A large deviation principle for the normalized excursion of α-stable Lévy processes without negative jumps

by

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Abstract – We establish a large deviation principle for the normalized excursion and bridge of an α-stable Lévy process without negative jumps, with $1 < \alpha < 2$. Based on this, we derive precise asymptotics for the tail distributions of functionals of the normalized excursion and bridge, in particular, the area and maximum functionals. We advocate the use of the Skorokhod M1 topology, rather than the more usual J1 topology, as we believe it is better suited to large deviation principles for Lévy processes in general.

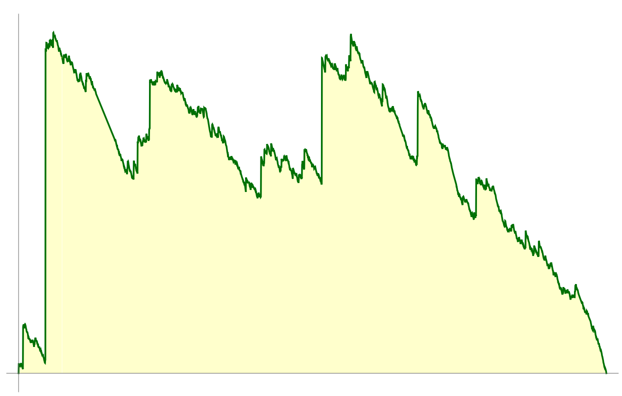

Figure 1 – Simulation of the area under the normalized excursion of a $\frac{4}{3}$-stable Lévy process without negative jumps.

Contents

1. Introduction

Fix $\alpha \in (1, 2)$. Let $L = (L_t, t \geqslant 0)$ be an α-stable Lévy process without negative jumps, having Laplace transform

$$\mathbb{E}\left[\exp(-\lambda L_t)\right] = \exp(t\lambda^{\alpha}), \qquad \lambda, t > 0.$$

Our main goal in this paper is to establish a strong large deviation principle for the normalized bridges and excursions of the process L. This contributes to the rich study of functional large deviation principles (LDP) for Lévy processes and related processes, which has its roots in the classical theorem of Cramér and its extension to random walks, see Chapter 5.1 in [8], and references therein. For a Lévy process $(X(t), t \geqslant 0)$, the natural setting is to consider the family of renormalized processes

$$X_T = (X(Tt)/T, 0 \leqslant t \leqslant 1). \tag{1.1}$$

The case where X is Brownian motion is addressed by a famous theorem of Schilder [27], who showed a large deviation principle with speed T in the space of continuous functions, where the rate function is the Dirichlet energy. Other Lévy processes are addressed in the landmark paper by Lynch and Sethuraman [19], which was extended in various directions in particular by Borovkov, Mogul'skiĭ and others, see for instance [5, 6, 15, 16, 20, 21] and references therein. However, the vast majority of the results in the above references assume that the Lévy process has a vanishing Gaussian part, as well as the Cramér condition that $\mathbb{E}\left[e^{\lambda X(1)}\right] < \infty$ for every λ in a non-empty neighborhood of 0. These conditions imply that the trajectories of the Lévy process have finite variation almost surely, and that the law of $X(1)$ has exponential tails. Various situations may occur when the Cramér condition does not hold. The references [11, 12] consider the case of stretched exponential tails, while [25] considers the situation where the tail of $X(1)$ is regularly varying. In these cases, large deviation principles hold with sublinear speeds, and even logarithmic speed when the tails are regularly varying.

The case of stable Lévy processes with no negative jumps is in a sense a boundary case of the works mentioned above, due to the asymmetric nature of the tails of L_1: we have

$$\mathbb{P}\left(L_1 > x\right) \asymp \frac{C_\alpha}{x^\alpha}, \qquad \mathbb{P}\left(L_1 < -x\right) \asymp \exp(-c_\alpha x^{\alpha'}), \qquad x \to \infty,$$

where $\alpha' = \alpha/(\alpha - 1) \in (2, \infty)$ is the conjugate exponent of α, $C_\alpha = -\frac{1}{\Gamma(1-\alpha)}$ and $c_\alpha = (\alpha - 1)/\alpha^{\alpha'}$. Heuristically, although it is "easy" for the process to go up, it is "costly" for it to go down, and large deviation probabilities may have different speeds depending on whether the events involved allow the process to "go down" or not.

However, considering bridges and excursions of such processes is a way to root them at a given value at time 1, which prevents the process from "going up too much". This phenomenon is well-known and was already exploited in [1, 17], in particular in the study of the heights of random trees. However, to our knowledge it has not been used to derive an actual LDP result for stable excursions and bridges, and we fill this gap in the present work. Note that LDPs for bridge-like random walks and Lévy processes were considered under the Cramér condition in [4].

Although we believe our results should have extensions to a much larger class of Lévy process bridges and excursions, we focus here on the particular case of stable processes. In this case, precise estimates are known for the transition densities and the entrance law of the excursion measure, which allow us to provide a rather straightforward extension of the method of proof presented by Serlet in [28]. However, our proof differs from the latter in a crucial aspect. As is often the case in LDP theory, the choice of an appropriate topology on the path space is an important matter. In [19], the authors derived a strong LDP for Lévy processes as in (1.1) in a "weak" topology, and observed that the rate function is not good (in the sense that it does not have compact level sets) in the natural Skorokhod J1 topology. In a series of papers, Borovkov and Mogul'skiĭ improved these results by considering local versions of the LDP in the J1 topology, or by working in the completion of the Skorokhod J1 metric. Unsurprisingly, similar questions arise in our context. Indeed, since it is much less costly for the process L to go up rather than to go down, a similar property also holds for its bridges and excursions, and this implies that in the large deviation regimes considered in this paper (and in stark contrast to [25]), we cannot distinguish between the situation where the process performs one big jump, or two jumps of half the size at extremely close locations, precluding exponential tightness in the J1 topology.

Fortunately, Skorokhod introduced three other possible topologies, called M1, J2 and M2, on spaces of càdlàg functions, and M1 will turn out to suit our purposes, with a very minor adaptation. This will allow us to introduce a distance function dist that induces a slight variant of the M1 topology and makes $(\mathbb{D}[0,1], \mathrm{dist})$ a Polish space in which a strong LDP holds for the excursion and bridge of the process L. We note that Mogul'skiĭ and others [6, 21] also considered the weaker M2 topology, but in the context of processes satisfying the Cramér condition. The M1 topology was also used by O'Brien [23] to prove LDPs for the processes $(\mathsf{L} \vee 1)^{\varepsilon}$ as $\varepsilon \downarrow 0$, but these large deviations regimes are very different from the one considered here.

It might be the case that a weak LDP holds for $\varepsilon \mathbb{e}$ in the J1 topology, or for the unconditioned scaled processes $\varepsilon \mathsf{L}$ in either the J1 topology or in our modified M1 topology, but we do not pursue these questions here.

1.1. Main results.

Let $\mathbb{e} = (\mathbb{e}_t, 0 \leqslant t \leqslant 1)$ be the normalized excursion of L above its past infimum [7], and let $\mathbb{b}^{(x)} = (\mathbb{b}^{(x)}_t, 0 \leqslant t \leqslant t)$ be the bridge of L from 0 to $x \in \mathbb{R}$ with unit duration. We set $\mathbb{b} = \mathbb{b}^{(0)}$. These processes satisfy a.s. $\mathbb{e}_0 = \mathbb{e}_1 = \mathbb{b}^{(x)}_0 = 0$, $\mathbb{b}^{(x)}_1 = x$, and $\mathbb{e}_t > 0$ for every $t \in (0,1)$. We view $\mathbb{e}$ and $\mathbb{b}^{(x)}$ as random variables in the space $\mathbb{D}[0,1]$ of "càdlàg" functions $f : [0,1] \to \mathbb{R}$, that is, functions which are right-continuous at every point $t \in [0,1)$ and have left-limits at every point $t \in (0,1]$. We denote by $f(t+) = f(t)$ and $f(t-)$ the right- and left-limits of $f \in \mathbb{D}[0,1]$ at t, whenever applicable. We turn $\mathbb{D}[0,1]$ into a measurable space by equipping it with the σ-algebra generated by the evaluation maps $f \mapsto f(t)$ for $t \in [0,1]$.

Let us recall the definition of a large deviation principle. If S is a topological space, a *rate function* is a lower semicontinuous function $I : S \to [0,\infty]$, *i.e.* a function such that the level sets $\mathcal{L}_I(c) = \{x \in S : I(x) \leqslant c\}$ are closed for every $c \geqslant 0$. A rate function is called *good* if the sets $\mathcal{L}_I(c), c \geqslant 0$ are compact.

Definition 1.1. Let $\beta > 0$ be fixed. A family $(X_\varepsilon)_{\varepsilon>0}$ of random elements in the space S (endowed with the completed Borel σ-algebra) is said to satisfy the **large deviation principle** (LDP) with speed $\varepsilon^{-\beta}$, and rate function I, if for every Borel set $A \subseteq S$,

$$-\inf_{x\in\mathring{A}} I(x) \leqslant \liminf_{\varepsilon\downarrow 0} \varepsilon^\beta \log \mathbb{P}(X_\varepsilon \in A) \leqslant \limsup_{\varepsilon\downarrow 0} \varepsilon^\beta \log \mathbb{P}(X_\varepsilon \in A) \leqslant -\inf_{x\in\bar{A}} I(x).$$

We define a rate function in the following way. Assume that $f \in \mathbb{D}[0,1]$ has bounded variation, meaning that it can be written as the difference $g - h$ of two non-decreasing functions $g, h \in \mathbb{D}[0,1]$. This decomposition is not unique; however, it becomes unique if we further require that $g(0) = 0$ and that the Stieltjes measures dg and dh are mutually singular. This "minimal" decomposition is classically called the *Jordan decomposition* of f, and we write $g = f_\uparrow, h = f_\downarrow$. We denote by $H^{(\alpha)}$ the subspace of $\mathbb{D}[0,1]$ of functions f with bounded variation such that $f_\downarrow$ is an absolutely continuous function with derivative $f'_\downarrow \in \mathbb{L}^{\alpha'}[0,1]$, where $\alpha' = \alpha/(\alpha-1) \in (2,\infty)$ is the conjugate exponent of α. Set

$$D_{\mathrm{ex}}[0,1] \coloneqq \left\{f \in \mathbb{D}[0,1] :\ f(1) = 0,\ f \geqslant 0\right\}, \tag{1.2}$$

(note that we do not impose the usual condition that $f(0) = 0$) and define

$$H_{\mathrm{ex}} = H^{(\alpha)} \cap D_{\mathrm{ex}}[0,1]. \tag{1.3}$$

Let us define the rate function $I_{\mathbb{e}} : \mathbb{D}[0,1] \to [0,\infty]$ by the formula

$$I_{\mathbb{e}}(f) = \begin{cases} c_\alpha \int_0^1 |f'_\downarrow(s)|^{\alpha'}\, ds & \text{if } f \in H_{\mathrm{ex}}, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.4}$$

Alternatively, we may define $I_{\mathbb{e}}$ as follows for nonnegative functions f with bounded variation and such that $f(1) = 0$. Write $f = f_{\mathrm{ac}} + f_{\mathrm{sing}}$ as a sum of an absolutely continuous part and a singular part, and let $f_{\mathrm{sing}} = f_{\mathrm{sing}\uparrow} - f_{\mathrm{sing}\downarrow}$ be the Jordan decomposition of f_{sing}. Then we have

$$I_{\mathbb{e}}(f) = c_\alpha \int_0^1 (f'_{\mathrm{ac}}(s))_-^{\alpha'}\, ds + \infty \cdot f_{\mathrm{sing}\downarrow}(1), \tag{1.5}$$

where $(f'_{\mathrm{ac}}(s))_-$ denotes the negative part of $f'_{\mathrm{ac}}(s)$, and we let $I_{\mathbb{e}}(f) = \infty$ if f is not nonnegative, or if f does not satisfy $f(1) = 0$, or if f does not have bounded variation. In this way, we note that the shape of the rate function is exactly that involved in [19, Theorem 5.1] (although this theorem does not apply in our context) and the other references mentioned earlier in this introduction.

We defer a discussion of the topology until Section 2.2, where we will introduce a distance dist on the set $\mathbb{D}[0,1]$. We can now state our main result.

Theorem 1.2 (LDP for the normalized excursion $\mathbb{e}$). *The laws of* $(\varepsilon\mathbb{e}_t)_{t\in[0,1]}$ *satisfy an LDP in* $(\mathbb{D}[0,1], \mathrm{dist})$ *as* $\varepsilon \downarrow 0$ *with speed* $\varepsilon^{-\alpha'}$ *and good rate function* $I_{\mathbb{e}}$.

We also have the following result, proved in Section 6.3.

Proposition 1.3. *The rate function* $I_{\mathbb{e}}$ *is not a good rate function for the Skorokhod J1 topology. Moreover, the laws of* $(\varepsilon \mathbb{e}_t)_{t\in[0,1]}, 0 < \varepsilon < 1$, *are not exponentially tight in* $\mathbb{D}[0,1]$ *endowed with the Skorokhod J1 topology.*

Recall that in a Polish space, an LDP with a good rate function implies exponential tightness (see [19, Lemma 2.6] or [9]). However, since the rate function $I_{\mathbb{e}}$ is not good, this does not rule out the possibility that $\varepsilon\mathbb{e}$ satisfies the LDP in the J1 topology, and we do not know whether this property holds or not. Note that this problem has been the topic of several references dealing with random walks and Lévy processes under the Cramér condition, including [5, 19, 20], and at present there is no complete answer to this question in that context either. However, we believe that the M1 topology is a more natural choice in this context, since it is arguably a strong topology for which the rate functions are better behaved.

Theorem 1.2 allows us to deduce general LDPs for functionals of the normalized stable excursion. This extends the results of [10] dealing with Brownian excursions to the case of stable excursions, and was the initial motivation for the present work. Define the sets

$$K^{(\alpha)} = \left\{ f \in H^{(\alpha)} : \|f'_{\downarrow}\|_{\alpha'} \leqslant 1 \right\}, \qquad K_{\mathrm{ex}} = K^{(\alpha)} \cap D_{\mathrm{ex}}[0,1]. \tag{1.6}$$

It will be shown in Lemma 6.1 below that K_{ex} is a compact subset of $(\mathbb{D}[0,1], \mathrm{dist})$. This will imply, using the contraction principle, the following logarithmic asymptotics for the right tails of functionals of $\mathbb{e}$.

Theorem 1.4 (Logarithmic asymptotics for the right tails of functionals of $\mathbb{e}$). *Let* Φ *be a continuous nonnegative functional* $D_{\mathrm{ex}}[0,1] \to \mathbb{R}_+$ *which is also positive-homogeneous in the sense that* $\Phi(\lambda f) = \lambda\Phi(f)$ *for every* $f \in D_{\mathrm{ex}}[0,1]$ *and* $\lambda \geqslant 0$, *and not identically* 0 *on* K_{ex}. *Define* $X = \Phi(\mathbb{e})$ *and let*

$$\gamma_\Phi := \max\left\{\Phi(f) : f \in K_{\mathrm{ex}}\right\}.$$

Then $\varepsilon\Phi(\mathbb{e})$ *satisfies an LDP in* $\mathbb{R}_+$ *as* $\varepsilon \downarrow 0$ *with speed* $\varepsilon^{-\alpha'}$ *and good rate function* $J_\Phi(x) = c_\alpha \left(\frac{x}{\gamma_\Phi}\right)^{\alpha'}$. *In particular,*

$$-\log \mathbb{P}(X > x) \sim c_\alpha \left(\frac{x}{\gamma_\Phi}\right)^{\alpha'} \qquad \textit{as } x \to +\infty\,. \tag{1.7}$$

Using [13, Theorem 4.5], we have that (1.7) implies the following asymptotics for the Laplace transform and the moments:

$$\log \mathbb{E}\left[e^{tX}\right] \sim (\gamma_\Phi t)^\alpha \qquad \text{as } t \to +\infty, \tag{1.8}$$

$$\mathbb{E}\left[X^n\right]^{1/n} \sim \alpha^{\frac{1}{\alpha}} \gamma_\Phi \left(\frac{n}{e}\right)^{1/\alpha'} \qquad \text{as } n \to +\infty. \tag{1.9}$$

Taking as a particular case the functions $\Phi(f) = \int_0^1 f(s)\,ds$ and $\Phi(f) = \sup_{s\in[0,1]} f(s)$, we obtain the following result, which improves [24, Corollary 1.2] by pinning down the precise constants.

