Scaling limits of critical random trees and graphs

Christina Goldschmidt
Department of Statistics and Lady Margaret Hall
PART I: RANDOM TREES

[Based on work of Aldous, Duquesne, Le Gall, Le Jan, ...]
Galton–Watson trees

Consider a Galton–Watson branching process with offspring distribution $p = (p_k)_{k \geq 0}$ such that $p_0 + p_1 < 1$. We may associate with it a family tree $T$. 

![Diagram of a Galton–Watson tree](image)
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Motivation: many natural combinatorial models of random trees may be recovered by taking specific offspring distributions, for example,

- Poisson(1): uniform labelled trees
- Geometric(1/2): uniform plane/ordered trees
- \( p_0 = p_2 = 1/2 \): uniform (complete) binary trees (\( n \) odd).

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Functional encoding

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Fix a tree $T$ with $|T| = n$. Let $\nu(i), 0 \leq i \leq n - 1$ be the vertex-labels in lexicographic order and write $d(u, v)$ for the graph distance between two vertices in the tree.
Height process

Let $G(k) = d(v(0), v(k))$ for $0 \leq k \leq n - 1$, the generation of vertex $v(k)$.
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It’s easy to recover the tree from its height process.
Depth-first walk

Let $C(k)$ be the number of children of $v(k)$, for $0 \leq k \leq n - 1$, let $S(0) = 0$ and for $1 \leq k \leq n$, 

$$S(k) = \sum_{i=0}^{k-1} (C(i) - 1).$$
Depth-first walk

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Observe that the depth-first walk must hit \(-1\) at step \(n\), since

\[ \sum_{i=0}^{n-1} C(i) = n - 1 \text{ i.e. } \sum_{i=0}^{n-1} (C(i) - 1) = -1. \]
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Moreover, it must be at 0 or above until then, since there must be a non-negative number of other nodes left to explore.
Height process and depth-first walk

The height process (and therefore the tree) may be recovered from the depth-first walk via

\[ G(k) = \# \left\{ 0 \leq j \leq k - 1 : S(j) = \min_{j \leq \ell \leq k} S(\ell) \right\}. \]
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**Essential idea:** whenever the depth-first walk enters a new subtree, it remains above its value at the start of the subtree until it leaves the subtree, when it goes one step lower. So instants \( j \) such that \( S(j) = \min_{j \leq \ell \leq k} S(\ell) \) correspond to subtrees that we have entered but not yet finished exploring by the time we visit \( v(k) \). But the number of such instants is the same as the generation of \( v(k) \).
Galton–Watson forests

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Instead, consider a sequence of i.i.d. Galton–Watson trees. It is convenient to start the depth-first walk for the $i$th tree from $-i + 1$, so that at the end of each tree the depth-first walk attains a new minimum. If we do this then defining

$$G(k) = \# \left\{ 0 \leq j \leq k - 1 : S(j) = \min_{j \leq \ell \leq k} S(\ell) \right\}$$

as before yields a process which is at 0 every time we visit the root vertex of a component.
Galton–Watson forests

Since the numbers of children of the different vertices are i.i.d., 
$(S(k))_{k\geq 0}$ is a random walk with step-sizes $C(k) - 1$, $k \geq 0$. Since $\mathbb{E}[C(0)] = 1$, this random walk is centred. (In contrast, the law of the height process is much harder to describe.)
Scaling limits

Standard (generalised) functional central limit theorems give the following.

1. Suppose that $\sigma^2 = \sum_{k=0}^{\infty} (k-1)^2 p_k < \infty$. Then
   \[
   \frac{1}{\sqrt{n}} (S(\lfloor nt \rfloor), t \geq 0) \xrightarrow{d} \sigma(B_t), t \geq 0,
   \]
   where $B$ is a standard Brownian motion.

2. Suppose that $p_k \sim c k^{-(1+\alpha)}$ as $k \to \infty$ for some $\alpha \in (1, 2)$. Then
   \[
   \frac{1}{n^{1/\alpha}} (S(\lfloor nt \rfloor), t \geq 0) \xrightarrow{d} C(L_t), t \geq 0,
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   where $L$ is a spectrally positive $\alpha$-stable Lévy process.

