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Lecture 4: Behaviour near the extinction time in self-similar fragmentations with negative index

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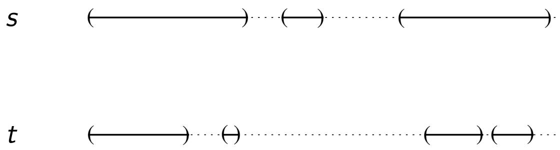
Joint work with Bénédicte Haas (Université Paris-Dauphine).

Interval fragmentations

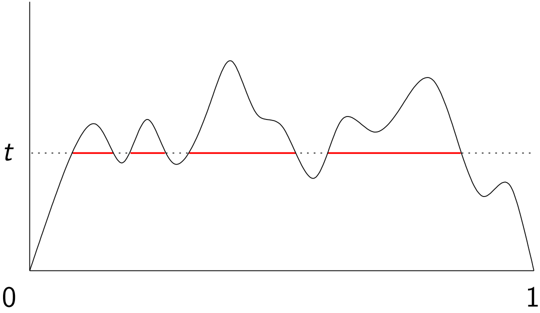
Recall that an interval fragmentation is a process $(O(t), t \geq 0)$ taking values in the set of open subsets of $(0, 1)$ such that $O(t) \subseteq O(s)$ whenever $0 \leq s \leq t$.



Interval fragmentations derived from excursions

The interval fragmentation associated to an excursion f is given by

$$O(t) := \{x \in [0, 1] : f(x) > t\}.$$



Ranked fragmentations

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Here, $F(t)$ takes values in the set

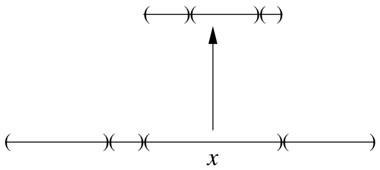
$$\mathcal{S}^{\downarrow} = \left\{ \mathbf{s} = (s_1, s_2, \ldots) : s_1 \geq s_2 \geq \ldots \geq 0, \sum_{i=1}^{\infty} s_i \leq 1 \right\}.$$

Self-similar fragmentations

Suppose we have an index of self-similarity $\alpha \in \mathbb{R}$ and a dislocation measure ν on $\mathcal{S}^{\downarrow}$.

Heuristically, a self-similar fragmentation is a ranked fragmentation such that when we have a block of size x, it splits at rate $x^{\alpha}\nu(\mathcal{S}^{\downarrow})$ into blocks of sizes $(xs_1, xs_2, \ldots)$, where $\mathbf{s} = (s_1, s_2, \ldots)$ is sampled according to the distribution $\nu(\cdot)/\nu(\mathcal{S}^{\downarrow})$.

At rate x^{α}:



Negative index and extinction

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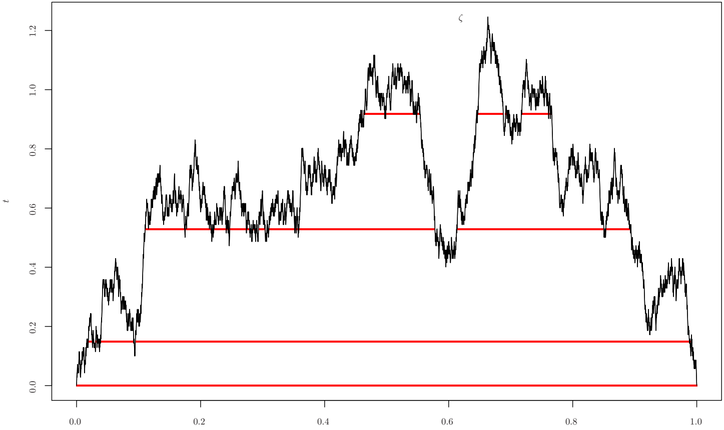
Problem: to understand how a general self-similar fragmentation with $\alpha < 0$ behaves as it approaches ζ.

The Brownian fragmentation

Take a standard Brownian excursion $(e(x), 0 \leq x \leq 1)$ and consider the associated interval fragmentation $(O(t), t \geq 0)$.