Corollary 1.5 (Logarithmic asymptotics for the right tails of the area under $\mathbb{e}$). *Set*

$$\mathcal{A}_{\mathrm{ex}} = \int_0^1 \mathbb{e}_t \,\mathrm{d}t.$$

Then it holds that

$$-\log \mathbb{P}\left(\mathcal{A}_{\mathrm{ex}} > x\right) \sim c_\alpha(\alpha+1)^{\frac{1}{\alpha-1}} x^{\alpha'} \qquad \text{as } x \to +\infty, \tag{1.10}$$

$$\log \mathbb{E}\left[e^{t\mathcal{A}_{\mathrm{ex}}}\right] \sim \frac{t^\alpha}{\alpha+1} \qquad \text{as } t \to +\infty, \tag{1.11}$$

$$\mathbb{E}\left[\mathcal{A}_{\mathrm{ex}}^n\right]^{1/n} \sim \left(\frac{\alpha}{\alpha+1}\right)^{\frac{1}{\alpha}} \left(\frac{n}{e}\right)^{1/\alpha'} \qquad \text{as } n \to +\infty. \tag{1.12}$$

Corollary 1.6 (Logarithmic asymptotics for the right tails of the supremum of $\mathbb{e}$). *It holds that*

$$-\log \mathbb{P}\left(\sup_{0\leqslant t\leqslant 1} \mathbb{e}_t > x\right) \sim c_\alpha x^{\alpha'} \qquad \text{as } x \to +\infty, \tag{1.13}$$

$$\log \mathbb{E}\left[e^{t\sup_{0\leqslant s\leqslant 1}\mathbb{e}_s}\right] \sim t^\alpha \qquad \text{as } t \to +\infty, \tag{1.14}$$

$$\mathbb{E}\left[\left(\sup_{0\leqslant t\leqslant 1} \mathbb{e}_t\right)^n\right]^{1/n} \sim \alpha^{1/\alpha}\left(\frac{n}{e}\right)^{1/\alpha'} \qquad \text{as } n \to +\infty. \tag{1.15}$$

1.2. Large deviation principles for bridges.

Theorems 1.2 and 1.4 have counterparts for bridges of the Lévy process L. For $a \in \mathbb{R}$, we let

$$D^{(a)}_{\mathrm{br}}[0,1] \coloneqq \{f \in \mathbb{D}[0,1] : \; f(1) = a\},$$
$$H^{(a)}_{\mathrm{br}} \coloneqq H^{(\alpha)} \cap D^{(a)}_{\mathrm{br}}[0,1].$$

We may now state the main results concerning the stable Lévy bridge. In this statement and the rest of the paper, for $a \in \mathbb{R}$, we let $a_- = a \vee 0$ and $a_- = (-a)_+$ be the positive and negative parts of a.

Theorem 1.7 (LDP for the stable bridge $\mathbb{b}^{(a)}$). *Let $(a_\varepsilon)_{\varepsilon>0}$ be such that $\varepsilon a_\varepsilon \to a$ as $\varepsilon \to 0$. Then the laws of $(\varepsilon \mathbb{b}^{(a_\varepsilon)}_t)_{t\in[0,1]}$ satisfy a LDP in $(\mathbb{D}[0,1], \mathrm{dist})$ as $\varepsilon \downarrow 0$ with speed $\varepsilon^{-\alpha'}$ and good rate function $I_{\mathbb{b},a}$ defined by*

$$I_{\mathbb{b},a}(f) = \begin{cases} c_\alpha\left(\int_0^1 |f'_\downarrow(s)|^{\alpha'}\,\mathrm{d}s - (a_-)^{\alpha'}\right) & \text{if } f \in H^{(a)}_{\mathrm{br}}, \\ +\infty & \text{otherwise.}\end{cases} \tag{1.16}$$

We obtain an analogue of Theorem 1.4 for bridges. Let

$$K_{\mathrm{br}} \coloneqq K^{(\alpha)} \cap D^{(0)}_{\mathrm{br}}[0,1].$$

Again, K_{br} is a compact subset of $(\mathbb{D}[0,1], \mathrm{dist})$, and the following logarithmic asymptotics hold for the right tails of functionals of $\mathbb{b}$.

Theorem 1.8 (Logarithmic asymptotics for the right tails of functionals of $\mathbb{b}$). *Let Φ be a continuous nonnegative functional $D^{(0)}_{\mathrm{br}}[0,1] \to \mathbb{R}_+$ which is also positive-homogeneous in the sense that $\Phi(\lambda f) = \lambda\Phi(f)$ for every $f \in D^{(0)}_{\mathrm{br}}[0,1]$ and $\lambda \geqslant 0$, and not identically 0 on K_{br}. Define $X = \Phi(\mathbb{b})$ and let*

$$\gamma_\Phi \coloneqq \max\{\Phi(f) : f \in K_{\mathrm{br}}\}.$$

Then $\varepsilon\Phi(\mathbb{b})$ satisfies an LDP in $\mathbb{R}_+$ as $\varepsilon \downarrow 0$ with speed $\varepsilon^{-\alpha'}$ and good rate function $J^{\mathbb{b}}_\Phi(x) = c_\alpha \left(\frac{x}{\gamma_\Phi}\right)^{\alpha'}$. In particular,

$$-\log \mathbb{P}(X > x) \sim c_\alpha \left(\frac{x}{\gamma_\Phi}\right)^{\alpha'} \qquad \text{as } x \to +\infty.$$

As an application, using the same proof as for Corollary 1.6, we may reprove an exact logarithmic asymptotic for the right tails of the supremum of the stable Lévy bridge obtained by Kortchemski in [17, Corollary 13].

Corollary 1.9 ([17, Corollary 13]). *We have*

$$-\log \mathbb{P}\left(\sup_{0\leqslant t\leqslant 1} \mathbb{b}_t > x\right) \sim c_\alpha x^{\alpha'}.$$

1.3. Outline of the proofs and organization of the paper. The proofs for excursions and bridges are very much alike, but some extra technicalities arise for excursions, so we focus mostly on this case, and deal with bridges in Section 7. Section 2 will recall the basics of stable processes, bridges and excursions, as well as the results on the M1 topology that will be needed in this paper.

In order to prove Theorem 1.2, we first establish the LDP for the finite-dimensional marginals of $\varepsilon\mathbb{e}$ (Proposition 3.1). This relies on the explicit form of the finite-dimensional marginals of the stable excursion in terms of stable densities and related quantities, which is recalled in Section 2.1. The key input is the following estimate for stable densities $p_t(x) = \mathbb{P}(L_t \in dx)/dx$, [26, Equation (14.35)]),

$$\begin{cases} p_1(x) = C_\alpha x^{-\alpha-1}\big(1 + O(x^{-\alpha})\big) & \text{as } x \to +\infty \\ p_1(-x) = c''_\alpha x^{\frac{2-\alpha}{2\alpha-2}} \exp\big(-c_\alpha x^{\alpha'}\big)\big(1 + O(x^{-\alpha'})\big) & \text{as } x \to +\infty. \end{cases} \tag{1.17}$$

The asymmetry of these two asymptotic behaviors will play a key role. This will imply that $\varepsilon\mathbb{e}$ satisfies a large deviation principle for the weak topology on $\mathbb{D}[0,1]$ of pointwise convergence at continuity points of the limit. This result is proved in Section 5, which is also devoted to the identification of the rate function. In order to prove an LDP in $(\mathbb{D}[0,1], \mathrm{dist})$, we show that the laws of $(\varepsilon\mathbb{e}, \varepsilon \in (0,1))$ are exponentially tight in this space. This relies on a Kolmogorov-type criterion which we prove in Section 2.2.3, and apply to our present context is Section 4. Finally, Theorem 1.4 is proved in Section 6.1 using the ideas of Fill and Janson [10], who treated the Brownian case.

Acknowledgement. Thanks are due to Loïc Chaumont for an interesting conversation around stable excursions.

2. Preliminaries

2.1. Excursions and bridges of stable Lévy processes without negative jumps. We begin by recalling the definitions of stable excursions and bridges, for which we mainly refer to [7]. We denote by $\mathbb{P}_x$ the law under which the canonical càdlàg process $(L_t, t \geqslant 0)$ is a stable Lévy process without negative jumps with exponent α, started from x, and we set $\mathbb{P} = \mathbb{P}_0$. We let $(\mathcal{F}_t)_{t\geqslant 0}$ be the natural filtration. We denote by $(p_t)_{t\geqslant 0}$ the continuous transition semigroup density of L under $\mathbb{P}_x$, which possesses the scaling property

$$p_t(x) = t^{-1/\alpha} p_1(t^{-1/\alpha} x).$$

Let $\mathcal{E}$ be the excursion space, which is defined by

$$\mathcal{E} = \left\{ \omega \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}_+) : \omega(0) = 0 \text{ and } \zeta(\omega) = \sup\{t > 0,\ \omega(t) > 0\} \in (0, \infty) \right\}.$$

Denote by $\underline{n}$ the Itô measure of L above its past infimum. The σ-finite "law" of the lifetime under $\underline{n}$ has been calculated by Monrad and Silverstein (see [22, Lemma 3.2])

$$\underline{n}(t < \zeta) = \Gamma\left(1 - \frac{1}{\alpha}\right)^{-1} t^{-1/\alpha}.$$

For $\lambda > 0$, we define the scaling operator $S^{(\lambda)}$ on $\mathcal{E}$ by

$$S^{(\lambda)}(\omega) = (\lambda^{1/\alpha} \omega_{t/\lambda},\ t \geqslant 0).$$

Then there exists a unique collection of probability measures $(\underline{n}^{(t)}, t > 0)$ on $\mathcal{E}$ such that

(i) for every $t > 0$, $\underline{n}^{(t)}(\zeta = t) = 1$;

(ii) for every $\lambda > 0$ and $t > 0$, we have $S^{(\lambda)}(\underline{n}^{(t)}) = \underline{n}^{(\lambda t)}$;

(iii) for every measurable subset A of $\mathcal{E}$,

$$\underline{n}(A) = \int_0^\infty \frac{ds}{\alpha \Gamma\left(1 - \frac{1}{\alpha}\right) s^{1+\frac{1}{\alpha}}} \underline{n}^{(s)}(A).$$

The probability distribution $\underline{n}^{(1)}$ on càdlàg paths with unit lifetime is called the law of the normalized excursion of L.

We denote by $\mathbb{P}_x^{(0,\infty)}$ the law of the process $(L, \mathbb{P}_x)$, $x > 0$, killed when it leaves $[0, \infty)$, so that

$$\mathbb{P}_x^{(0,\infty)}(A,\ t < \zeta) \coloneqq \mathbb{P}\left(A,\ t < \tau_{(-\infty,0)}\right), \quad t \geqslant 0, \quad A \in \mathcal{F}_t.$$

We denote by $(p_t^{(0,\infty)}(x, \cdot))_{t\geqslant 0}$ the transition semigroup under $\mathbb{P}_x^{(0,\infty)}$. The measure $\underline{n}$ is Markovian with semigroup $(p_t^{(0,\infty)}(x, \cdot))_{t\geqslant 0}$ under $\mathbb{P}_x$, which means that if F is measurable and nonnegative, and $\theta_t f = f(t + \cdot)$ is the shift operator, then

$$\underline{n}\big(\mathbb{1}_A\, F \circ \theta_t\, \mathbb{1}_{\{t<\zeta\}}\big) = \underline{n}\big(\mathbb{1}_A\, \mathbb{E}_{L_t}^{(0,\infty)}[F]\, \mathbb{1}_{\{t<\zeta\}}\big), \quad t \geqslant 0, \quad A \in \mathcal{F}_t. \tag{2.1}$$

We denote by $(q_x(t))_{t\geqslant 0}$ the density of the first hitting time of 0 under $\mathbb{P}_x^{(0,\infty)}$. Thanks to the absence of negative jumps, the density $(q_x(t))_{t\geqslant 0}$ can be related to the law of L through the relation (see for instance [2, Corollary VII.3])

$$q_x(t) = \frac{x}{t} p_t(-x).$$

Hence, it satisfies the following scaling property

$$q_x(t) = x^{-\alpha} q_1(x^{-\alpha} t).$$

Let $(j_t)_{t\geqslant 0}$ be the density of the entrance law under the measure $\underline{n}$, defined by the fact that, for every $t > 0$,

$$\underline{n}(f(L_t)\mathbb{1}_{\{t<\zeta\}}) = \int_0^\infty f(x) j_t(x)\, dx,$$

where f is an arbitrary bounded Borel function. Recall that for all $t > 0$, j_t is an integrable function in $\mathbb{L}^\infty(\mathbb{R}_+)$, and we may choose it so that it satisfies the following scaling property

$$j_t(x) = t^{-2/\alpha} j_1(t^{-1/\alpha} x) \tag{2.2}$$

(see [22, Lemma 3.2]). Combining the above expressions, we may show that the law of the normalized excursion has a density with respect to the Lebesgue measure. Indeed, if $f : \mathbb{R} \to \mathbb{R}_+^*$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ are two nonnegative measurable functions, we then have

$$\begin{aligned} \underline{n}\left(f(L_t) g(\zeta) \mathbb{1}_{\{t<\zeta\}}\right) &= \int_t^\infty ds g(s) \int_{\mathbb{R}} f(x) j_t(x) q_x(s-t)\, dx \\ &= \int_t^\infty \frac{g(s)}{\alpha\Gamma\left(1-\frac{1}{\alpha}\right) s^{1+\frac{1}{\alpha}}} \underline{n}^{(s)}(f(L_t))\, ds. \end{aligned}$$

This implies that for all $s > 0$,

$$\underline{n}^{(s)}(f(L_t)) = \alpha\Gamma\left(1-\frac{1}{\alpha}\right) \int_{\mathbb{R}_+} f(x) j_t(x) q_x(s-t)\, dx.$$

In particular, when $s = 1$ the law of the normalized excursion is then

$$\underline{n}^{(1)}(f(L_t)) = \alpha\Gamma\left(1-\frac{1}{\alpha}\right) \int_{\mathbb{R}_+} f(x) j_t(x) q_x(1-t)\, dx.$$

Using a similar argument and the Markov property, we can compute

$$\underline{n}^{(1)}(f(L_{t_1}, \dots, L_{t_n})) = \alpha\Gamma\left(1-\frac{1}{\alpha}\right) \int_{\mathbb{R}_+^n} j_{t_1}(x_1) \prod_{i=1}^{n-1} p_{t_{i+1}-t_i}^{(0,\infty)}(x_i, x_{i+1}) q_{x_n}(1-t_n)\, dx_1 \cdots dx_n. \tag{2.3}$$

This gives the finite-dimensional marginals for the excursion process $\mathbb{e}$, so that the left hand-side of the preceding equation may also be written as $\mathbb{E}\left[f(\mathbb{e}_{t_1}, \dots, \mathbb{e}_{t_n})\right]$. In particular, for every $t \in (0,1)$, the law of $(\mathbb{e}_{t+s}, 0 \leqslant s \leqslant 1-t)$ can be obtained from the following formula, valid for every non-negative measurable F:

$$\mathbb{E}\left[F(\mathbb{e}_{t+s}, 0 \leqslant s \leqslant 1-t)\right] = \alpha\Gamma\left(1-\frac{1}{\alpha}\right) \int_{\mathbb{R}_+} j_t(x) q_x(1-t) \mathbb{E}_x^{1-t}[F(L)], \tag{2.4}$$

where $\mathbb{E}_x^\delta$ is the law of the first-passage bridge of duration $\delta > 0$ started from $x > 0$. The latter is defined by absolute continuity for every $\delta' > 0$ and $\mathcal{F}_{\delta'}$-measurable $F \geqslant 0$ by the formula

$$\mathbb{E}_x^\delta[F] = \mathbb{E}_x\left[F \mathbb{1}_{\{T_0 > \delta'\}} \frac{q_{L_{\delta'}}(\delta - \delta')}{q_x(\delta)}\right]. \tag{2.5}$$

In a similar but simpler way, the law of the bridge $\mathbb{b}^{(a)}$ has finite-dimensional marginals given by

$$\mathbb{E}\left[f(\mathbb{b}^{(a)}_{t_1}, \ldots, \mathbb{b}^{(a)}_{t_n})\right] = \frac{1}{p_1(a)} \int_{\mathbb{R}^n} f(x_1, \ldots, x_n) \prod_{i=0}^{n} p_{t_{i+1}-t_i}(x_{i+1} - x_i)\, dx_1 \cdots dx_n, \tag{2.6}$$

where $0 = t_0 < t_1 < \ldots < t_n < 1 = t_{n+1}$, and by convention we let $x_0 = 0$ and $x_{n+1} = a$ in the above integral.