3. More general settings (with $n$-dependent offspring distributions) give rise to more general spectrally positive Lévy processes in the limit.
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Interpretation

Recall that the depth-first walk attains a new minimum every time it starts exploring a new component. In the limit, the excursions above the running infimum should encode limiting “trees”. The height process gives us a way to deal with them as metric spaces.

[Figure 6.6: Simulations of the continuous setting: construction of the stable lamination.]

[Pictures by Igor Kortchemski]
The height process is, however, more complicated. We have

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The limiting analogue \((H_t, t \geq 0)\) is defined as a (suitably normalised) local time at level 0 of the process

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In the Brownian and stable cases, the height process is continuous.
1. In the Brownian case, it turns out that

$$\frac{1}{\sqrt{n}} (G(\lfloor nt \rfloor), t \geq 0) \overset{d}{\to} \frac{2}{\sigma} (H_t, t \geq 0),$$

where $H_t$ is a reflected Brownian motion.

2. More generally, in the $\alpha$-stable case, we get

$$n^{-\frac{(\alpha-1)}{\alpha}} (G(\lfloor nt \rfloor), t \geq 0) \overset{d}{\to} C(H_t, t \geq 0).$$

Idea: excursions of the limiting height process above 0 code limiting trees ($\mathbb{R}$-trees), the tallest of which have heights of order $n^{\frac{\alpha-1}{\alpha}}$, $\alpha \in (1, 2]$. 
(Interpret distances vertically)
$\alpha$-stable trees ($\alpha = 1.1$ and $\alpha = 1.5$)

[Pictures by Igor Kortchemski]
PART II: RANDOM GRAPHS:

the Erdős–Rényi universality class
The Erdős–Rényi random graph

The simplest model of a random graph: take $n$ labelled vertices, join any pair by an edge independently with probability $p \in [0, 1]$.

$\text{Let } p = \frac{c}{n}.$

$\text{For } c > 1, \text{ there is a giant component consisting of order } n \text{ vertices with high probability.}$

$\text{For } c < 1, \text{ there are only small components, of size at most } O(\log n).$

$\text{At } c = 1, \text{ the critical case, the largest components have size on the order of } n^{2/3} \text{ and are “tree-like” in the sense that they only have a small number of edges more than a tree.}$

Modern proofs of this phase transition essentially involve comparing the components to branching processes.
The Erdős–Rényi random graph

[Erdős & Rényi (1960)]

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[Erdoes & Renyi (1960)]

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Critical random graph: depth-first walk

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Start from the vertex labelled 1. It has a $\text{Bin}(n - 1, 1/n) \approx \text{Po}(1)$ number of neighbours. Use the labels to obtain an ordering on the neighbours, and then proceed in a depth-first manner.
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As before, let

\[ S^n(k) = \sum_{i=0}^{k-1} (C^n(i) - 1), \quad 0 \leq k \leq n, \]

where \( C^n(i) \) is the number of children of the \( i \)th vertex explored in depth-first order.
Depth-first walk

As long as we have explored $o(n)$ vertices, it remains the case that the number of children of a vertex is approximately $\text{Po}(1)$, although as we eat away at the vertices, there are fewer and fewer possible neighbours. This effect appears in the limit as a negative drift.
**Depth-first walk**

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**Theorem (Aldous (1997), breadth-first)**

\[
\frac{1}{n^{1/3}} \left( S^n(\lfloor tn^{2/3} \rfloor), t \geq 0 \right) \overset{d}{\to} \left( B_t - \frac{t^2}{2}, t \geq 0 \right).
\]

[Picture by Louigi Addario-Berry]
Component sizes and surplus edges

We start a new component every time we create a new minimum. Let

\[ Z_t := B_t - \frac{t^2}{2} - \inf_{0 \leq s \leq t} \left( B_s - \frac{s^2}{2} \right), \quad t \geq 0. \]

This represents the limiting rescaled number of vertices seen but not fully explored at time \( t \).
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This represents the limiting rescaled number of vertices seen but not fully explored at time \( t \). Every time \( Z \) hits 0, a new component begins.

[Picture by Louigi Addario-Berry]
Aldous also showed that the edges forming cycles arise as a point process which in the limit is Poisson with intensity given by $Z_t$ at time $t$.

We may think of the Poisson points as occurring with intensity 1 in the area under the graph of $Z$. 

[Picture by Louigi Addario-Berry]
Let $\mathbf{C}^n = (C_1^n, C_2^n, \ldots)$ be the sizes of the components, listed in decreasing order, and $\mathbf{S}^n = (S_1^n, S_2^n, \ldots)$ the corresponding numbers of surplus edges.