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Theorem. (Bertoin (2002)) $(F(t), t \geq 0)$ is a self-similar fragmentation of index $\alpha = -1/2$ and binary dislocation measure specified by $\nu(s_1 + s_2 < 1) = 0$ and

$$\nu(s_1 \in \mathrm{d}x) = \frac{2}{\sqrt{2\pi x^3(1-x)^3}} \mathbb{I}_{[1/2,1]}(x)\mathrm{d}x.$$

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Negative index

Note that if a self-similar interval fragmentation is to have a representation in terms of an excursion, it will necessarily have negative index. The extinction time is then the maximum of the excursion:

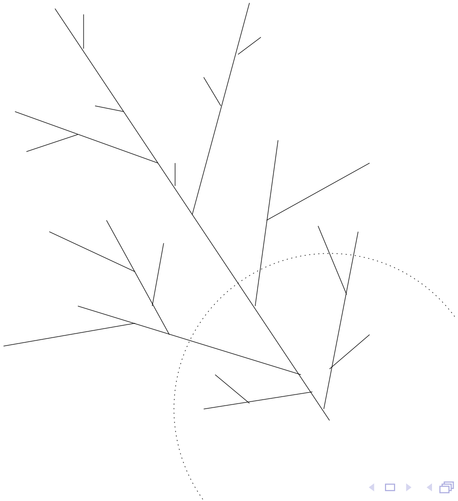
$$\zeta = \max_{0 \leq x \leq 1} f(x).$$

The Brownian fragmentation and the CRT

Recall that the Brownian CRT $\mathcal{T}$ is the real tree encoded by $(2e(x), 0 \leq x \leq 1)$. There is a natural "uniform" measure μ on the tree, which is the image of Lebesgue measure on $[0, 1]$ on $\mathcal{T}$.

The Brownian fragmentation and the CRT

Another (equivalent) way of thinking of $F(t)$ is as the sequence of μ-masses of the subtrees formed by discarding the parts of the Brownian CRT at distance less than $2t$ from the root.



Self-similar fragmentations and trees

It is natural to want to associate a genealogical tree (with edge-lengths) to a fragmentation process.

We root the tree at a node $\emptyset$ representing the initial block $(0, 1)$; its offspring are nodes representing the blocks which result from the first split, and so on. We let distances in the tree represent times in the fragmentation. In this way, the most recent common ancestor of two blocks is the block which completely included both for the last time.

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Think of the Brownian fragmentation, expressed as the masses of the subtrees at distance further than $2t$ from the root in the Brownian CRT. The splitting events occur precisely at the branch-points of the tree. It follows that the tree which describes the genealogy of the fragmentation is the Brownian CRT itself.

Self-similar fragmentations with negative index and CRT's

Take any (ranked) self-similar fragmentation F with $\alpha < 0$, $\nu(\sum_{i=1}^{\infty} s_i < 1) = 0$ and $c = 0$.

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Haas and Miermont (2004) have shown that there exists a continuum random tree $(\mathcal{T}_F, d_F)$ with mass measure μ_F such that, if we write $F'(t)$ for the sequence of μ_F-masses of the connected components of $\{v \in \mathcal{T}_F : d_F(\rho, v) > t\}$, then F' has the same distribution as F.

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Moreover, if $\nu(\mathcal{S}^{\downarrow}) = \infty$, then there exists a continuous random function H_F which is the height function for the CRT $(\mathcal{T}_F, d_F)$. In that case, we can put

$$O(t) = \{x \in [0,1] : H_F(x) > t\}$$

and then the associated ranked fragmentation is F'.

A canonical interval fragmentation

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There is, however, a special class of CRT's where we can say much more; these are the stable trees.

Stable trees

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$$\frac{1}{\gamma_n} \left(\sum_{i=1}^{n} Z_i - n \right) \xrightarrow{d} S,$$

where $\mathbb{E}\left[\exp(-\lambda S)\right] = \exp(\lambda^{\beta})$.

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Since the offspring distribution must have finite mean (in order to be critical), the parameter β of the stable law must lie in $(1, 2)$.

Stable trees

Now think of a Galton-Watson forest constructed of such trees.

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Theorem. (Duquesne and Le Gall (2002)) As $n \to \infty$,

$$\frac{\gamma_n}{n}(X(\lfloor nt \rfloor), t \geq 0) \xrightarrow{d} (\xi(t), t \geq 0),$$

where ξ is a certain β-stable Lévy process with no negative jumps and

$$\frac{\gamma_n}{n}(H(\lfloor nt \rfloor), t \geq 0) \xrightarrow{d} (\eta(t), t \geq 0),$$

where η is the stable height process.