2.2. Basic results on the Skorokhod M1 topology. In this section, we introduce define the distance dist on $\mathbb{D}[0,1]$ and study some of its key properties.

By convention, for $f \in \mathbb{D}[0,1]$, we let $f(0-) = 0$, which is a way to "root" the function f at 0: by contrast, we adopt the convention $f(1+) = f(1)$.

For two real numbers $x, y \in \mathbb{R}$, the real interval $[x \wedge y, x \vee y]$ will more simply be denoted by $[x, y]$, even when $x > y$.

2.2.1. The space $(\mathbb{D}[0,1], \mathrm{dist})$. For $f \in \mathbb{D}[0,1]$, we define the *augmented graph* of f *rooted at* 0, to be the set

$$\Gamma_0(f) = \left\{(t,x) : t \in [0,1], x \in [f(t-), f(t)]\right\} \subset [0,1] \times \mathbb{R}.$$

Note in particular that $\Gamma_0(f)$ contains the segments $\{0\} \times [0, f(0)]$ and $\{1\} \times [f(1-), f(1)]$. For $(t,x), (u,y) \in \Gamma_0(f)$, we write $(t,x) \preceq (u,y)$ if $t < u$ or if $t = u$ and $|x - f(t-)| \leqslant |y - f(t-)|$. This defines a total order on $\Gamma_0(f)$. We say that a function $r \in [0,1] \mapsto (t(r), x(r))$ is a *parametric representation* of $\Gamma_0(f)$ if it is an increasing bijection from $([0,1], \leqslant)$ to $(\Gamma_0(f), \preceq)$, and we write $\Pi(f)$ for the set of all parametric representations of $\Gamma_0(f)$. For $f_1, f_2 \in D[0,1]$, we let

$$\mathrm{dist}(f_1, f_2) = \inf\left\{\sup_{r \in [0,1]} |t_1(r) - t_2(r)| \vee |x_1(r) - x_2(r)| : (t_1, x_1) \in \Pi(f_1), (t_2, x_2) \in \Pi(f_2)\right\}.$$

This indeed defines a distance function that makes $(\mathbb{D}[0,1], \mathrm{dist})$ a Polish space[1]. We refer to [30, Theorems 12.3.1 and 12.8.1] with the slight modification that the convention taken in this and other classical references is that $f(0-) = f(0)$ rather than $f(0-) = 0$, and that f is supposed to be continuous at 0 and 1. Had we chosen the former convention, the topology induced by dist would be the so-called *Skorokhod M1 topology* introduced by Skorokhod in [29], which is a weaker variant of the more classical J1 topology. Our choice of convention that $f(0-) = 0$ allows a sequence of functions that jump "right after time 0" to be possibly convergent in our topology. For example, one has $\mathrm{dist}(\mathbb{1}_{[1/n,1]}, \mathbb{1}_{[0,1]}) \to 0$

[1]Note, however, that the distance dist is not complete, see [30, Section 12.8].

as $n \to \infty$, while the sequence $\mathbb{1}_{[1/n,1]}$ is not convergent in the classical M1 topology, and in fact our topology is strictly weaker than the M1 topology. On the other hand, observe that $\mathbb{1}_{[0,1/n]}$ is not convergent in $(\mathbb{D}[0,1], \mathrm{dist})$.

2.2.2. The M*-oscillation.* In this section, we introduce an oscillation function that will serve as a substitute in $(\mathbb{D}[0,1], \mathrm{dist})$ for the classical modulus of continuity. For $x \in \mathbb{R}$ and $A \subset \mathbb{R}$ we let $d(x, A) = \inf\{|x - y| : y \in A\}$ be the distance from x to A. Note the elementary inequalities

$$d(x, A) - d(y, A) \leqslant d(x, y)\,, \qquad d(x, A) - d(x, B) \leqslant d_H(A, B) \tag{2.7}$$

where $x \in \mathbb{R}$ and $A, B \subset \mathbb{R}$, and $d_H(A, B) = \sup_{x \in A} d(x, B) \vee \sup_{y \in B} d(y, A)$ is the Hausdorff distance between A and B.

For $x, y, z \in \mathbb{R}$, we set

$$\mathsf{M}(x, y, z) = d(y, [x, z]) = (y - x)_+ \wedge (y - z)_+ + (y - x)_- \wedge (y - z)_-.$$

The M-oscillation of a function $f \in \mathbb{D}[0,1]$ is the function defined for $\delta > 0$ by

$$w_{\mathsf{M}}(f, \delta) = \sup\left\{\mathsf{M}(f(t_1-), f(t), f(t_2)) : 0 \leqslant t_1 < t < t_2 \leqslant 1, |t_2 - t_1| < \delta\right\}.$$

The choice of the left-limit at t_1 in the first term might appear unnatural at first sight, because of the fact that f has left limits. In fact, it is only needed when $t_1 = 0$, because of our rooting convention $f(0-) = 0$. So in fact, $w_{\mathsf{M}}(f, \delta)$ is the maximum of the two quantities

$$\sup\left\{\mathsf{M}(f(t_1), f(t), f(t_2)) : 0 \leqslant t_1 < t < t_2 \leqslant 1, |t_2 - t_1| < \delta\right\}$$

and

$$\sup\left\{\mathsf{M}(0, f(t), f(t_2)) : 0 < t < t_2 < \delta\right\},$$

and if $f(0) = 0$, then $w_{\mathsf{M}}(f, \delta)$ is equal to the first quantity.

Theorem 2.1. *Let* $\mathcal{D}$ *be a fixed countable subset of* $[0,1]$ *containing* 1*. Let* $K \subset \mathbb{D}[0,1]$ *be such that*

$$\sup\left\{|f(q)| : f \in K, q \in \mathcal{D}\right\} < \infty$$

and

$$\lim_{\delta \downarrow 0} \sup\left\{w_{\mathsf{M}}(f, \delta) : f \in K\right\} = 0.$$

Then K *is a relatively compact subset of* $(\mathbb{D}[0,1], \mathrm{dist})$.

This theorem can be found in Chapter 12 of Whitt [30], with the minor difference that our space of functions starts with an “initial jump” (recall that $f(0-) = 0$ by our convention).

2.2.3. An exponential tightness criterion. Our second result gives a sufficient condition to check exponential tightness in $(\mathbb{D}[0,1], \mathrm{dist})$ for a family of random processes.

Theorem 2.2 (Exponential Tightness in $(\mathbb{D}[0,1], \mathrm{dist})$). *Let $\alpha > 1$ and $\{X_{(\varepsilon)}, \varepsilon > 0\}$ be a family of $\mathbb{D}[0,1]$-valued stochastic processes. We assume that $X_{(\varepsilon)}(0) = 0$ for every $\varepsilon > 0$.*

1. *Suppose that there exist two constants $c, C \in (0, \infty)$ such that for every $\lambda > 0$, and every $0 \leqslant t_1 \leqslant t \leqslant t_2 \leqslant 1$,*
$$\mathbb{E}\left[\exp(\lambda M(X_{(\varepsilon)}(t_1), X_{(\varepsilon)}(t), X_{(\varepsilon)}(t_2)))\right] \leqslant c \exp\left(C(\lambda\, \varepsilon)^{\alpha}|t_2 - t_1|\right),$$
then it holds that for every $\beta \in (0, 1/\alpha)$,
$$\lim_{N\to\infty} \limsup_{\varepsilon\downarrow 0} \varepsilon^{\alpha'} \log \mathbb{P}\left(\bigcup_{n>N} \left\{ w_M(X_{(\varepsilon)}, 2^{-n}) > 2\frac{2^{-n\beta}}{1-2^{-\beta}} \right\}\right) = -\infty.$$
2. *Suppose further that for some countable dense set $\mathcal{D}$ of $[0,1]$ containing 1, the family of random variables $\{X_{(\varepsilon)}(q), \varepsilon > 0\}$ is exponentially tight with speed $\varepsilon^{-\alpha'}$, for every $q \in \mathcal{D}$. Then the laws of $X_{(\varepsilon)}$ as $\varepsilon \downarrow 0$ are exponentially tight in $(\mathbb{D}[0,1], \mathrm{dist})$, with speed $\varepsilon^{-\alpha'}$.*

Proof. For 1., we follow and adapt the approach of Billingsley [3]. Let $D_n = \{k2^{-n}, 0 \leqslant k \leqslant 2^n\}$ be the dyadic numbers of level n. Then for $\beta > 0$ and $\lambda > 0$, by Markov's inequality,
$$\mathbb{P}\left(M(X_{(\varepsilon)}(k2^{-n}), X_{(\varepsilon)}((k+1)2^{-n}), X_{(\varepsilon)}((k+2)2^{-n}) > 2^{-n\beta}\right) \leqslant c\exp(-\lambda 2^{-n\beta} + C(\lambda\varepsilon)^{\alpha} 2^{-n+1})$$
which by optimizing over $\lambda > 0$ and taking a union bound yields
$$\begin{aligned}\mathbb{P}\left(\exists k \in \{0,1,\dots,2^n-2\} : M(X_{(\varepsilon)}(k2^{-n}), X_{(\varepsilon)}((k+1)2^{-n}), X_{(\varepsilon)}((k+2)2^{-n})) > 2^{-n\beta}\right)& \\ \leqslant 2^n c\exp(-C'(\alpha)\varepsilon^{-\alpha'} 2^{n(1-\alpha\beta)/(\alpha-1)})&.\end{aligned}$$
Setting $A_n = \max\{M(f(k2^{-n}, (k+1)2^{-n}, (k+2)2^{-n})), 0 \leqslant k \leqslant 2^n - 2\}$, this shows that $\mathbb{P}\left(A_n > 2^{-n\beta}\right) \leqslant 2^n c\exp(-C'(\alpha)\varepsilon^{-\alpha'} 2^{n(1-\alpha\beta)/(\alpha-1)})$. Next, let $f \in \mathbb{D}[0,1]$ and, for $I \subset [0,1]$, let
$$\mathcal{L}(I) = \sup\left\{M(f(t_1), f(t), f(t_2)) : t_1, t, t_2 \in I, t_1 \leqslant t \leqslant t_2\right\}.$$
Fix $n \geqslant 1$ and $k \in \{0, 1, \dots, 2^n - 2\}$. We aim to provide bounds on $\mathcal{L}([k2^{-n}, (k+2)2^{-n}])$. To this end, by right-continuity, it suffices to bound uniformly the quantities $M(f(t_1), f(t), f(t_2))$ for $t_1 \leqslant t \leqslant t_2$ in $[k2^{-n}, (k+2)2^{-n}] \cap \bigcup_{m\geqslant 0} D_m$. For $m \geqslant n$, let
$$B_m = \max\left\{M(f(t_1), f(t), f(t_2)) : k2^{-n} \leqslant t_1 \leqslant t \leqslant t_2 \leqslant (k+2)2^{-n}, t_1, t, t_2 \in D_m\right\},$$
so that $\mathcal{L}([k2^{-n}, (k+2)2^{-n}])$ is the increasing limit of B_m as $m \to \infty$. The key observation is that, for every $m \geqslant n$,
$$B_m \leqslant B_{m-1} + 2A_m\,. \tag{2.8}$$

To check this, let us assume that $t_1 < t < t_2$ are in D_m and achieve the maximum defining B_m. If $f(t)$ lies between $f(t_1)$ and $f(t_2)$ then this means that $B_m = 0$ and there is nothing to prove. Otherwise, we may assume without loss of generality that $f(t_2) \leqslant f(t_1) < f(t)$, the other cases being symmetric, so that $B_m = f(t) - f(t_1)$. Note that if $t \in D_m \setminus D_{m-1}$, then $t - 2^{-m}, t + 2^{-m}$ belong to D_{m-1}. Moreover, it must hold that $f(t - 2^{-m}) \vee f(t + 2^{-m}) \leqslant f(t)$, as otherwise, for instance if $f(t - 2^{-m}) > f(t)$, then we would have

$$M(f(t_1), f(t - 2^{-m}), f(t_2)) = f(t - 2^{-m}) - f(t_1) > f(t) - f(t_1) = M(f(t_1), f(t), f(t_2)),$$

and this would contradict the assumption that $M(f(t_1), f(t), f(t_2))$ is maximal over points $t_1 < t < t_2$ in D_m. This implies that $M(f(t - 2^{-m}), f(t), f(t + 2^{-m})) = (f(t) - f(t - 2^{-m})) \wedge (f(t) - f(t + 2^{-m}))$, and therefore, we may choose $t' \in \{t - 2^{-m}, t + 2^{-m}\}$ such that $|f(t) - f(t')| \leqslant A_m$. If $t \in D_{m-1}$, we let $t' = t$.

We define t_1' in a similar way, setting it to be t_1 if the latter belongs to D_{m-1}. If $t_1 \in D_m \setminus D_{m-1}$, on the other hand, then $t_1 \pm 2^{-m}$ belong to D_{m-1}. We note that $f(t_1 - 2^{-m}) \wedge f(t_1 + 2^{-m}) \geqslant f(t_1)$, as otherwise this would again contradict the maximality of $M(f(t_1), f(t), f(t_2))$ over points in D_m. So we may choose $t_1' \in \{t \pm 2^{-m}\}$ in such a way that $|f(t_1') - f(t_1)| \leqslant A_m$.