Theorem (Aldous (1997))

As $n \to \infty$, $(n^{-2/3}C_n, S_n) \to (C, S)$, where the convergence of the component sizes is in $\ell_{\downarrow}^2$. 
Let $\mathbf{C}^n = (C_1^n, C_2^n, \ldots)$ be the sizes of the components, listed in decreasing order, and $\mathbf{S}^n = (S_1^n, S_2^n, \ldots)$ the corresponding numbers of surplus edges. Let $\mathbf{C} = (C_1, C_2, \ldots)$ be the lengths of the excursions above 0 of $Z$ listed in decreasing order, and let $\mathbf{S} = (S_1, S_2, \ldots)$ be the corresponding numbers of Poisson points.
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**Theorem (Aldous (1997))**

As $n \to \infty$,

$$\left(n^{-2/3} \mathbf{C}^n, \mathbf{S}^n\right) \xrightarrow{d} (\mathbf{C}, \mathbf{S}),$$

where the convergence of the component sizes is in $\ell_2^\downarrow$.
The excursions encode spanning subtrees, and the points of the Poisson process tell us where to make vertex-identifications.
Metric space scaling limit

[Addario-Berry, Broutin & G. (2012)]

\((Z_t)_{t \geq 0}\) (the drifting Brownian motion reflected at its running infimum) has a time-inhomogeneous excursion measure at 0.
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$(Z_t)_{t \geq 0}$ (the drifting Brownian motion reflected at its running infimum) has a time-inhomogeneous excursion measure at 0. However, the inhomogeneity manifests itself in the selection of the lengths of the excursions only.
Metric space scaling limit

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\((Z_t)_{t \geq 0}\) (the drifting Brownian motion reflected at its running infimum) has a time-inhomogeneous excursion measure at 0. However, the inhomogeneity manifests itself in the selection of the lengths of the excursions only. Conditionally on having length \(x\), an excursion \(\tilde{e}\) of \((Z_t)_{t \geq 0}\) above 0 has law determined by

\[
\mathbb{E} [f(\tilde{e})] = \frac{\mathbb{E} [f(e) \exp (\int_0^x e(u)du)]}{\mathbb{E} [\exp (\int_0^x e(u)du)]},
\]

where \(e\) is a Brownian excursion of length \(x\).
(\(Z_t\))_{t \geq 0} (the drifting Brownian motion reflected at its running infimum) has a time-inhomogeneous excursion measure at 0. However, the inhomogeneity manifests itself in the selection of the lengths of the excursions only. Conditionally on having length \(x\), an excursion \(\tilde{e}\) of \((Z_t)_{t \geq 0}\) above 0 has law determined by

\[
\mathbb{E}[f(\tilde{e})] = \frac{\mathbb{E}[f(e) \exp(\int_0^x e(u)du)]}{\mathbb{E}[\exp(\int_0^x e(u)du)]},
\]

where \(e\) is a Brownian excursion of length \(x\).

Conditionally on \(\tilde{e}\), we get a Poisson number of vertex-identifications with mean

\[
\int_0^x \tilde{e}(u)du.
\]

Each identifies a random leaf with a uniformly-chosen point down the backbone to the root.
Metric space scaling limit

[Addario-Berry, Broutin & G. (2012)]
Universality

The same objects have been shown to occur as the scaling limit in a variety of settings.
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Component sizes (and in some cases surpluses):

- Critical percolation on random regular graphs: Nachmias & Peres (2010)
Universality

Metric structure:

- Very general, encompassing all of the above models; framework based on scaling exponents and approximation by the multiplicative coalescent: Bhamidi, Sen & X. Wang (2014+), Bhamidi, Broutin, Sen & X. Wang (2014+)
Universality

Metric structure:

▷ Very general, encompassing all of the above models; framework based on scaling exponents and approximation by the multiplicative coalescent: Bhamidi, Sen & X. Wang (2014+), Bhamidi, Broutin, Sen & X. Wang (2014+)

See Shankar Bhamidi’s talk, Continuum scaling limits of critical inhomogeneous random graph models, on Thursday afternoon in the Interacting particle systems and their scaling limits session.
Conjectural Erdős–Rényi universality class