Stable trees

The stable height process η is obtained from ξ via the analogue of the relationship between the discrete height process and the depth-first walk:

$$H(i) = \#\left\{0 \le j \le i-1 : X(j) = \min_{j \le k \le i} X(k)\right\}.$$

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(The correct definition is the local time at level 0 of the process $\hat{\sigma}^{(t)} - \hat{\eta}^{(t)}$, where

$$\hat{\eta}^{(t)}(s) = \eta(t) - \eta((t-s)-)$$

and

$$\hat{\sigma}^{(t)}(s) = \sup_{r \le s} \hat{\eta}^{(t)}(r)$$

for all $t \ge 0$.)

A single stable tree

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As before, it is possible to consider instead a single Galton-Watson tree conditioned to have total progeny n. Duquesne (2005) shows that we again get convergence of the height process of the conditioned Galton-Watson tree to an excursion $(e(t), 0 \leq t \leq 1)$ of the β-stable height process. This codes a single β-stable tree.

Stable fragmentations

Miermont (2003) generalized the construction of the Brownian fragmentation by replacing the Brownian excursion by an excursion $(e(x), 0 \leq x \leq 1)$ of the height process associated with the stable tree of index $\beta \in (1, 2)$.

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For $\beta \in (1, 2)$, it turns out that the derived fragmentation $(F(t), t \geq 0)$ is self-similar of index $\alpha = 1/\beta - 1$. Here, the dislocation measure ν only puts mass on infinite sequences $\mathbf{s}$, so that every split is into infinitely many pieces.

Exctinction for stable fragmentations

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It turns out that for an excursion of the β-stable height process, $\beta \in (1,2)$, this maximum is attained at a unique point x_*.

Let $O_*(t)$ be the interval component of $O(t)$ containing x_* and let $F_*(t)$ be its length. We will refer to both as the last fragment.

Question:

How does the stable fragmentation process behave as it approaches its extinction time?

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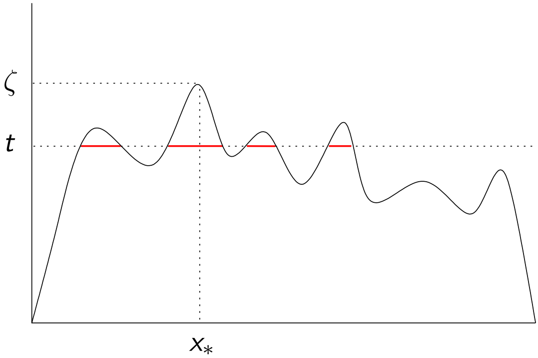
Clearly, $F(t) \to (0, 0, \dots)$ as $t \to \zeta$. But how does $F((\zeta - t)^+)$ scale as $t \to 0$?



Results

Theorem. Let $(F(t), t \geq 0)$ be the ranked stable fragmentation with index of self-similarity $\alpha \in [-1/2, 0)$. Then

$$t^{1/\alpha} F((\zeta - t)^+) \xrightarrow{d} F_\infty \quad \text{as } t \to 0,$$

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Results

Corollary. Recall that $(F_*(t), t \geq 0)$ is the last fragment process. Then

$$t^{1/\alpha} F_*((\zeta - t)^+) \xrightarrow{d} \zeta_*^{1/\alpha} \quad \text{as } t \to 0,$$

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In particular, if $\alpha = -1/2$, this is the usual notion of a size-biased version of ζ.

Proof for the Brownian case on blackboard!

Future work

In future work we will treat the case of a general self-similar fragmentation. In the stable case, we very much relied on an excursion theory approach. However, in general we do not have access to an excursion theory so our methods will be different. We nonetheless conjecture that in generic cases $t^{1/\alpha}$ is the correct re-scaling for non-trivial limiting behaviour.

Reference

C. Goldschmidt and B. Haas,
Behavior near the extinction time in self-similar fragmentations I: the stable case,
to appear in *Annales de l'Institut Henri Poincaré, 2009*

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Thank you for listening!