Finally, we define t_2' in the following way. If $t_2 \in D_{m-1}$, we let $t_2' = t_2$ as usual. If $t_2 \in D_m \setminus D_{m-1}$, we have two situations. If $f(t_2 - 2^{-m}) \wedge f(t_2 + 2^{-m}) \geqslant f(t_2)$ then we may again choose $t_2' \in \{t_2 \pm 2^{-m}\}$ in such a way that $|f(t_2') - f(t_2)| \leqslant A_m$. In this case, we notice that $d_H([f(t_1), f(t_2)], [f(t_1'), f(t_2')]) \leqslant A_m$, so that

$$d(f(t), [f(t_1'), f(t_2')]) \geqslant d(f(t), [f(t_1), f(t_2)]) - A_m \tag{2.9}$$

by (2.7). Otherwise, we choose $t_2' \in \{t_2 - 2^{-m}, t_2 + 2^{-m}\}$ in such a way that $f(t_2') \leqslant f(t_2)$. In this case, we have $d_H([f(t_1), f(t_2')], [f(t_1'), f(t_2')]) \leqslant A_m$ so that, again by (2.7),

$$d(f(t), [f(t_1'), f(t_2')]) \geqslant d(f(t), [f(t_1), f(t_2')]) - A_m = d(f(t), [f(t_1), f(t_2)]) - A_m,$$

so that (2.9) holds in every case. Therefore, for this choice of t_1', t', t_2' and by (2.7), we obtain

$$\begin{aligned} B_{m-1} \geqslant M(f(t_1'), f(t'), f(t_2')) = d(f(t'), [f(t_1'), f(t_2')]) \\ \geqslant d(f(t), [f(t_1'), f(t_2')]) - A_m \\ \geqslant d(f(t), [f(t_1), f(t_2)]) - 2A_m = B_m - 2A_m\,, \end{aligned}$$

so that (2.8) holds. By taking a limit, (2.8) implies that $\mathcal{L}([k2^{-n}, (k+2)2^{-n}]) \leqslant B_{n-1} + 2\sum_{m \geqslant n} A_m$ where we note that $B_{n-1} = 0$. Furthermore, we note that

$$w_M(f, 2^{-n}) \leqslant \max_{0 \leqslant k \leqslant 2^n - 2} \mathcal{L}([k2^{-n}, (k+2)2^{-n}])$$

because three numbers within distance 2^{-n} can all be fitted into the same interval $[k2^{-n}, (k+2)2^{-n}]$ for some k. We deduce that

$$w_M(f, 2^{-n}) \leqslant 2 \sum_{m \geqslant n} A_m.$$

Finally, let

$$K_N = \bigcap_{n\geqslant N} \left\{ f \in \mathbb{D}[0,1] : w_M(f, 2^{-n}) \leqslant 2\frac{2^{-n\beta}}{1-2^{-\beta}} \right\},$$

so that

$$\begin{aligned}
\mathbb{P}\left(X_{(\varepsilon)} \notin K_N\right) &\leqslant \sum_{n\geqslant N} \mathbb{P}\Big(2\sum_{m\geqslant n} A_m \geqslant 2\frac{2^{-n\beta}}{1-2^{-\beta}}\Big) \\
&\leqslant \sum_{n\geqslant N}\sum_{m\geqslant n} \mathbb{P}\left(A_m \geqslant 2^{-m\beta}\right) \\
&\leqslant \sum_{n\geqslant N}\sum_{m\geqslant n} c2^m \exp(-C'(\alpha)\varepsilon^{-\alpha'}2^{m(1-\alpha\beta)/(\alpha-1)}) \\
&\leqslant C''2^N \exp(-C'(\alpha)\varepsilon^{-\alpha'}2^{N(1-\alpha\beta)/(\alpha-1)}),
\end{aligned}$$

for some universal constant $C'' = C''(\alpha) > 0$. We finally deduce that

$$\limsup_{\varepsilon\downarrow 0} \varepsilon^{\alpha'} \log \mathbb{P}\left(X_{(\varepsilon)} \notin K_N\right) \leqslant -C'2^{N(1-\alpha\beta)/(\alpha-1)},$$

which converges to $-\infty$ as $N \to \infty$.

It remains to prove 2. Notice that for every choice of $0 = t_0 < t_1 < \ldots < t_k = 1$ in $\mathcal{D}$ with $\max\{t_i - t_{i-1} : 1 \leqslant i \leqslant k\} < \delta$, it holds that

$$\sup_{t\in[0,1]} |X_{(\varepsilon)}(t)| \leqslant \max\left\{|X_{(\varepsilon)}(t_i)| : 1 \leqslant i \leqslant k\right\} + w_M(X_{(\varepsilon)}, \delta),$$

so that

$$\mathbb{P}\left(\sup_{t\in[0,1]} |X_{(\varepsilon)}(t)| > A\right) \leqslant \sum_{i=1}^{k} \mathbb{P}\left(|X_{(\varepsilon)}(t_i)| > A/2\right) + \mathbb{P}\left(w_M(X_{(\varepsilon)}, \delta) > A/2\right).$$

From the fact that the $X_{(\varepsilon)}(t_i), 1 \leqslant i \leqslant k$ are exponentially tight, and by 1., we obtain the existence of $A_N \in (0, \infty)$ such that

$$\limsup_{\varepsilon\downarrow 0} \varepsilon^{\alpha'} \log \mathbb{P}\left(\sup_{t\in\mathcal{D}} |X_{(\varepsilon)}(t)| > A_N\right) < -N.$$

We deduce that the relatively compact sets of $\mathbb{D}[0,1]$ given by $\{f \in \mathbb{D}[0,1] : \sup_{t\in\mathcal{D}} |f(t)| \leqslant A_N\} \cap K_N$ fulfill the definition of exponential tightness. □

3. Large deviations for the finite-dimensional marginal distributions

In this section we prove the following proposition.

Proposition 3.1 (LDP for the marginals of $\mathbb{e}$). *Let $\sigma = (t_1, \ldots, t_n)$, where $0 < t_1 < \cdots < t_n < 1$, be fixed. Under $\mathbb{P}$ the laws of $\varepsilon(\mathbb{e}_{t_1}, \ldots, \mathbb{e}_{t_n})$ satisfy a LDP in $\mathbb{R}^n$ with speed $\varepsilon^{-\alpha'}$ and good rate function*

$$J_\sigma(x_1, \ldots, x_n) = \begin{cases} c_\alpha \sum_{i=1}^{n-1} (t_{i+1} - t_i) \left(\frac{(x_i - x_{i+1})_+}{t_{i+1}-t_i}\right)^{\alpha'} & \text{if } x_1, \ldots, x_n \in \mathbb{R}_+ \\ \infty & \text{otherwise.} \end{cases}$$

The fact that J_σ is a good rate function on $\mathbb{R}^n$ is easy to see. Indeed, it is clearly continuous, and for every $c > 0$, $J(x_1, \dots, x_n) \leqslant c$ implies $(x_i - x_{i+1})_+ \leqslant c'$ for $1 \leqslant i \leqslant n$ and $x_n \leqslant c'$, where c' is some positive number depending only on $t_1, \dots, t_n, c$ that $x_i \leqslant x_{i+1} + c'$ that $\max_{1 \leqslant i \leqslant n} x_i \leqslant nc'$, and the level sets of J_σ are therefore compact.

3.1. Estimates for transition densities. We will need some crucial estimates for the tails of the transition densities, $p_t(x)$, and for the density of the entrance law, $j_t(x)$. In this section, we will make use of positive, finite universal constants c_1, c_2 depending only on α, but whose values may vary from line to line, and of non-universal constants c, C depending on some extra parameters that will always be specified.

First, [26, Equation (14.35)] entails that for every $x \geqslant 0$, we have

$$c_1 \exp\left(-c_\alpha x^{\alpha'}\right) \leqslant p_1(-x) \leqslant c_2(1 + x^{\frac{2-\alpha}{2\alpha-2}}) \exp\left(-c_\alpha x^{\alpha'}\right) \tag{3.1}$$

and [26, Equation (14.34)] entails that

$$c_1(1+x)^{-\alpha-1} \leqslant p_1(x) \leqslant c_2(1+x)^{-\alpha-1}. \tag{3.2}$$

By the scaling relations for $p_t(x)$, we deduce that for every $x > 0$ and $t \in (0, 1]$,

$$c_1 \exp\left(-c_\alpha \left(\frac{x}{t^{1/\alpha}}\right)^{\alpha'}\right) \leqslant p_t(-x) \leqslant \frac{c_2}{t^{1/\alpha}} \left(1 + \left(\frac{x}{t^{1/\alpha}}\right)^{\frac{2-\alpha}{2\alpha-2}}\right) \exp\left(-c_\alpha \left(\frac{x}{t^{1/\alpha}}\right)^{\alpha'}\right) \tag{3.3}$$

and

$$c_1(1 + x/t^{1/\alpha})^{-\alpha-1} \leqslant p_t(x) \leqslant c_2(1+x)^{-\alpha-1}. \tag{3.4}$$

In particular, note that for every fixed $\eta \in (0, 1)$ and $x_0 > 0$, we have, for any $t \in (0, 1)$ and $x \geqslant x_0$,

$$p_t(-x) \leqslant C(\eta, x_0) \exp\left(-(1-\eta)c_\alpha \left(\frac{x}{t^{1/\alpha}}\right)^{\alpha'}\right). \tag{3.5}$$

A similar bound holds for $q_x(t) = \frac{x}{t} p_t(-x)$, with possibly different constants.

Next, by [22, Formula (3.20)], it holds that there exists a positive constant $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and for $0 < a < b < +\infty$,

$$c_1 \frac{\varepsilon^{\alpha+1}}{a^\alpha t^{\frac{1-\alpha}{\alpha}}} \left(1 - \left(\frac{a}{b}\right)^\alpha\right) \leqslant \int_{a/\varepsilon}^{b/\varepsilon} j_t(y)\, dy \leqslant c_2 \frac{\varepsilon^{\alpha+1}}{a^\alpha t^{\frac{1-\alpha}{\alpha}}} \left(1 - \left(\frac{a}{b}\right)^\alpha\right). \tag{3.6}$$

These estimates will allow us to evaluate the densities involved in (2.3) in the large deviations regime. First we give an explicit formula for $p_t^{(0,\infty)}(x, y)$.

Lemma 3.2. *Let $t > 0$ and $x, y > 0$. Then*

$$p_t^{(0,\infty)}(x, y) = p_t(y - x) - \int_0^t q_x(s) p_{t-s}(y)\, ds. \tag{3.7}$$

Proof. Let $f : \mathbb{R} \to \mathbb{R}_+$ be a measurable function. On the one hand, we have by definition that

$$\mathbb{E}_x^{(0,\infty)}[f(L_t)] = \int_{\mathbb{R}} f(y) p_t^{(0,\infty)}(x,y)\, dy.$$

On the other hand, if we denote by T_0 the first hitting time of 0 by $(L_t)_{t\geqslant 0}$, and $L_{t-s}^{(s)} = L_{s+(t-s)}$ for $s < t$, we then have

$$\begin{aligned}
\mathbb{E}_x^{(0,\infty)}[f(L_t)] &= \mathbb{E}_x\left[f(L_t)\mathbb{1}_{\{T_0>t\}}\right] \\
&= \mathbb{E}_x[f(L_t)] - \mathbb{E}_x\left[f(L_t)\mathbb{1}_{\{T_0<t\}}\right] \\
&= \int_{\mathbb{R}} f(y)p_t(y-x)\,dy - \mathbb{E}_x\left[\mathbb{1}_{\{T_0<t\}}\mathbb{E}\left[f(L_{t-T_0}^{(T_0)}) \mid \mathcal{F}_{T_0}\right]\right] \\
&= \int_{\mathbb{R}} f(y)p_t(y-x)\,dy - \int_0^t q_x(s)\mathbb{E}\left[f(L_{t-s}^{(s)})\right] ds \\
&= \int_{\mathbb{R}} f(y)p_t(y-x)\,dy - \int_0^t q_x(s)\int_{\mathbb{R}} f(y)p_{t-s}(y)\,dy\,ds \\
&= \int_{\mathbb{R}} f(y)\left\{p_t(y-x) - \int_0^t q_x(s)p_{t-s}(y)\, ds\right\} dy,
\end{aligned}$$

where we used the Markov property in the third equality. Thus Equation (3.7) follows. ☐

Using the scaling properties of $p_t(x)$ and $q_x(t)$, we may deduce from Lemma 3.2 a bound on the error when we approximate $p_t^{(0,\infty)}(x,y)$ by $p_t(y-x)$, as follows.

Lemma 3.3. *For any fixed* $t > 0$ *and* $\eta > 0$*, there exists* $C = C(\eta, t) > 0$ *such that for every* $x, y > 0$*,*

$$\int_0^t q_x(s)p_{t-s}(y)\, ds \leqslant C \exp\left(-c_\alpha(1-\eta)\left(\frac{x^\alpha}{t}\right)^{\frac{1}{\alpha-1}}\right). \tag{3.8}$$

Proof. Using the scaling relations for $p_t(x)$, we have

$$\begin{aligned}
\int_0^t q_x(s)p_{t-s}(y)\,ds &= \int_0^{t/2} \frac{x}{s}p_s(-x)p_{t-s}(y)\,ds + \int_{t/2}^t \frac{x}{s}p_s(-x)p_{t-s}(y)\,ds \\
&\leqslant \|p_{t/2}\|_\infty \int_0^{t/2} \frac{x}{s}p_s(-x)\,ds + \|p_1\|_\infty \int_{t/2}^t \frac{ds}{(t-s)^{1/\alpha}}\frac{x}{s}p_s(-x),
\end{aligned}$$

and then, using (3.3), we get

$$\begin{aligned}
\int_0^t q_x(s)p_{t-s}(y)\,ds \leqslant{}& c_2\|p_{t/2}\|_\infty \int_0^{t/2} \frac{x\,ds}{(t/2)^\alpha s^{1+1/\alpha}}(1+(x/s^{1/\alpha})^{\frac{2-\alpha}{2\alpha-2}})\exp\left(-c_\alpha(x/s^{1/\alpha})^{\alpha'}\right) \\
&+ c_2\|p_1\|_\infty \frac{x}{(t/2)^{1+1/\alpha}}\left(1+\left(\frac{x}{(t/2)^{1/\alpha}}\right)^{\frac{2-\alpha}{2\alpha-2}}\right)\exp\left(-c_\alpha\left(\frac{x}{t^{1/\alpha}}\right)^{\alpha'}\right)\int_{t/2}^t \frac{ds}{(t-s)^{1/\alpha}}.
\end{aligned}$$

The second term is of the desired form, while, by performing a change of variables $u = s^{-\alpha'/\alpha}$, it is straightforward to see that the first term is negligible compared to the second. ☐

Proof of Proposition 3.1. Since J_σ is a good rate function on $\mathbb{R}^n$, [28, Lemma 5] shows that it suffices to prove that, for every open subset $G \subset \mathbb{R}^n$,

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{\alpha'} \log \mathbb{P}\left(\varepsilon(\mathbb{e}_{t_1}, \ldots, \mathbb{e}_{t_n}) \in G\right) = -\inf_G J_\sigma . \tag{3.9}$$

Since J_σ is infinite on $\mathbb{R}^n \setminus \mathbb{R}^n_+$, it suffices to consider open sets G of $\mathbb{R}^n_+$ (with the induced topology), and this is what we do from now on. For convenience let us write

$$\Psi_\varepsilon(x_1, \ldots, x_n) \coloneqq \prod_{i=1}^{n-1} p^{(0,\infty)}_{t_{i+1}-t_i}\left(\frac{x_i}{\varepsilon}, \frac{x_{i+1}}{\varepsilon}\right) \times q_{x_n/\varepsilon}(1-t_n).$$

Using (2.3), we can write

$$\mathbb{P}\left(\varepsilon\left(\mathbb{e}_{t_1}, \ldots, \mathbb{e}_{t_n}\right) \in G\right) = C\varepsilon^n \int_G dx_1 \ldots dx_n \, j_{t_1}\left(\frac{x_1}{\varepsilon}\right) \Psi_\varepsilon(x_1, \ldots, x_n),$$

where $C = C(\alpha) > 0$ is a positive constant depending only on α.