The Erdős–Rényi random graph can be thought of as a mean-field model for percolation on a finite graph. It is conjectured that for a wide variety of finite base graphs $G_n$ which are sufficiently “high dimensional”, although the percolation critical point will be model-dependent, the behaviour in the vicinity of that critical point should essentially be the same as in the Erdős–Rényi model.
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The Erdős–Rényi random graph is a poor model for many real-world networks. In particular, there is a lot of interest in modelling situations where we observe power-law degree distributions.
Outside the Erdős–Rényi universality class

There has been much recent work on a particular model for inhomogeneous random graphs (the Norros–Reittu model) with parameters chosen to give power-law degrees. Analogous results to those we obtained in the Erdős–Rényi setting have been developed in a series of papers by Bhamidi, van der Hofstad, van Leeuwaarden and Sen, and in work in progress by Broutin, Duquesne & M. Wang.
Outside the Erdős–Rényi universality class

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The limit spaces they obtain are certain tilted inhomogeneous continuum random trees [Aldous & Pitman (2000)] again with a finite number of additional vertex-identifications. The approach via the height process used for the Erdős–Rényi random graph doesn’t work here, since there is currently no convergence result for the height processes in this context.
PART III: RANDOM GRAPHS:

i.i.d. degrees with power-law tails

[Based on work in progress with Guillaume Conchon-Kerjan (ENS)]
Random graphs with given degrees

Consider a graph $G_n$ chosen uniformly at random from those such that the vertex set is $\{1, 2, \ldots, n\}$ and vertex $i$ has degree (number of neighbours) $d_i$. 
Standard method for generating a (multi)graph on \( n \) vertices with given degrees \( d_1, d_2, \ldots, d_n \).

Suppose \( d_i \geq 1 \) for all \( 1 \leq i \leq n \) and \( \ell_n = \sum_{i=1}^{n} d_i \) is even.

Assign \( d_i \) “half-edges” or “stubs” to the vertex labelled \( i \). Number the stubs in an arbitrary way from 1 to \( \ell_n \). Now pair the half-edges uniformly at random to form edges.
Configuration model

Example: \( n = 5 \) and \( d_1 = 3, \ d_2 = 2, \ d_3 = 1, \ d_4 = 4, \ d_5 = 2. \)
Configuration model

Example: $n = 5$ and $d_1 = 3$, $d_2 = 2$, $d_3 = 1$, $d_4 = 4$, $d_5 = 2$. 
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This procedure can give rise to loops or multiple edges, in which case we have a multigraph. But if we condition the graph to have no loops or multiple edges (to be simple), then it is uniformly chosen from the set of graphs with these degrees.
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(It’s not always the case that a particular degree sequence with even sum can give a simple graph, so this conditioning may not always be valid. This will not be problematic in the context we consider.)
Suppose that we have i.i.d. random degrees, $D_1, D_2, \ldots, D_n$ having finite variance, and let $\gamma = \mathbb{E} [D(D - 1)] / \mathbb{E} [D]$. 

Important point: we can generate the matching of the half-edges edge by edge, in any order that is convenient. In particular, rather than first sampling the graph and then exploring it, we will find it useful to generate the graph step-by-step as we explore it.
Suppose that we have i.i.d. random degrees, $D_1, D_2, \ldots, D_n$ having finite variance, and let $\gamma = \mathbb{E}[D(D - 1)] / \mathbb{E}[D]$. (We can resolve the problem of $\sum_{i=1}^{n} D_i$ potentially being odd by taking $n$ to have degree $D_n + 1$ in that case; this has an asymptotically negligible effect on the graph.)
Configuration model with i.i.d. degrees

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Then, as $n \to \infty$,

$$\mathbb{P}(G_n \text{ is simple}) \to \exp(-\gamma/2 - \gamma^2/4).$$
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Configuration model with i.i.d. degrees

[Molloy & Reed (1995)]

Recall that $\gamma = \mathbb{E}[D(D-1)] / \mathbb{E}[D]$. The critical point for the emergence of a giant component is $\gamma = 1$. 

