribution of the (random) value $\lambda \in \mathbb{R}$ (with $p = 1/n + \lambda n^{-4/3}$) at which the diameter of the random graph is maximized? What is the distribution of this diameter?

The proof of our main result relies on a careful analysis of a depth-first exploration process of the graph which yields a "canonical" spanning forest and a way to add surplus edges according to the appropriate distribution. The forest is made of non-uniform trees that are biased in favor of those with a large area. In the limit, these trees rescale to continuum random trees encoded by tilted Brownian excursions. We have limited our analysis of these excursions to a minimum, but it seems likely that much more can be said, which might in turn yield results for the structure of the graphs or the behavior of other graph exploration algorithms.

In this paper, we have very much relied upon the depth-first viewpoint. Grégory Miermont has suggested to us that, at least intuitively, there should be an analogous breadth-first approach to the study of a limiting component, in which one might think of the shortcuts as being made "horizontally" across a generation rather than "vertically" along paths to the root. The advantage of the depth-first walk is that it converges to the same excursion as the height process of the depth-first tree. The rescaled breadth-first walk, however, converges to the same limit as the rescaled *height profile* (i.e. the number of vertices at each height) of a "breadth-first tree", which contains less information and, in particular, does not code the structure of that tree. As a result, it seems that it would be much harder to derive a metric space construction of a limiting component using the breadth-first viewpoint. It may, nonetheless, be the case that the breadth-first perspective is better adapted to answering certain questions about the limiting components where only the profile matters.

In a companion paper [3], we describe an alternative construction of the limit object which has the cycle structure of connected components at its heart: a connected component may be described as a multigraph (which gives the cycles), onto which trees are pasted. Together with the results of this paper, the perspective of [3] yields many limiting distributional results about sizes and lengths in critical random graphs.

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