We start with the lower bound. For a given $\delta > 0$, there exists $(y_1, \ldots, y_n) \in G$ such that $J_\sigma(y_1, \ldots, y_n) \leqslant \inf_G J_\sigma + \delta$, and we may assume without loss of generality that $y_1, \ldots, y_n$ are pairwise distinct and all lie in $(0, \infty)$. Then, there exists a hypercube $\mathcal{Q}_\delta = \prod_{i=1}^n (a_i, b_i) \subseteq G$ containing $(y_1, \ldots, y_n)$ such that the intervals $[a_i, b_i] \subset (0, \infty)$ are pairwise disjoint, and such that for all $(x_1, \ldots, x_n) \in \mathcal{Q}_\delta$, we have

$$J_\sigma(x_1, \ldots, x_n) \leqslant \inf_G J_\sigma + \delta.$$

Let us now consider the terms $p^{(0,\infty)}_{t_{i+1}-t_i}(x_i/\varepsilon, x_{i+1}/\varepsilon)$ involved in the definition of $\Psi_\varepsilon(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n \in Q_\delta$. Fix $\eta \in (0, 1)$. From (3.4) and Lemmas 3.2 and 3.3, for every i such that $y_i > y_{i+1}$, we may bound

$$p^{(0,\infty)}_{t_{i+1}-t_i}\left(\frac{x_i}{\varepsilon}, \frac{x_{i+1}}{\varepsilon}\right) \geqslant c_1 \exp\left(-\frac{c_\alpha}{\varepsilon^{\alpha'}}\left(\frac{(b_i - a_{i+1})^\alpha}{t_{i+1}-t_i}\right)^{\frac{1}{\alpha-1}}\right) - C(\eta) \exp\left(-\frac{c_\alpha}{\varepsilon^{\alpha'}}(1-\eta)\left(\frac{a_i^\alpha}{t_{i+1}-t_i}\right)^{\frac{1}{\alpha-1}}\right),$$

and for every i such that $y_i < y_{i+1}$,

$$p^{(0,\infty)}_{t_{i+1}-t_i}\left(\frac{x_i}{\varepsilon}, \frac{x_{i+1}}{\varepsilon}\right) \geqslant c_1\left(1 + \frac{b_{i+1} - a_i}{\varepsilon(t_{i+1}-t_i)^{1/\alpha}}\right)^{-\alpha-1} - C(\eta) \exp\left(-\frac{c_\alpha}{\varepsilon^{\alpha'}}(1-\eta)\left(\frac{a_i^\alpha}{t_{i+1}-t_i}\right)^{\frac{1}{\alpha-1}}\right).$$

Also, we may bound

$$q_{x_n/\varepsilon}(1-t_n) \geqslant c_1 \frac{a_n}{1-t_n} \exp\left(-\frac{c_\alpha}{\varepsilon^{\alpha'}}\left(\frac{b_n^\alpha}{1-t_n}\right)^{\frac{1}{\alpha-1}}\right).$$

Therefore, by choosing η small enough so that $(1-\eta)a_i^{\alpha'} \geqslant (b_i - a_{i+1})^{\alpha'}$ for every i such that $y_i > y_{i+1}$, we see that for every ε small enough, $\mathbb{P}\left((\mathbb{e}_{t_1}, \ldots, \mathbb{e}_{t_n}) \in G\right)$ is bounded from below by a quantity of the form

$$c\varepsilon^n \tilde{\Psi}_\varepsilon \int_{a_1}^{b_1} j_{t_1}\left(\frac{x_1}{\varepsilon}\right) dx_1$$

where, for some constant c depending only on $t_1, \dots, t_n$, $a_1, \dots, a_n$ and $b_1, \dots, b_n$, and η,

$$\tilde{\Psi}_\varepsilon = c \prod_{i=1}^{n} \left(\exp\left(-\frac{c_\alpha}{\varepsilon^{\alpha'}} \left(\frac{(b_i - a_{i+1})^\alpha}{t_{i+1} - t_i} \right)^{\frac{1}{\alpha-1}} \right) \mathbb{1}_{\{y_i > y_{i+1}\}} + \varepsilon^{\alpha+1} \mathbb{1}_{\{y_i < y_{i+1}\}} \right),$$

with the convention that $y_{n+1} = a_{n+1} = 0$. Using the asymptotics (3.6), we finally obtain

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^{\alpha'} \log \mathbb{P}\left(\varepsilon(\mathbb{e}_{t_1}, \dots, \mathbb{e}_{t_n}) \in G\right) \geqslant -c_\alpha \sum_{i=1}^{n} \left(\frac{(b_i - a_{i+1})_+^\alpha}{t_{i+1} - t_i} \right)^{\frac{1}{\alpha-1}}.$$

By letting a_i and b_i tend to y_i, we obtain

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^{\alpha'} \log \mathbb{P}\left(\varepsilon(\mathbb{e}_{t_1}, \dots, \mathbb{e}_{t_n}) \in G\right) \geqslant -J_\sigma(y_1, \dots, y_n) \geqslant -\inf_G J - \delta,$$

and since δ was arbitrary, we may conclude that

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^{\alpha'} \log \mathbb{P}\left(\varepsilon(\mathbb{e}_{t_1}, \dots, \mathbb{e}_{t_n}) \in G\right) \geqslant -\inf_G J.$$

The corresponding upper bound is obtained by similar arguments. It is clear that the upper bound holds if $\inf_G J = 0$, so that we may assume $\inf_G J > 0$. By Lemma 3.2, we have $p_t^{(0,\infty)}(x, y) \leqslant p_t(y - x)$. Therefore,

$$\Psi_\varepsilon(x_1, \dots, x_n) \leqslant C x_n \prod_{i=1}^{n-1} p_{t_{i+1} - t_i}\left(\frac{x_{i+1} - x_i}{\varepsilon} \right) p_{1 - t_n}\left(-\frac{x_n}{\varepsilon} \right),$$

where C is a positive and finite constant that depends only on $t_1, \dots, t_n$.

Let $\eta \in (0, 1/2)$ be a fixed constant. Now observe that for $t > 0$, and $x \in \mathbb{R}$, we have (*)

$$p_t(x) \leqslant p_t(x) \mathbb{1}_{\{x < 0\}} + \|p_t\|_\infty \mathbb{1}_{\{x \geqslant 0\}} \leqslant C \exp\left(-c_\alpha (1 - \eta) \frac{(x_-)^{\alpha'}}{t^{\frac{1}{\alpha-1}}} \right),$$

where the constant C depends only on η and t, but not on x. A similar bound holds for $x p_t(x)$, possibly with a different constant C. Thus we may write for all $(x_1, \dots, x_n) \in G$, with our usual convention that $x_{n+1} = 0$ and $t_{n+1} = 1$, and for a constant C that depends on $\eta, t_1, \dots, t_n$ but not on $x_1, \dots, x_n$,

$$\begin{aligned} \Psi_\varepsilon(x_1, \dots, x_n) \leqslant & C \exp\left(-\frac{(1 - \eta)}{\varepsilon^{\alpha'}} J_\sigma(x_1, \dots, x_n) \right) \\ \leqslant & C \exp\left(-\frac{1 - 2\eta}{\varepsilon^{\alpha'}} \inf_G J \right) \exp\left(-\frac{\eta}{\varepsilon^{\alpha'}} J_\sigma(x_1, \dots, x_n) \right). \end{aligned}$$

Since $j_{t_1} \in \mathbb{L}^\infty(\mathbb{R}_+)$, we obtain after changing x_i/ε into x_i in the integral,

$$\mathbb{P}\left(\varepsilon(\mathbb{e}_{t_1}, \dots, \mathbb{e}_{t_n}) \in G\right) \leqslant C \exp\left(-\frac{1 - 2\eta}{\varepsilon^{\alpha'}} \inf_G J \right) \int_{\mathbb{R}_+^n} dx_1 \dots dx_n \exp\left(-\eta J_\sigma(x_1, \dots, x_n) \right)$$

and the last integral is finite. This implies that

$$\limsup_{\varepsilon \downarrow 0} \varepsilon^{\alpha'} \log \mathbb{P}\left(\varepsilon(\mathbb{e}_{t_1}, \dots, \mathbb{e}_{t_n}) \in G\right) \leqslant -(1 - 2\eta) \inf_G J.$$

Since this is true for all $\eta > 0$, we get the upper bound

$$\limsup_{\varepsilon \downarrow 0} \varepsilon^{\alpha'} \log \mathbb{P}\left(\varepsilon(\mathbb{e}_{t_1}, \dots, \mathbb{e}_{t_n}) \in G\right) \leqslant -\inf_G J.$$

□

4. Exponential tightness for the normalized excursion

In this section we prove the following proposition.

Proposition 4.1 (Exponential tightness of $\mathbb{e}$). *Under $\mathbb{P}$, the laws of $(\varepsilon \mathbb{e}_t)_{t\in[0,1]}$ as $\varepsilon \downarrow 0$ are exponentially tight in $(\mathbb{D}[0,1], \mathrm{dist})$ with speed $\varepsilon^{-\alpha'}$.*

In order to prove this result, we want to apply Theorem 2.2. Note that we already have exponential tightness for $(\varepsilon\mathbb{e}(q), \varepsilon > 0)$ for every $q \in [0,1]$, as a consequence of Proposition 3.1 for $n = 1$. It turns out, however, that the criterion given in Theorem 2.2 cannot immediately be used to obtain exponential tightness over the whole interval $[0,1]$. We must instead treat the intervals $[0, 1-\delta]$ and $[1-\delta, 1]$ separately.

Lemma 4.2. *For every $\delta \in (0,1)$, there exists a constant $C = C(\alpha, \delta) \in (0,\infty)$ such that for every $s, t, u \in [0, 1-\delta]$ with $s \leqslant t \leqslant u$ and $\lambda \geqslant 0$,*

$$\mathbb{E}\left[\exp\left(\lambda M(\mathbb{e}_s, \mathbb{e}_t, \mathbb{e}_u)\right)\right] \leqslant C \exp\left((u-s)\lambda^\alpha\right).$$

Proof. We split the expectation into five terms:

$$\begin{aligned}\mathbb{E}\left[\exp\left(\lambda M(\mathbb{e}_s, \mathbb{e}_t, \mathbb{e}_u)\right)\right] &= \mathbb{E}\left[e^{\lambda(\mathbb{e}_s - \mathbb{e}_t)} \mathbb{1}_{\{\mathbb{e}_t \leqslant \mathbb{e}_s \leqslant \mathbb{e}_u\}}\right] + \mathbb{E}\left[e^{\lambda(\mathbb{e}_t - \mathbb{e}_u)} \mathbb{1}_{\{\mathbb{e}_s \leqslant \mathbb{e}_u \leqslant \mathbb{e}_t\}}\right] \\ &\quad + \mathbb{E}\left[e^{\lambda(\mathbb{e}_t - \mathbb{e}_s)} \mathbb{1}_{\{\mathbb{e}_u \leqslant \mathbb{e}_s \leqslant \mathbb{e}_t\}}\right] + \mathbb{E}\left[e^{\lambda(\mathbb{e}_u - \mathbb{e}_t)} \mathbb{1}_{\{\mathbb{e}_t \leqslant \mathbb{e}_u \leqslant \mathbb{e}_s\}}\right] \\ &\quad + \mathbb{P}\left(\{\mathbb{e}_s \leqslant \mathbb{e}_t \leqslant \mathbb{e}_u\} \cup \{\mathbb{e}_u \leqslant \mathbb{e}_t \leqslant \mathbb{e}_s\}\right) \\ &\leqslant 2\mathbb{E}\left[e^{\lambda(\mathbb{e}_s - \mathbb{e}_t)} \mathbb{1}_{\{\mathbb{e}_t \leqslant \mathbb{e}_s\}}\right] + 2\mathbb{E}\left[e^{\lambda(\mathbb{e}_t - \mathbb{e}_u)} \mathbb{1}_{\{\mathbb{e}_u \leqslant \mathbb{e}_t\}}\right] + 1.\end{aligned}$$

We see that the two expectation terms on the last line are of the same form $\mathbb{E}\left[e^{\lambda(\mathbb{e}_a - \mathbb{e}_b)} \mathbb{1}_{\{\mathbb{e}_b \leqslant \mathbb{e}_a\}}\right]$ where $a \leqslant b$ with $b - a \leqslant u - s$. For such a, b, letting $c = \alpha\Gamma(1-1/\alpha)$, we have

$$\begin{aligned}\mathbb{E}\left[e^{\lambda(\mathbb{e}_a - \mathbb{e}_b)} \mathbb{1}_{\{\mathbb{e}_b \leqslant \mathbb{e}_a\}}\right] &= c \int_0^\infty dx \int_0^\infty dy\, j_a(x) p_{b-a}^{(0,\infty)}(x,y) q_y(1-b) e^{\lambda(x-y)} \mathbb{1}_{\{y \leqslant x\}} \\ &\leqslant c \int_0^\infty dz\, p_{b-a}(-z) e^{\lambda z} \int_z^\infty j_a(x) q_{x-z}(1-b)\, dx,\end{aligned}$$

where we have used the fact that $p_{b-a}^{(0,\infty)}(x,y) \leqslant p_{b-a}(y-x)$ and a change of variables. We claim that the last integral in x is uniformly bounded over $z \geqslant 0$, $0 \leqslant a < b \leqslant 1-\delta$. If we can prove this claim, then this will imply the existence of a finite constant such that

$$\mathbb{E}\left[e^{\lambda(\mathbb{e}_a - \mathbb{e}_b)} \mathbb{1}_{\{\mathbb{e}_b \leqslant \mathbb{e}_a\}}\right] \leqslant C\mathbb{E}\left[e^{-\lambda L_{b-a}}\right] = C \exp\left((b-a)\lambda^\alpha\right) \leqslant C \exp\left((u-s)\lambda^\alpha\right),$$

for every $a, b \in [0, 1-\delta]$ with $a \leqslant b$ and $b - a \leqslant u - s$, which gives the result.

To prove the claim, note that

$$\begin{aligned}\int_z^\infty j_a(x) q_{x-z}(1-b)\,dx &= \int_z^\infty j_a(x) \frac{x-z}{1-b} p_{1-b}(z-x)\,dx \\ &\leqslant \frac{\|p_\delta\|_\infty}{\delta} \int_0^\infty x j_a(x)\,dx \\ &= \frac{\|p_\delta\|_\infty}{\delta} \int_0^\infty x j_1(x)\,dx,\end{aligned}$$

where in the last display we have used the scaling relation (2.2) that implies that the integral does not depend on a. Letting $\bar{J}_1(x) = \int_x^\infty j_a(y)\,dy$, we may integrate by parts and get an upper bound which is within a multiplicative constant of

$$\left[x\bar{J}_1(x)\right]_0^\infty + \int_0^\infty \bar{J}_1(x)\,dx.$$

Now by [22, (3.20)], we have that $\bar{J}_1(x) \sim \bar{c}x^{-\alpha}$ as $x \to \infty$ for some finite constant $\bar{c}$, and the desired uniform upper bound follows. □

Our next lemma shows that $\mathbb{e}$ is exponentially well-behaved near time 1.