Intuition: imagine exploring the graph, as usual in a depth-first manner, starting from an arbitrarily-chosen vertex. The first half-edge we look at connects to a vertex chosen with probability proportional to its degree, and this is true whenever we look to connect another half-edge. Assuming that we have only looked at a small number of vertices, the chosen degree should have law close to the size-biased distribution $P(D^* = k) = kP(D = k) \mathbb{E}[D]$, $k \geq 1$. So the "offspring distribution" to which we should compare is the law of $D^* - 1$ which has $\mathbb{E}[D^* - 1] = \mathbb{E}[D^2] \mathbb{E}[D] - 1 = \mathbb{E}[D(D-1)] \mathbb{E}[D] = \gamma$. 
Configuration model with i.i.d. degrees

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\[
\mathbb{P}(D^* = k) = \frac{k \mathbb{P}(D = k)}{\mathbb{E}[D]}, \quad k \geq 1.
\]
Configuration model with i.i.d. degrees

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So the “offspring distribution” to which we should compare is the law of $D^* - 1$ which has

$$\mathbb{E}[D^* - 1] = \frac{\mathbb{E}[D^2]}{\mathbb{E}[D]} - 1 = \frac{\mathbb{E}[D(D - 1)]}{\mathbb{E}[D]} = \gamma.$$
We have i.i.d. degrees $D_1, D_2, \ldots, D_n$ with law $\nu$ such that

1. $\mathbb{P}(D_1 \geq 1) = 1$
2. $\gamma = \frac{\mathbb{E}[D_1(D_1 - 1)]}{\mathbb{E}[D_1]} = 1$
3. $\mathbb{P}(D_1 = k) \sim ck^{-(\alpha+2)}$ as $k \to \infty$, for some $c > 0$, $\alpha \in (1, 2)$.

Write $\mu = \mathbb{E}[D_1]$ (our conditions imply that $\mu \in (1, 2)$).
\[ \alpha = 1.2 \]
$\alpha = 1.5$

[Picture by Delphin Sénizergues]
$\alpha = 1.8$

[Picture by Delphin Sénizergues]
Sample the degrees $D_1, D_2, \ldots, D_n$ and then start from a vertex $v(0)$ chosen with probability proportional to its degree.

For $k \geq 0$, proceed as follows.
Depth-first exploration

[Riordan (2012); Joseph (2014)]

$v(k)$

$v(i), i < k$
Depth-first exploration

[Riordan (2012); Joseph (2014)]

\[ v(k), \quad v(i), \quad i < k \]
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[Riordan (2012); Joseph (2014)]
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[Riordan (2012); Joseph (2014)]
Depth-first exploration
[Riordan (2012); Joseph (2014)]

Important point: in any case, we see the vertices in size-biased order of degree: $(\hat{D}_1^n, \hat{D}_2^n, \ldots, \hat{D}_n^n)$. 
Approximate depth-first walk

Let $\tilde{S}^n(0) = 0$ and

$$\tilde{S}^n(k) = \sum_{i=1}^{k} (\hat{D}^n_i - 2), \quad k \geq 1.$$
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This is an approximation in two ways:

1. for the vertex at the start of a component, the number of children is actually $\hat{D}_i^n$ rather than $\hat{D}_i^n - 1$;
2. it ignores the possibility of surplus edges.

Neither is problematic in the limit.
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1. for the vertex at the start of a component, the number of children is actually $\hat{D}^n_i$ rather than $\hat{D}^n_i - 1$;
2. it ignores the possibility of surplus edges.

Neither is problematic in the limit.

Indeed, it is possible to show that there are only $O(1)$ surplus edges in the first $O(n^{\alpha/(\alpha+1)})$ steps.
Theorem (Joseph (2014))

\[ n^{-1/(\alpha+1)} \left( \tilde{S}^n([tn^{\alpha/(\alpha+1)}]), t \geq 0 \right) \xrightarrow{d} (\tilde{L}_t, t \geq 0), \]

where \( \tilde{L} \) is the process with independent increments characterised by its Laplace transform

\[
\mathbb{E}
\begin{bmatrix}
\exp(-\lambda \tilde{L}_t)
\end{bmatrix}
= \exp \left( \int_0^t ds \int_0^\infty dx (e^{-\lambda x} - 1 + \lambda x) \frac{c}{\mu x^{\alpha+1}} e^{-xs/\mu} - \lambda C_\alpha \frac{t^\alpha}{\mu^\alpha} \right),
\]

where \( C_\alpha = \frac{c \Gamma(2-\alpha)}{\alpha(\alpha-1)} \).
Component sizes

Let $\mathbf{C}^n = (C^n_1, C^n_2, \ldots)$ be the ordered component sizes of the multigraph $G_n$, and let $\mathbf{C} = (C_1, C_2, \ldots)$ be the ordered lengths of excursions of $\tilde{L}$ above its running infimum.

Theorem (Joseph (2014))

$$n^{-\alpha/(\alpha+1)} \mathbf{C}^n \xrightarrow{d} \mathbf{C}$$

as $n \to \infty$, in $\ell^\downarrow_2$. 

Note: this is the same scaling as in [Bhamidi, van der Hofstad & van Leeuwaarden (2012)], but a different limit.
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as $n \to \infty$, in $\ell_2^\perp$.

Note: this is the same scaling as in [Bhamidi, van der Hofstad & van Leeuwaarden (2012)], but a different limit.
Absolute continuity relations

Let $D_1^*, D_2^*, \ldots$ be i.i.d. with law $k \nu_k / \mu$, $k \geq 1$ (the true size-biased degree distribution) and let $S(0) = 0$ and

$$S(k) = \sum_{i=1}^{k} (D_i^* - 2), \quad k \geq 1.$$
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Since

$$\mathbb{E} [D_1^* - 2] = 0 \quad \text{and} \quad k\nu_k/\mu \sim \frac{c}{\mu} k^{-(\alpha+1)} \quad \text{as} \quad k \to \infty,$$

$S$ is a random walk in the domain of attraction of a spectrally positive $\alpha$-stable Lévy process $L$. 
Absolute continuity relations

[Conchon-Kerjan & G. (in progress)]

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$S$ is a random walk in the domain of attraction of a spectrally positive $\alpha$-stable Lévy process $L$, with Laplace transform

$$\mathbb{E}[\exp(-\lambda L_t)] = \exp \left( t \int_{0}^{\infty} dx \left( e^{-\lambda x} - 1 + \lambda x \frac{c}{\mu x^{\alpha+1}} \right) \right) = \exp \left( C_{\alpha} \lambda^\alpha t/\mu \right), \quad \lambda \geq 0.$$
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$$= \exp \left( C_\alpha \lambda^\alpha t/\mu \right), \quad \lambda \geq 0.$$ 

$L$ encodes a forest of stable trees.
Absolute continuity relations

Proposition

For every $t \geq 0$, we have the following absolute continuity relation: for every suitable test-functional $F$,

$$
\mathbb{E} \left[ F \left( \tilde{L}_s, 0 \leq s \leq t \right) \right] = \mathbb{E} \left[ \exp \left( - \frac{1}{\mu} \int_0^t s dL_s - C_\alpha \frac{t^{\alpha+1}}{\left( \alpha + 1 \right) \mu^{\alpha+1}} \right) F(L_s, 0 \leq s \leq t) \right].
$$
Absolute continuity relations

There is also a discrete analogue: for $m < n$,

$$
\mathbb{E} \left[ f(\hat{D}_1^n, \hat{D}_2^n, \ldots, \hat{D}_m^n) \right] = \mathbb{E} \left[ \phi_m^n(D_1^*, D_2^*, \ldots, D_m^*) f(D_1^*, D_2^*, \ldots, D_m^*) \right]
$$

where for $m = \lfloor tn^{\alpha/(\alpha+1)} \rfloor$,

$$
\phi_m^n(D_1^*, D_2^*, \ldots, D_m^*) \xrightarrow{d} \exp \left( -\frac{1}{\mu} \int_0^t s dL_s - C_\alpha \frac{t^{\alpha+1}}{(\alpha + 1)\mu^{\alpha+1}} \right).
$$
Height processes

Let

$$\tilde{G}^n(k) = \# \left\{ 0 \leq j \leq k - 1 : \tilde{S}^n(j) = \min_{j \leq \ell \leq k} \tilde{S}^n(\ell) \right\}$$
Height processes

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\[ \tilde{G}^n(k) = \# \left\{ 0 \leq j \leq k - 1 : \tilde{S}^n(j) = \min_{j \leq \ell \leq k} \tilde{S}^n(\ell) \right\} \]
and define a height process \( \tilde{H} \) via
\[
\mathbb{E} \left[ f(\tilde{L}_u, \tilde{H}_u, 0 \leq u \leq t) \right] = \mathbb{E} \left[ \exp \left( -\frac{1}{\mu} \int_0^t s dL_s - \frac{C_\alpha t^{\alpha+1}}{(\alpha + 1)\mu^{\alpha+1}} \right) f(L_u, H_u, 0 \leq u \leq t) \right],
\]
where \( L \) and \( H \) are a spectrally positive \( \alpha \)-stable Lévy process and the corresponding height process, respectively.
Height processes