Lemma 4.3. *For every* $\lambda, \gamma > 0$*, there exists* $\delta \in (0,1)$ *such that*

$$-\limsup_{\varepsilon \downarrow 0} \varepsilon^{\alpha'} \log \mathbb{P}\left(\sup_{1-\delta \leqslant t \leqslant 1} \varepsilon \mathbb{e}_t \geqslant \gamma\right) \geqslant \lambda.$$

Proof. For all $\delta > 0$, we have

$$\mathbb{P}\left(\sup_{1-\delta \leqslant t \leqslant 1} \varepsilon \mathbb{e}_t \geqslant \gamma\right) = \mathbb{P}\left(\varepsilon \mathbb{e}_{1-\delta} \geqslant \gamma\right) + \mathbb{P}\left(\sup_{1-\delta < t \leqslant 1} \varepsilon \mathbb{e}_t \geqslant \gamma, \varepsilon \mathbb{e}_{1-\delta} < \gamma\right).$$

From Proposition 3.1,

$$\limsup_{\varepsilon \downarrow 0} \varepsilon^{\alpha'} \log \mathbb{P}\left(\varepsilon \mathbb{e}_{1-\delta} \geqslant \gamma\right) \leqslant -c_\alpha \left(\frac{\gamma^\alpha}{\delta}\right)^{\frac{1}{\alpha-1}}. \tag{4.1}$$

Let us prove now a similar bound for the second probability. By (2.4), we may recast it as

$$\mathbb{P}\left(\sup_{1-\delta < t \leqslant 1} \varepsilon \mathbb{e}_t \geqslant \gamma, \varepsilon \mathbb{e}_{1-\delta} < \gamma\right) = \alpha\Gamma\left(1 - \frac{1}{\alpha}\right) \int_0^{\gamma/\varepsilon} dx\, j_{1-\delta}(x) q_x(\delta) \mathbb{P}_x^\delta \left(\sup_{0 \leqslant t \leqslant \delta} L \geqslant \gamma/\varepsilon\right). \tag{4.2}$$

Now note that $\mathbb{P}_x^\delta(\sup_{0 \leqslant t \leqslant \delta} L \geqslant \gamma/\varepsilon)$ is the limit of $\mathbb{P}_x^\delta(\sup_{0 \leqslant t \leqslant \delta'} L \geqslant \gamma/\varepsilon)$ as $\delta' \uparrow \delta$. By the absolute continuity relation (2.5) and an elementary martingale argument, the latter can be rewritten as

$$\mathbb{E}_x\left[\mathbb{1}_{\{T_0 > S\}} \mathbb{1}_{\{S < \delta'\}} \frac{q_{L_S}(\delta - S)}{q_x(\delta)}\right],$$

where S denotes the stopping time $\inf\{t \geqslant 0 : L_t > \gamma/\varepsilon\}$. Finally, for every $\eta \in (0,1)$, we may use (3.5) to obtain

$$q_{\gamma/\varepsilon}(\delta - S) \leqslant C(\eta, \gamma) \exp\left(-(1-\eta)c_\alpha \left(\frac{\gamma^\alpha}{\varepsilon^\alpha \delta}\right)^{\frac{1}{\alpha-1}}\right).$$

Since this bound does not depend on δ', plugging it into the previous expectation gives

$$\mathbb{P}\left(\sup_{1-\delta<t\leqslant 1} \varepsilon e_t \geqslant \gamma, \varepsilon e_{1-\delta} < \gamma\right) \leqslant \alpha\Gamma\left(1-\frac{1}{\alpha}\right)\int_0^{\gamma/\varepsilon} dx\, j_{1-\delta}(x) C(\eta,\gamma) \exp\left(-(1-\eta)c_\alpha\left(\frac{\gamma^\alpha}{\varepsilon^\alpha\delta}\right)^{\frac{1}{\alpha-1}}\right).$$

Since $j_{1-\delta}$ is integrable, the desired bound follows. □

Proof of Proposition 4.1. Fix $\lambda > 0$. By Lemma 4.3, for every $n \geqslant 1$, there exists a $\delta_n \in (0,1)$ such that for every $\varepsilon > 0$ small enough

$$\mathbb{P}\left(\sup_{1-2\delta_n<t\leqslant 1} \varepsilon e \geqslant \frac{1}{2^n}\right) \leqslant \frac{\exp\left(-\lambda\varepsilon^{-\alpha'}\right)}{2^n}.$$

For this choice of δ_n, by Lemma 4.2 and Theorem 2.2, for every $n \geqslant 0$ there exists a compact set $K_\lambda^{(n)}$ of $\mathbb{D}[0,1]$ such that for every $\varepsilon > 0$ small enough,

$$\mathbb{P}\left((\varepsilon e_t^{(n)}, 0 \leqslant t \leqslant 1-\delta_n) \notin K_\lambda^{(n)}\right) \leqslant \frac{\exp\left(-\lambda\varepsilon^{-\alpha'}\right)}{2^n},$$

where $e^{(n)}$ is the process $(e_{t\wedge(1-\delta_n)}, 0 \leqslant t \leqslant 1)$. We conclude by noting that the set K_λ of functions $f \in \mathbb{D}[0,1]$ such that for every $n \geqslant 0$, $\big(f(t\wedge(1-\delta_n)), 0 \leqslant t \leqslant 1\big) \in K_\lambda^{(n)}$ and $\sup_{1-2\delta_n\leqslant t\leqslant 1}|f(t)| \leqslant 2^{-n}$ is relatively compact, and, by the above, satisfies $\mathbb{P}(\varepsilon e \notin K_\lambda) \leqslant 2\exp(-\lambda\varepsilon^{-\alpha'})$. □

5. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We will do this by combining the exponential tightness with a weaker form of the LDP, as we now explain. The *weak topology* $\mathcal{W}$ on $\mathbb{D}[0,1]$ is the topology generated by the basis of neighborhoods of the form

$$N(f, t_1, \ldots, t_k, \varepsilon_1, \ldots, \varepsilon_k) = \left\{g \in \mathbb{D}[0,1] : |g(t_i) - f(t_i)| < \varepsilon_i, 1 \leqslant i \leqslant k\right\},$$

where $f \in \mathbb{D}[0,1]$, $\varepsilon_1, \ldots, \varepsilon_k > 0$, and $t_1, \ldots, t_k$ are elements of $[0,1]$ *that are continuity points of* f. Here, by convention, 0 is a continuity point of f if and only if $f(0) = 0$, which is consistent with our convention that $f(0-) = 0$. Clearly this defines a Hausdorff topology, since two different elements of $\mathbb{D}[0,1]$ necessarily differ at some common continuity point. It is easy to see that a sequence $(f_n, n \geqslant 0)$ that converges to a limit f in $(\mathbb{D}[0,1], \mathrm{dist})$ also converges pointwise at every continuity point of f, and therefore converges to f in the topological space $(\mathbb{D}[0,1], \mathcal{W})$. Consequently, the weak topology is coarser than the topology of $(\mathbb{D}[0,1], \mathrm{dist})$. Therefore, by [8, Corollary 4.2.6], and by the exponential tightness established in Proposition 4.1, Theorem 1.2 will follow from the following statement.

Proposition 5.1. *The laws of* $(\varepsilon e_t)_{t\in[0,1]}$ *satisfy an LDP in* $(\mathbb{D}[0,1], \mathcal{W})$ *as* $\varepsilon \downarrow 0$ *with speed* $\varepsilon^{-\alpha'}$ *and good rate function* I_e.

The remainder of this section is thus devoted to the proof of this proposition, which follows the approach of Lynch and Sethuraman [19] closely.

5.1. Facts about the rate function. Denote by $\mathfrak{S}$ the set of finite subdivisions of $[0,1]$. For $\sigma = (t_1, \dots, t_n)$, where $0 < t_1 < \cdots < t_n < 1$, recall that for $x_1, \dots, x_n \in \mathbb{R}_+$, we let

$$J_\sigma(x_1, \dots, x_n) = c_\alpha \sum_{i=1}^{n} (t_{i+1} - t_i) \left(\frac{(x_i - x_{i+1})_+}{t_{i+1} - t_i} \right)^{\alpha'}$$

where, by convention, $x_{n+1} = 0$ and $t_{n+1} = 1$. We let $J_\sigma(x_1, \dots, x_n) = \infty$ if one of the x_i's is negative. To ease notation, for $f : [0,1] \to \mathbb{R}$, we let $I^\sigma_{\mathbb{e}}(f) = J_\sigma(f(t_1), \dots, f(t_n))$. By the Dawson-Gärtner theorem [8, Theorem 4.6.1], it follows from Propositions 3.1 and 4.1 that the laws of $(\varepsilon \mathbb{e}_t)_{t \in [0,1]}$ satisfy a LDP in $\mathbb{R}^{[0,1]}$ (with the product topology) as $\varepsilon \downarrow 0$, with speed $\varepsilon^{-\alpha'}$ and good rate function

$$\widetilde{I}_{\mathbb{e}}(f) = \sup_{\sigma \in \mathfrak{S}} J_\sigma(f(t_1), \dots, f(t_n)). \tag{5.1}$$

We cannot immediately make use of this, since the domain of this rate function is not a space of càdlàg functions. However, let us prove some properties of the rate function $\widetilde{I}_{\mathbb{e}}$ and, in particular, that its restriction to $\mathbb{D}[0,1]$ coincides with the rate function $I_{\mathbb{e}}$ given in Theorem 1.2. To this end, we prove the following proposition.

Proposition 5.2. *A function* $f \in \mathbb{D}[0,1]$ *with* $f \geqslant 0$ *and* $f(0) = 1$ *is in* H_{ex} *if and only if*

$$M_{\mathbb{e}}(f) \coloneqq \sup_{\sigma \in \mathfrak{S}} I^\sigma_{\mathbb{e}}(f) < \infty.$$

In this case, we have

$$M_{\mathbb{e}}(f) = I_{\mathbb{e}}(f)$$

and, consequently, the functions $I_{\mathbb{e}}$ *and* $\widetilde{I}_{\mathbb{e}}$ *coincide on* $\mathbb{D}[0,1]$.

Proof. This statement should be compared with [19, Theorem 3.2], where the proof uses a martingale argument. We provide another elementary proof here, based on the Lebesgue differentiation theorem instead. For convenience, let $\Lambda(x) = c_\alpha (x_-)^{\alpha'}$ for all $x \in \mathbb{R}$.

Let $f \in H_{\text{ex}}$, and write $f = f_\uparrow - f_\downarrow$ for its Jordan decomposition with absolutely continuous $f_\downarrow$, such that $f'_\downarrow \in \mathbb{L}^{\alpha'}[0,1]$. Let $\sigma = (t_1, \dots, t_n)$ be a subdivision of $[0,1]$. Here and below, we adopt the notational convention that $t_0 = 0$ and $t_{n+1} = 1$. Then

$$\begin{aligned}
c_\alpha \sum_{i=0}^{n} \left(\frac{\big(f(t_i) - f(t_{i+1})\big)_+^{\alpha}}{t_{i+1} - t_i} \right)^{\frac{1}{\alpha - 1}} &= \sum_{i=0}^{n} (t_{i+1} - t_i) \Lambda \left(\frac{f(t_{i+1}) - f(t_i)}{t_{i+1} - t_i} \right) \\
&\leqslant \sum_{i=0}^{n} (t_{i+1} - t_i) \Lambda \left(\frac{f_\downarrow(t_i) - f_\downarrow(t_{i+1})}{t_{i+1} - t_i} \right) \\
&= c_\alpha \sum_{i=0}^{n} (t_{i+1} - t_i) \left(\frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} f'_\downarrow(s)\, ds \right)^{\alpha'} \\
&\leqslant c_\alpha \sum_{i=0}^{n} \int_{t_i}^{t_{i+1}} f'_\downarrow(s)^{\alpha'}\, ds
\end{aligned}$$

$$= \quad c_\alpha \int_0^1 f'_\downarrow(s)^{\alpha'} \, ds,$$

where we used the fact that $f_\uparrow$ and Λ are non-increasing in the first inequality, and applied Jensen's inequality in the second inequality. Since this is true for any subdivision $\sigma \in \mathfrak{S}$, we get the first bound $M_{e}(f) \leqslant \int_0^1 f'_\downarrow(s)^{\alpha'} \, ds < \infty$.

Conversely, assume that $f \in H_{ex}$ is not of bounded variation and fix $A > 0$. Then there exists a subdivision $\sigma = (t_1, \dots, t_n)$ such that

$$A < \sum_{i=0}^{n} |f(t_{i+1}) - f(t_i)| = \sum_{i=0}^{n} \big(f(t_i) - f(t_{i+1})\big)_+ + \sum_{i=1}^{n} \big(f(t_i) - f(t_{i+1})\big)_-.$$

Furthermore

$$f(0) = \sum_{i=0}^{n} \big(f(t_i) - f(t_{i+1})\big) = \sum_{i=0}^{n} \big(f(t_i) - f(t_{i+1})\big)_+ - \sum_{i=0}^{n} \big(f(t_i) - f(t_{i+1})\big)_-.$$

This implies that

$$\sum_{i=0}^{n} \big(f(t_i) - f(t_{i+1})\big)_+ \geqslant \frac{A + f(0)}{2}.$$

Therefore,

$$\begin{aligned} \frac{A + f(0)}{2} \leqslant \sum_{i=0}^{n} \big(f(t_i) - f(t_{i+1})\big)_+ &= \sum_{i=0}^{n} (t_{i+1} - t_i)^{1/\alpha} \frac{\big(f(t_i) - f(t_{i+1})\big)_+}{(t_{i+1} - t_i)^{1/\alpha}} \\ &\leqslant \left(\sum_{i=0}^{n} (t_{i+1} - t_i) \right)^{1/\alpha} \left(\sum_{i=0}^{n} \frac{\big(f(t_i) - f(t_{i+1})\big)_+^{\alpha'}}{(t_{i+1} - t_i)^{\frac{1}{\alpha - 1}}} \right)^{1/\alpha'}, \end{aligned}$$

by Hölder's inequality, and this entails that $M_{e}(f) = \infty$. By the contrapositive, this implies that if $M_{e}(f) < \infty$, then f has bounded variation. Therefore, assuming that $M_{e}(f) < \infty$, we may write $f = f_\uparrow - f_\downarrow$ for the Jordan decomposition of f, with $f_\uparrow, f_\downarrow$ nondecreasing and such that $f_\uparrow(0) = 0$ and $df_\uparrow \perp df_\downarrow$. We proceed by contradiction. Suppose that $f_\downarrow$ is not absolutely continuous. Then there exists $\varepsilon > 0$ such that for all $k \geqslant 1$, there exists an open set of the form $U_k = \bigsqcup_{i=1}^{n(k)} (s_i^{(k)}, t_i^{(k)})$ with

$$\sum_{i=1}^{n(k)} (t_i^{(k)} - s_i^{(k)}) < \frac{1}{k} \quad \text{and} \quad \sum_{i=1}^{n(k)} \big((f_\downarrow(t_i^{(k)}) - f_\downarrow(s_i^{(k)}) \big) > 2\varepsilon.$$

Moreover since $df_\downarrow \perp df_\uparrow$, there exists a measurable set B such that $df_\downarrow(B^c) = df_\uparrow(B) = 0$, and by regularity of the measures Leb, $df_\downarrow$ and $df_\uparrow$ applied to the set $B \cap U_k$, we may find open sets $V_k^{(1)}, V_k^{(2)}$ containing $B \cap U_k$ such that

$$\text{Leb}(V_k^{(1)}) < \frac{1}{k} \qquad \text{and} \qquad df_\uparrow(V_k^{(2)}) < \varepsilon,$$

and by setting $V_k = V_k^{(1)} \cap V_k^{(2)}$, we see that these two inequalities remain true with V_k in place of $V_k^{(1)}$ and $V_k^{(2)}$ respectively, while

$$df_\downarrow(V_k) \geqslant df_\downarrow(B \cap U_k) = df_\downarrow(U_k) > 2\varepsilon.$$

By writing the open set V_k as the limit of finite unions of open intervals, we deduce that we may choose the family of intervals $\{(s_i^{(k)}, t_i^{(k)}),\ 1 \leqslant i \leqslant n(k)\}$ in such a way that

$$\sum_{i=1}^{n(k)} \left(f_\uparrow(t_i^{(k)}) - f_\uparrow(s_i^{(k)}) \right) < \varepsilon.$$

Then on the one hand we have

$$\begin{aligned} \sum_{i=1}^{n(k)} \left(f(s_i^{(k)}) - f(t_i^{(k)}) \right)_+ &= \sum_{i=1}^{n(k)} \left(f_\downarrow(t_i^{(k)}) - f_\downarrow(s_i^{(k)}) - (f_\uparrow(t_i^{(k)}) - f_\uparrow(s_i^{(k)})) \right)_+ \\ &\geqslant \sum_{i=1}^{n(k)} \left(f_\downarrow(t_i^{(k)}) - f_\downarrow(s_i^{(k)}) \right) - \sum_{i=1}^{n(k)} \left(f_\uparrow(t_i^{(k)}) - f_\uparrow(s_i^{(k)}) \right) \\ &\geqslant \varepsilon. \end{aligned} \tag{5.2}$$

On the other hand, by Hölder's inequality,

$$\begin{aligned} \sum_{i=1}^{n(k)} \left(f(s_i^{(k)}) - f(t_i^{(k)}) \right)_+ &\leqslant \left(\sum_{i=1}^{n(k)} (t_i^{(k)} - s_i^{(k)}) \right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^{n(k)} \frac{\left(f(s_i^{(k)}) - f(t_i^{(k)}) \right)_+^{\alpha'}}{(t_i^{(k)} - s_i^{(k)})^{\frac{1}{\alpha-1}}} \right)^{1/\alpha'} \\ &\leqslant \frac{1}{c_\alpha k^{1/\alpha}} M_e(f). \end{aligned} \tag{5.3}$$

But (5.2) and (5.3) combined contradict the assumption that $M_e(f) < +\infty$. Thus $f_\downarrow$ is absolutely continuous.