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\]
where \( L \) and \( H \) are a spectrally positive \( \alpha \)-stable Lévy process and the corresponding height process, respectively. Using Duquesne & Le Gall’s theorem we can considerably strengthen Joseph’s result:

**Theorem**

\[
\left( n^{-\frac{1}{\alpha+1}} \tilde{S}^n(\lfloor un^{\alpha/(\alpha+1)} \rfloor), n^{-\frac{\alpha-1}{\alpha+1}} \tilde{G}^n(\lfloor un^{\alpha/(\alpha+1)} \rfloor), 0 \leq u \leq t \right) \\
\xrightarrow{d} \left( \tilde{L}_u, \tilde{H}_u, 0 \leq u \leq t \right).
\]
The change of measure acts on the excursions of the Lévy process to give that an excursion of length $x$ of $\tilde{L}$ above its infimum is such that

$$E[f(\tilde{e})] = E[f(e) \exp(1/\mu \int_0^x e(u) \, du)] E[\exp(1/\mu \int_0^x e(u) \, du)],$$

where $e$ is an excursion of $L$ above its infimum, conditioned to have length $x$. So the limit spanning trees are tilted stable trees. (Recall that the random quantity in the exponential martingale is $-1/\mu \int_0^t s \, dL_s = -tL_t + 1/\mu \int_0^t L_s \, ds$ and note that $L_t = 0$ at the beginning and end of each excursion.)
Metric space scaling limit: the stable graph

This will enable us to deduce the convergence of the metric structure of the depth-first spanning trees.

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where $e$ is an excursion of $L$ above its infimum, conditioned to have length $x$.

(Recall that the random quantity in the exponential martingale is

$$-\frac{1}{\mu} \int_0^t s dL_s = -\frac{tL_t}{\mu} + \frac{1}{\mu} \int_0^t L_s ds$$

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\mathbb{E} \left[ f(\tilde{e}) \right] = \frac{\mathbb{E} \left[ f(e) \exp \left( \frac{1}{\mu} \int_0^x e(u) \, du \right) \right]}{\mathbb{E} \left[ \exp \left( \frac{1}{\mu} \int_0^x e(u) \, du \right) \right]},
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\]

and note that \( L_t = 0 \) at the beginning and end of each excursion.)
Neither multiple edges nor loops occur until $n^\alpha/(\alpha+1)$ steps of the exploration have occurred, so conditioning the graph to be simple does not affect the distribution of the large components.
Metric space scaling limit: the stable graph

The surplus edges can again be shown to occur as a Poisson point process with unit intensity in the area under the graph of

$$
\left( \tilde{L}_t - \inf_{0 \leq s \leq t} \tilde{L}_s, t \geq 0 \right).
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$$
\left( \tilde{L}_t - \inf_{0 \leq s \leq t} \tilde{L}_s, t \geq 0 \right).
$$

In the limit, the vertex-identifications are from leaves to hubs (branch-points of infinite degree).
Consequences

Distributional and geometric information about the limiting spaces may be deduced from knowledge of the stable trees.
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Distributional and geometric information about the limiting spaces may be deduced from knowledge of the stable trees.

For example, the Hausdorff dimension of the limiting metric spaces is $\frac{\alpha}{\alpha - 1}$ almost surely.
There is a beautiful construction of the Brownian CRT via line-breaking, due to Aldous. In [Addario-Berry, Broutin & G. (2010)], we showed that a closely related line-breaking construction can be used to build a limit component in the Erdős–Rényi random graph. In [G. & Haas (2015)], we proved a (more complicated) line-breaking construction for the stable trees. I expect that there will be a related construction of the components of the stable graph.
Perspectives: generalisations

The absolute continuity relation holds for a broad class of spectrally positive Lévy processes which may be used to encode a forest, which suggests that these results should be generalisable beyond the stable setting.
Open problem

How can one relate the limits obtained by Bhamidi, van der Hofstad, van Leeuwaarden and Sen in the setting of the Norros-Reittu model to the stable graph? Can one obtain the stable graph by averaging?
Thank you for listening!