Let us now prove that $\int_0^1 f_\downarrow'(s)^{\alpha'}\, ds \leqslant M_e(f)$, which will prove that $f \in H_{ex}$, and that if $f \in H_{ex}$, then $M_e(f) = \int_0^1 f_\downarrow'(s)^{\alpha'}\, ds$. For $n \geqslant 1$, define

$$f^{(n)}(t) = n\left(f\left(\frac{\lfloor (n+1)t \rfloor}{n} \right) - f\left(\frac{\lfloor nt \rfloor}{n} \right) \right) \text{ for } t \in [0,1), \quad f^{(n)}(1) = n\left(f(1) - f\left(1 - \frac{1}{n}\right) \right).$$

By the Lebesgue decomposition theorem, we may write $f_\uparrow = f_{\uparrow ac} + f_{\uparrow sing}$, where $f_{\uparrow ac}$ is an absolutely continuous function and $f_{\uparrow sing}$ is such that $df_{\uparrow sing}$ is singular with respect to the Lebesgue measure. By the Lebesgue differentiation theorem, for Lebesgue-almost every $t \in [0,1]$ we have

$$f^{(n)}(t) \underset{n\to+\infty}{\longrightarrow} f_{\uparrow ac}'(t) - f_\downarrow'(t).$$

Considering the subdivision $\sigma_n = \left(0, \frac{1}{n}, \frac{2}{n}, \dots, 1\right)$, we then have

$$\begin{aligned} M_e(f) &\geqslant \liminf_{n\to+\infty} J_{\sigma_n}(f) \\ &= \liminf_{n\to+\infty} \sum_{i=1}^{n} \frac{1}{n} \Lambda\left(f^{(n)}\left(\frac{i}{n}\right) \right) \\ &= \liminf_{n\to+\infty} \int_0^1 \Lambda\left(f^{(n)}(s) \right) ds. \end{aligned}$$

By Fatou's lemma, we thus get

$$M_{\mathbb{e}}(f) \geqslant \int_0^1 \Lambda\left(f'_{\uparrow ac}(s) - f'_{\downarrow}(s)\right)\, ds.$$

Recall that $df_{\uparrow} \perp df_{\downarrow}$, so that we also have $df_{\uparrow ac} \perp df_{\downarrow}$. But by the Lebesgue differentiation theorem, we have $df_{\uparrow ac}(s) = f'_{\uparrow ac}(s)\, ds$ and $df_{\downarrow}(s) = f'_{\downarrow}(s)\, ds$, so that the sets $\{f'_{\uparrow ac} > 0\}$ and $\{f'_{\downarrow} > 0\}$ intersect in a set of zero Lebesgue measure. Since $f'_{\uparrow ac} \geqslant 0$ Lebesgue-a.e., we have $\Lambda(f'_{\uparrow ac}) = 0$, which implies that

$$\int_0^1 \Lambda\left(f'_{\uparrow ac}(s) - f'_{\downarrow}(s)\right)\, ds = \int_0^1 \Lambda\left(-f'_{\downarrow}(s)\right)\, ds = c_\alpha \int_0^1 f'_{\downarrow}(s)^{\alpha'}\, ds,$$

which concludes the proof. □

In passing, we note that the reasoning at the end of this proof explains why we may express the rate function $I_{\mathbb{e}}$ in the alternative form (1.5).

Lemma 5.3. *The function* $I_{\mathbb{e}}$ *is a good rate function on the spaces* $(\mathbb{D}[0,1], \mathcal{W})$ *and* $(\mathbb{D}[0,1], \mathrm{dist})$.

Proof. Since the weak topology is not first-countable, we must initially use nets to characterise the lower-semicontinuity of $I_{\mathbb{e}}$. We first need to show that if (f_λ) is a net that converges to f in the weak topology, with $f_\lambda \geqslant 0$ and $f_\lambda(1) = f(1) = 0$ for every λ, then $\liminf_\lambda I_{\mathbb{e}}(f_\lambda) \geqslant I_{\mathbb{e}}(f)$. Let $\sigma = (t_1, \ldots, t_k)$ be a subdivision of continuity points of f, so that $f_\lambda(t_i)$ converges to $f(t_i)$ for $1 \leqslant i \leqslant k$, and $I^\sigma_{\mathbb{e}}(f_\lambda) = J_\sigma(f_\lambda(t_1), \ldots, f_\lambda(t_k))$ converges to $I^\sigma_{\mathbb{e}}(f) = J_\sigma(f(t_1), \ldots, f(t_k))$. Since $I_{\mathbb{e}}(f_\lambda) \geqslant I^\sigma_{\mathbb{e}}(f_\lambda)$ by Proposition 5.2, this implies that $\liminf_\lambda I_{\mathbb{e}}(f_\lambda) \geqslant I^\sigma_{\mathbb{e}}(f)$. Applying Proposition 5.2 once again allows us to conclude that $I_{\mathbb{e}}$ is a rate function on $(\mathbb{D}[0,1], \mathcal{W})$, and therefore also on $(\mathbb{D}[0,1], \mathrm{dist})$.

Let us now prove that $I_{\mathbb{e}}$ is good on $(\mathbb{D}[0,1], \mathrm{dist})$, which will imply the result. Fix $c \in (0, \infty)$, and then pick $f \in \mathbb{D}[0,1]$ with $I_{\mathbb{e}}(f) \leqslant c$ so that, in particular, $f(1) = 0$ and f has bounded variation. Let $s \leqslant t$ be in $[0,1]$. Then, by Hölder's inequality,

$$f(s) - f(t) \leqslant f_{\downarrow}(t) - f_{\downarrow}(s) = \int_s^t f'_{\downarrow}(u)\, du \leqslant (c/c_\alpha)(t-s)^{1/\alpha}.$$

Since $f(1) = 0$, this implies that f is uniformly bounded and, moreover, that for every $s \leqslant t \leqslant u$ we have

$$M\big(f(s), f(t), f(u)\big) \leqslant 2\big((f(s) - f(t))\mathbb{1}_{\{f(t) \leqslant f(s)\}} + (f(t) - f(u))\mathbb{1}_{\{f(u) \leqslant f(t)\}}\big) \leqslant 4(c/c_\alpha)(u-s)^{1/\alpha}.$$

The conclusion now follows from Theorem 2.1. □

Next, for $A \subset \mathbb{D}[0,1]$, and for $\sigma \in \mathfrak{S}$, we let

$$I^\sigma_{\mathbb{e}}(A) = \inf_{f \in A} I^\sigma_{\mathbb{e}}(f) \qquad \text{and} \qquad I_{\mathbb{e}}(A) = \inf_{f \in A} I_{\mathbb{e}}(f)\,.$$

Lemma 5.4. *For every closed set* F *of* $(\mathbb{D}[0,1], \mathcal{W})$*, we have*

$$I_{\mathbb{e}}(F) = \sup_{\sigma \in \mathfrak{S}} I^\sigma_{\mathbb{e}}(F)\,.$$

Proof. The proof follows that of [19, Theorem 3.5] closely. Since we know from Proposition 5.2 that $I^{\sigma}_{\mathbb{e}}(A) \leqslant I_{\mathbb{e}}(A)$ for every $\sigma \in \mathfrak{S}$ and every set A, let us assume, for a contradiction, that $\sup_{\sigma\in\mathfrak{S}} I^{\sigma}_{\mathbb{e}}(F) < c < I_{\mathbb{e}}(F)$ for some constant $c \in (0,\infty)$. For every subdivision $\sigma = (t_1,\dots,t_k) \in \mathfrak{S}$, we may find some element $f_\sigma \in \mathbb{D}[0,1]$ such that $I^{\sigma}_{\mathbb{e}}(f_\sigma) < c$. Let $\hat{f}_\sigma$ be the piecewise affine interpolation of the values of f_σ at times $0 < t_1 < \ldots < t_k < 1$, with $\hat{f}_\sigma(0) = \hat{f}_\sigma(1) = 0$ and of course $\hat{f}_\sigma(t_i) = f_\sigma(t_i), 1 \leqslant i \leqslant k$. Then, plainly,

$$I_{\mathbb{e}}(\hat{f}_\sigma) = I^{\sigma}_{\mathbb{e}}(\hat{f}_\sigma) = I^{\sigma}_{\mathbb{e}}(f_\sigma) < c$$

and, by Lemma 5.3, we obtain that $\{\hat{f}_\sigma, \sigma \in \mathfrak{S}\}$ forms a relatively compact family in $(\mathbb{D}[0,1], \mathcal{W})$ (and even in $(\mathbb{D}[0,1], \mathrm{dist})$). Let f_0 be a cluster point of this family, and $\sigma' = (t'_1,\dots,t'_l) \in \mathfrak{S}$ be a subdivision consisting of continuity points of f_0. We fix $\varepsilon > 0$ and consider the weak neighborhood of f_0 defined by

$$N_{\sigma',\varepsilon} = \left\{ f \in \mathbb{D}[0,1] : \max_{1\leqslant i\leqslant l} |f(t'_i) - f_0(t'_i)| < \varepsilon \right\} \in \mathcal{W}.$$

For any partition σ'' finer than σ', there exits an even finer σ such that $\hat{f}_\sigma \in N_{\sigma',\varepsilon}$, since f_0 is a cluster point. But since $\hat{f}_\sigma$ agrees with f_σ on σ, it follows that $f_\sigma \in N_{\sigma',\varepsilon}$ and, therefore, that f_0 is also a cluster point of $\{f_\sigma : \sigma \in \mathfrak{S}\} \subset F$. Since F is closed, we conclude that $f_0 \in F$, and that $I_{\mathbb{e}}(f_0) \leqslant c$ by lower semicontinuity of $I_{\mathbb{e}}$. This contradicts the assumption that $I_{\mathbb{e}}(F) > c$, and the result follows. □

We now have all the tools needed to prove Proposition 5.1. The proof is split into two lemmas which follow [19, Theorems 4.1 and 4.2] closely.

Lemma 5.5. *If F is a closed subset of $(\mathbb{D}[0,1], \mathcal{W})$, then*

$$\limsup_{\varepsilon\downarrow 0} \varepsilon^{\alpha'} \log \mathbb{P}(\varepsilon\mathbb{e} \in F) \leqslant -I_{\mathbb{e}}(F).$$

Proof. For any $\sigma = (t_1,\dots,t_k) \in \mathfrak{S}$, it holds that

$$\mathbb{P}(\varepsilon\mathbb{e} \in F) \leqslant \mathbb{P}\left(I^{\sigma}_{\mathbb{e}}(\varepsilon\mathbb{e}) \geqslant I^{\sigma}_{\mathbb{e}}(F)\right).$$

From the explicit form of $I^{\sigma}_{\mathbb{e}}(\varepsilon\mathbb{e})$, and the fact that $\varepsilon(\mathbb{e}_{t_1},\dots,\mathbb{e}_{t_k})$ satisfy an LDP with continuous rate function J_σ by Proposition 3.1, we obtain by the contraction principle that

$$\limsup_{\varepsilon\downarrow 0} \varepsilon^{\alpha'} \log \mathbb{P}(\varepsilon\mathbb{e} \in F) \leqslant -I^{\sigma}_{\mathbb{e}}(F).$$

We conclude using Lemma 5.4. □

Lemma 5.6. *If G is an open subset of $(\mathbb{D}[0,1], \mathcal{W})$, then*

$$\liminf_{\varepsilon\downarrow 0} \varepsilon^{\alpha'} \log \mathbb{P}(\varepsilon\mathbb{e} \in G) \geqslant -I_{\mathbb{e}}(G).$$

Proof. Without loss of generality, we may assume that $I(G) < \infty$. We then fix $\varepsilon > 0$ and select $f \in G$ such that $I_{\mathbb{e}}(f) < I_{\mathbb{e}}(G) + \varepsilon$. Then, we may find $\delta > 0$ and a subdivision $\sigma = (t_1,\dots,t_k)$ consisting of continuity points of f such that $\{g \in \mathbb{D}[0,1] : \max_{1\leqslant i\leqslant k} |g(t_i) - f(t_i)| < \delta\}$ is contained in G. We deduce that

$$\mathbb{P}(\varepsilon \mathbb{e} \in G) \geqslant \mathbb{P}\left(\varepsilon(\mathbb{e}_{t_1}, \dots, \mathbb{e}_{t_k}) \in G'\right)$$

where G' is the open set $\{(x_1, \dots, x_k) \in \mathbb{R}^k : \max_{1\leqslant i\leqslant k} |x_i - f(t_i)| < \delta\}$. By Proposition 3.1, we obtain that

$$\liminf_{\varepsilon\downarrow 0} \varepsilon^{\alpha'} \log \mathbb{P}(\varepsilon \mathbb{e} \in G) \geqslant - \inf_{(x_1,\dots,x_k)\in G'} J_\sigma(x_1, \dots, x_k) .$$

Letting $\delta \to 0$, we may conclude that

$$\liminf_{\varepsilon\downarrow 0} \varepsilon^{\alpha'} \log \mathbb{P}(\varepsilon \mathbb{e} \in G) \geqslant -I_{\mathbb{e}}^{\sigma}(f) \geqslant -I_{\mathbb{e}}(f) \geqslant -I_{\mathbb{e}}(G) - \varepsilon ,$$

as desired. □

6. Consequences of the LDP for stable excursions

In this section, we prove the remaining statements: Theorem 1.4, Corollaries 1.5 and 1.6, and Proposition 1.3.

6.1. Proof of Theorem 1.4. We follow the approach of Fill and Janson [10] closely. First, a direct consequence of the fact that $I_{\mathbb{e}}$ is a good rate function is the following.

Lemma 6.1. *The set K_{ex} defined at (1.6) is a compact subset of $\mathbb{D}[0,1]$.*

The argument for the proof of Theorem 1.4 is the same as [10, p.415]. We apply the contraction principle ([14, Theorem 27.11], [8, Theorem 4.2.1]) to the continuous functional $\Phi : D_{\mathrm{ex}}[0,1] \to \mathbb{R}_+$. This entails that $\varepsilon X = \Phi(\varepsilon \mathbb{e})$ satisfies an LDP in $[0,\infty)$ with good rate function whose value at $x > 0$ is given by

$$\begin{aligned}
\inf_{f\in H_{\mathrm{ex}} : \Phi(f)=x} c_\alpha \|f'_\downarrow\|_{\alpha'}^{\alpha'} &= \inf_{f\in H_{\mathrm{ex}} : \Phi(f)\neq 0} c_\alpha \left\| \frac{x f'_\downarrow}{\Phi(f)} \right\|_{\alpha'}^{\alpha'} \\
&= \inf_{f\in H_{\mathrm{ex}} : \Phi(f)\neq 0} c_\alpha \left(\frac{x}{\Phi(f)} \right)^{\alpha'} \|f'_\downarrow\|_{\alpha'}^{\alpha'} \\
&= \inf_{f\in H_{\mathrm{ex}} : \Phi(f)\neq 0} c_\alpha \left(\frac{x}{\Phi(f/\|f_\downarrow\|_{\alpha'})} \right)^{\alpha'} \\
&= \inf_{f\in H_{\mathrm{ex}} : \|f'_\downarrow\|_{\alpha'}=1, \Phi(f)\neq 0} c_\alpha \left(\frac{x}{\Phi(f)} \right)^{\alpha'} \\
&= c_\alpha \left(\frac{x}{\gamma_\Phi} \right)^{\alpha'} .
\end{aligned}$$

Taking $A = (1,\infty)$ and $\varepsilon = x^{-1}$ in the definition of an LDP proves (1.7). Finally, (1.8) and (1.9) follow from (1.7) by [13, Theorem 4.5]. This concludes the proof of Theorem 1.4. □

6.2. Applications. Corollaries 1.5 and 1.6 are obtained by applying Theorem 1.4 to the area and supremum functionals, which are both positive-homogeneous and continuous for the M1 topology.

6.2.1. Area under $\mathfrak{e}$. Let us compute the constant γ_Φ for the area under the normalized excursion $\mathcal{A}_{\text{ex}}$, corresponding to the functional $\Phi(f) = \int_0^1 f(s)\, ds$. So let

$$\gamma_\int = \max\left\{ \int_0^1 f(u) du : f \in K_{\text{ex}} \right\}.$$

Lemma 6.2 (Constant γ_Φ for $\mathcal{A}_{\text{ex}}$). *We have* $\gamma_\int = (\alpha+1)^{-1/\alpha}$.

Proof. We first find an upper bound. Let $f \in K_{\text{ex}}$. Note that, integrating by parts, we have

$$\begin{aligned}\int_0^1 f(s)\, ds &= \int_0^1 f_\uparrow(s)\, ds - \int_0^1 f_\downarrow(s)\, ds \\ &= \int_0^1 (f_\uparrow(s) - f_\downarrow(1))\, ds + \int_0^1 s f_\downarrow'(s)\, ds \\ &\leqslant \left(\int_0^1 |f_\downarrow'(s)|^{\alpha'}\, ds \right)^{1/\alpha'} \left(\int_0^1 s^\alpha\, ds \right)^{1/\alpha} \leqslant (\alpha+1)^{-1/\alpha},\end{aligned}$$

where in the third line we have used the fact that $f_\uparrow(1) - f_\downarrow(1) = f(1) = 0$, which entails that $f_\uparrow \leqslant f_\downarrow(1)$, and then Hölder's inequality. We obtain $\gamma_\int \leqslant (\alpha+1)^{-1/\alpha}$.

Now note that $f(s) = \frac{(\alpha+1)^{1/\alpha'}}{\alpha}(1-s^\alpha)$ lies in K_{ex} and is such that

$$\int_0^1 f(s)\, ds = (\alpha+1)^{-1/\alpha},$$

so that $\gamma_\int = (\alpha+1)^{-1/\alpha}$ is indeed the optimum. □

6.2.2. Supremum of $\mathfrak{e}$. We now compute the constant γ_Φ corresponding to the functional $\sup_{0\leqslant t\leqslant 1} f(t)$ which is continuous for the M1 Skorokhod topology.

Lemma 6.3 (Constant γ_Φ for $\sup \mathfrak{e}$). *We have* $\gamma_{\sup} = 1$.

Proof. First, notice that if $f \in K_{\text{ex}}$, then using the fact $f_\uparrow \leqslant f_\downarrow(1)$ we get that for all $t \geqslant 0$,

$$f(t) \leqslant f_\downarrow(1) - f_\downarrow(t) = \int_t^1 |f_\downarrow'(s)|\, ds \leqslant \int_0^1 |f_\downarrow'(s)|\, ds \leqslant \left(\int_0^1 |f_\downarrow'(s)|^{\alpha'}\, ds \right)^{1/\alpha'} = 1,$$

where we used Hölder's inequality. We thus obtain the upper bound $\gamma_{\sup} \leqslant 1$.

Now note that the function $f(t) = 1 - t$ lies in K_{ex} and satisfies $\sup f = 1$, so that $\gamma_{\sup} = 1$. □

6.3. Negative results for the Skorokhod J1 topology. Here we give a sketch of proof for Proposition 1.3. The idea is that it is costless for the process $\varepsilon\mathbb{e}$ to make macroscopic jumps within small time intervals, which prevents it from being concentrated in J1-compact sets. Fix $\delta > 0$. We note that

$$\mathbb{P}(\varepsilon\mathbb{e}_\delta \in [1,2], \varepsilon\mathbb{e}_{2\delta} \in [3,4]) = \alpha\Gamma\left(1-\frac{1}{\alpha}\right)\int_{1/\varepsilon}^{2/\varepsilon} dx_1\, j_\delta(x_1)\int_{3/\varepsilon}^{4/\varepsilon} dx_2\, p_\delta^{(0,\infty)}(x_1,x_2)q_{x_2}(1-2\delta)$$
$$\geqslant C(\delta)\varepsilon^{2\alpha+2}\exp(-c_\alpha(3/\varepsilon(1-2\delta))^{\alpha'}), \quad (6.1)$$

where we have used Lemmas 3.2, 3.3 and (3.2) to bound $p_\delta^{(0,\infty)}(x_1,x_2)$ uniformly from below by some constant times $\varepsilon^{\alpha+1}$, then (3.3) to bound $q_{x_2}(1-2\delta)$ uniformly from below by some constant times $\exp(-c_\alpha(3/\varepsilon(1-2\delta))^{\alpha'})$, and finally (3.6) to bound the remaining integral. Setting $\omega_{J1}(f,\eta) = \sup_{s<t<u, u-s\leqslant\eta}|f(u)-f(t)|\wedge|f(t)-f(s)|$, we obtain

$$\lim_{\delta\downarrow 0}\liminf_{\varepsilon\downarrow 0}\varepsilon^{\alpha'}\log\mathbb{P}(\omega_{J1}(\varepsilon\mathbb{e},2\delta)>1/2)\geqslant -3^{\alpha'}c_\alpha .$$

Let K be a compact subset of $\mathbb{D}[0,1]$ in the J1 topology, so that $\sup_{f\in K}\omega_{J1}(f,\delta)$ converges to 0 as $\delta\downarrow 0$. In particular, there exists δ_0 such that $\omega_{J1}(f,2\delta_0)\leqslant 1/2$ for every $f\in K$. Therefore,

$$\liminf_{\varepsilon\downarrow 0}\varepsilon^{\alpha'}\log\mathbb{P}(\varepsilon\mathbb{e}\notin K)\geqslant\liminf_{\varepsilon\downarrow 0}\varepsilon^{\alpha'}\log\mathbb{P}(\omega_{J1}(\varepsilon\mathbb{e},2\delta_0)>1/2)\geqslant -3^{\alpha'}c_\alpha ,$$

and so $(\varepsilon\mathbb{e})_{0<\varepsilon<1}$ cannot be exponentially tight.

7. LDP for the α-stable Lévy bridge

In this section, we adapt the proof of the LDP for the normalized excursion in order to get an LDP for the Lévy bridge. Roughly speaking the process $\mathbb{b}^{(a)}$, called the $(0,0)\to(1,a)$ bridge, is obtained by conditioning L to be equal to a at time 1. This is obviously a degenerate conditioning; however, it can be obtained by performing a space-time h-transform with respect to the function $\frac{p_{1-t}(a-L_t)}{p_1(a)}$ (see, for instance, [18]). This means that the law of $\mathbb{b}^{(a)}$ may be defined by

$$\mathbb{P}^{\mathrm{br}}(A) := \mathbb{E}\left[\frac{p_{1-t}(a-L_t)}{p_1(a)}\mathbb{1}_A\right], \quad \forall A\in\mathcal{F}_t , \quad t\in[0,1). \quad (7.1)$$

See [7] or [2, Chapter VIII] for a rigorous construction.

7.1. Large deviations for the finite-dimensional marginal distributions. This section is devoted to proving that the finite-dimensional marginals of $\mathbb{b}^{(a)}$ satisfy an LDP on $\mathbb{R}$.

Proposition 7.1 (LDP for the marginals of the stable bridge). *Fix $a\in\mathbb{R}$, and let $(a_\varepsilon)_{\varepsilon>0}$ be such that $\varepsilon a_\varepsilon\to a$ as $\varepsilon\to 0$. Let $\sigma=(t_1,\dots,t_n)$ be a finite subdivision of $[0,1]$. Under $\mathbb{P}$, the laws of $\varepsilon(\mathbb{b}_{t_1}^{(a_\varepsilon)},\dots,\mathbb{b}_{t_n}^{(a_\varepsilon)})$ satisfy an LDP in $\mathbb{R}^n$ with speed $\varepsilon^{-\alpha'}$ and good rate function*

$$J_{\mathbb{b},a}^\sigma(x_1,\dots,x_n) = c_\alpha\left(\left(\frac{(-x_1)_+^\alpha}{t_1}\right)^{\frac{1}{\alpha-1}} + \sum_{i=1}^{n-1}\left(\frac{(x_i-x_{i+1})_+^\alpha}{t_{i+1}-t_i}\right)^{\frac{1}{\alpha-1}} + \left(\frac{(x_n-a)_+^\alpha}{1-t_n}\right)^{\frac{1}{\alpha-1}} - (a_-)^{\alpha'}\right).$$

The proof of this proposition is similar to that of Proposition 3.1, but is technically simpler and we only explain where the argument differs. It is not difficult to check that $J_{\mathbb{b},a}$ is a good rate function, so [28, Lemma 5] applies. We use this in the same way as in the proof of Proposition 3.1, using the expression (2.6) for the marginal laws of the bridge, and the bounds (3.3) for the stable transition densities. The term $-(a_-)^{\alpha'}$ in the definition of $J_{\mathbb{b},a}$ arises from the contribution of the density $p_1(a_\varepsilon)$ in the denominator of (2.6): by (3.3) and (3.4), we have

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{\alpha'} \log p_1(a_\varepsilon) = -c_\alpha (a_-)^{\alpha'}.$$

7.2. Exponential tightness for the stable bridge.

Proposition 7.2 (Exponential tightness for the stable bridge). *Under $\mathbb{P}$, the laws of $(\varepsilon \mathbb{b}_t^{(a_\varepsilon)})_{t\in[0,1]}$ as $\varepsilon \downarrow 0$ are exponentially tight with speed $\varepsilon^{-\alpha'}$.*

Proposition 7.2 is a direct consequence of the tightness criterion in $(\mathbb{D}[0,1], \mathrm{dist})$ from Theorem 2.2 and the following lemma.

Lemma 7.3. *There exists a constant $C = C(\alpha) > 0$ such that for every $s,t,u \in [0,1]$ with $s \leqslant t \leqslant u$ and $\lambda \geqslant 0$,*

$$\mathbb{E}\left[\exp\left(\lambda M(\mathbb{b}_s^{(a_\varepsilon)}, \mathbb{b}_t^{(a_\varepsilon)}, \mathbb{b}_u^{(a_\varepsilon)})\right)\right] \leqslant C \exp\left((u-s)\lambda^\alpha\right). \tag{7.2}$$

Proof. Splitting the expectation into five terms as at the beginning of the proof of Proposition 4.1, we get the following bound

$$\mathbb{E}\left[\exp\left(\lambda M(\mathbb{b}_s^{(a_\varepsilon)}, \mathbb{b}_t^{(a_\varepsilon)}, \mathbb{b}_u^{(a_\varepsilon)})\right)\right] \leqslant 2\mathbb{E}\left[e^{\lambda(\mathbb{b}_s^{(a_\varepsilon)} - \mathbb{b}_t^{(a_\varepsilon)})} \mathbb{1}_{\{\mathbb{b}_t^{(a_\varepsilon)} \leqslant \mathbb{b}_s^{(a_\varepsilon)}\}}\right] + 2\mathbb{E}\left[e^{\lambda(\mathbb{b}_t^{(a_\varepsilon)} - \mathbb{b}_u^{(a_\varepsilon)})} \mathbb{1}_{\{\mathbb{b}_u^{(a_\varepsilon)} \leqslant \mathbb{b}_t^{(a_\varepsilon)}\}}\right] + 1.$$

We see that the last two terms are of the same form $\mathbb{E}\left[e^{\lambda(\mathbb{b}_\sigma^{(a_\varepsilon)} - \mathbb{b}_\rho^{(a_\varepsilon)})} \mathbb{1}_{\{\mathbb{b}_\rho^{(a_\varepsilon)} \leqslant \mathbb{b}_\sigma^{(a_\varepsilon)}\}}\right]$ where $\rho \leqslant \sigma$ with $\sigma - \rho \leqslant u - s$. For such ρ, σ, we have

$$\begin{aligned}
\mathbb{E}\left[e^{\lambda(\mathbb{b}_\sigma^{(a_\varepsilon)} - \mathbb{b}_\rho^{(a_\varepsilon)})} \mathbb{1}_{\{\mathbb{b}_\rho^{(a_\varepsilon)} \leqslant \mathbb{b}_\sigma^{(a_\varepsilon)}\}}\right] &= \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\, p_\rho(x) p_{\sigma-\rho}(y-x) \frac{p_{1-\sigma}(a_\varepsilon - y)}{p_1(a_\varepsilon)} e^{\lambda(x-y)} \mathbb{1}_{\{y\leqslant x\}} \\
&= \frac{1}{p_1(a_\varepsilon)} \int_0^\infty dz\, p_{\sigma-\rho}(a_\varepsilon - z) e^{\lambda z} \int_{-\infty}^{+\infty} dx\, p_\rho(x) p_{1-\sigma}(z-x),
\end{aligned}$$

where we have used the change of variables $z = x - y + a_\varepsilon$. It remains to show that the last integral in x is uniformly bounded over $z \geqslant 0$, $0 \leqslant \rho < \sigma \leqslant 1$. Indeed this gives the existence of a constant $C < \infty$ such that

$$\mathbb{E}\left[e^{\lambda(\mathbb{b}_\sigma^{(a_\varepsilon)} - \mathbb{b}_\rho^{(a_\varepsilon)})} \mathbb{1}_{\{\mathbb{b}_\rho^{(a_\varepsilon)} \leqslant \mathbb{b}_\sigma^{(a_\varepsilon)}\}}\right] \leqslant C\mathbb{E}\left[e^{-\lambda \mathbb{b}_{\sigma-\rho}^{(a_\varepsilon)}}\right] = C\exp\left((\sigma-\rho)\lambda^\alpha\right),$$

which gives the result.

However, by the cyclic invariance of the increments of $\mathbb{b}^{(\alpha)}$, which is a direct consequence of (2.6), we see that it suffices to check this boundedness assumption for $0 \leqslant \rho < \sigma \leqslant 1/2$ say. For such σ, ρ we have

$$\begin{aligned}\int_{-\infty}^{+\infty} \mathrm{d}x\, p_\rho(x) p_{1-\sigma}(z-x) &= (1-\sigma)^{-\frac{1}{\alpha}} \int_{-\infty}^{+\infty} \mathrm{d}x\, p_\rho(x) p_1\left(\frac{z-x}{(1-\sigma)^{\frac{1}{\alpha}}}\right) \\ &\leqslant 2^{\frac{1}{\alpha}} \|p_1\|_\infty \int_{-\infty}^{+\infty} \mathrm{d}x\, p_\rho(x) \\ &= 2^{\frac{1}{\alpha}} \|p_1\|_\infty \int_{-\infty}^{+\infty} \mathrm{d}x\, \rho^{-\frac{1}{\alpha}} p_1\left(\frac{x}{\rho^{\frac{1}{\alpha}}}\right) \\ &= 2^{\frac{1}{\alpha}} \|p_1\|_\infty \int_{-\infty}^{+\infty} \mathrm{d}x\, p_1(x),\end{aligned}$$

where the last integral does not depend on ρ. □

7.3. Proof of Theorem 1.7. We may finally prove Theorem 1.7 using Proposition 7.1 and Proposition 7.2. The scheme of proof is exactly the same as that in Section 5, and combines the exponential tightness in $(\mathbb{D}[0,1], \mathrm{dist})$ with an LDP in the weak topology $(\mathbb{D}[0,1], \mathcal{W})$, similar to Proposition 5.1. Therefore, we will give a brief account, only pointing out the places where the formulas differ.

We can easily adapt the proof of Proposition 5.2 to get the following proposition. For a subdivision $\sigma = (t_1, \ldots, t_n)$ and $f \in \mathbb{D}[0,1]$, define $I^\sigma_{\mathbb{b},a}(f) = J^\sigma_{\mathbb{b},a}(f(t_1), \ldots, f(t_n))$.

Proposition 7.4. *A function* $f \in \mathbb{D}[0,1]$ *is in* $H^{(a)}_{\mathrm{br}}$ *if and only if*

$$M_{\mathbb{b},a}(f) \coloneqq \sup_{\sigma \in \mathfrak{S}} I^\sigma_{\mathbb{b},a}(f) < \infty.$$

In this case, we have

$$M_{\mathbb{b},a}(f) = I_{\mathbb{b},a}(f).$$

Then, analogs of Lemmas 5.3, 5.4, 5.6 and 5.6 hold true with $I_{\mathbb{b},a}$ in place of $I_{\mathbb{e}}$, with exactly the same proofs. This ends the proof of Theorem 1.7.

Theorem 1.8 can then be deduced from Theorem 1.7 along the same lines as in Section 6.2.

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