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BEHAVIOR NEAR THE EXTINCTION TIME IN SELF-SIMILAR FRAGMENTATIONS II: FINITE DISLOCATION MEASURES

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We study a Markovian model for the random fragmentation of an object. At each time, the state consists of a collection of blocks. Each block waits an exponential amount of time with parameter given by its size to some power α, independently of the other blocks. Every block then splits randomly into sub-blocks whose relative sizes are distributed according to the so-called dislocation measure. We focus here on the case where $\alpha < 0$. In this case, small blocks split intensively, and so the whole state is reduced to "dust" in a finite time, almost surely (we call this the extinction time). In this paper, we investigate how the fragmentation process behaves as it approaches its extinction time. In particular, we prove a scaling limit for the block sizes which, as a direct consequence, gives us an expression for an invariant measure for the fragmentation process. In an earlier paper [*Ann. Inst. Henri Poincaré Probab. Stat.* **46** (2010) 338–368], we considered the same problem for another family of fragmentation processes, the so-called *stable fragmentations*. The results here are similar, but we emphasize that the methods used to prove them are different. Our approach in the present paper is based on Markov renewal theory and involves a somewhat unusual "spine" decomposition for the fragmentation, which may be of independent interest.

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1. Introduction and main results. We consider a Markovian model for the random fragmentation of a collection of blocks of some material, where the manner in which the fragmentation occurs is controlled solely by the masses of the blocks. More specifically, suppose that the current state consists of blocks of masses $m_1, m_2, \dots$ which are such that (for definiteness) $\mathbf{m} = (m_1, m_2, \dots)$ belongs to the state-space

$$\mathcal{S} := \left\{\mathbf{s} = (s_1, s_2, \dots) : s_1 \geq s_2 \geq \cdots \geq 0, \sum_{i=1}^{\infty} s_i < \infty\right\},$$

which is endowed with the ℓ_1-distance

$$d(\mathbf{s}, \mathbf{s}') = \|\mathbf{s} - \mathbf{s}'\|_1 := \sum_{i \geq 1} |s_i - s_i'| \qquad \text{for } \mathbf{s}, \mathbf{s}' \in \mathcal{S}.$$

The transition mechanism depends on two parameters: a real number α and a probability measure ν on $\mathcal{S}_1 := \{\mathbf{s} \in \mathcal{S} : \|\mathbf{s}\|_1 = 1\}$, and can be described as follows. The different blocks evolve independently. For $i \geq 1$, block i splits after an exponential time of mean $m_i^{-\alpha}$ into sub-blocks of masses $m_i \mathbf{S}$, where the random sequence $\mathbf{S} = (S_1, S_2, \dots)$ is distributed according to ν. To avoid "phantom" fragmentation events, we will always assume that $\nu(\mathbf{1}) = 0$, where the state $\mathbf{1} = (1, 0, \dots)$ consists of a single block of mass 1. We will then write

$$F(t) = (F_1(t), F_2(t), \dots) \in \mathcal{S}$$

for the state of the fragmentation process at time t, and $\mathbb{P}_{\mathbf{s}}$ for the law of $(F(t), t \geq 0)$ started from a state $\mathbf{s} \in \mathcal{S}$. By default, we will start our processes from the state $\mathbf{1}$, and we will write $\mathbb{P}$ instead of $\mathbb{P}_{\mathbf{1}}$. Whenever we write $(F(t), t \geq 0)$ without making explicit reference to its law, we implicitly assume $F(0) = \mathbf{1}$. It is clear that (whatever its starting point) $(F(t), t \geq 0)$ is a transient Markov process with a single absorbing state at $\mathbf{0} = (0, 0, \dots)$.

This model described in the previous paragraph is a *self-similar fragmentation process*, as introduced by Filippov in [17] and Bertoin in [8, 9]. We refer to the second pair of papers for a rigorous construction based on Poisson point processes. This construction gives a version of the fragmentation which is càdlàg for the topology of pointwise convergence. Proposition 1.9 of [11] shows, in addition, that the sum of the masses of the blocks is a continuous function almost surely. Hence, there exists a càdlàg version of the fragmentation for the ℓ_1-distance, which is the version we will always consider in this paper. More precisely, $(F(t), t \geq 0)$ is a càdlàg strong Markov process which possesses the following self-similarity property:

> $(F(t), t \geq 0)$ has the same distribution under $\mathbb{P}_{m\mathbf{1}}$ as $(mF(m^{\alpha}t), t \geq 0)$ has under $\mathbb{P}_{\mathbf{1}}$

(we will revisit a stronger version of this property in Proposition 2.1 below). Consequently, the parameter α is known as the *index of self-similarity*. The probability measure ν is called the *dislocation measure*. In [8, 9], Bertoin constructs a more general class of processes in which ν is allowed to be an infinite (but σ-finite) measure satisfying a certain integrability condition; roughly speaking, these processes are allowed to jump at a dense set of times. He also allows dislocation measures which do not preserve the original mass, and the possibility of deterministic erosion of the block masses, but we will not consider any of these variants further here.

Henceforth, we will restrict our attention to the case $\alpha < 0$. In this case, smaller blocks split (on average) faster than larger ones. Despite the fact that each splitting event preserves the total mass present in the system, the fragmentation exhibits the striking phenomenon of *loss of mass*, whereby splitting events accumulate in such a way that blocks are reduced in finite time to blocks of mass 0 (known as *dust*). This is reflected by the fact that the total mass $M(t) = \sum_{i \geq 1} F_i(t)$ decreases as time passes [so that the dust has mass $1 - M(t)$]. Moreover, if we define the *extinction time*,

$$\zeta = \inf\{t \geq 0 : F(t) = \mathbf{0}\},$$

then $\zeta < \infty$ almost surely; see [10]. The manner in which mass is lost has been studied in detail by Bertoin [10] and Haas [19, 20]. Our focus here is different: we aim to understand the behavior of the fragmentation process close to its extinction time.

In most of the sequel, we will impose a further condition on the dislocation measure ν: we will require it to be *nongeometric*. That is, for any $r \in (0, 1)$, we have

$$\nu\big(s_i \in r^{\mathbb{N}} \cup \{0\}, \forall i \geq 1\big) < 1$$

(where $\mathbb{N} := \{1, 2, \ldots\}$). Fragmentations with geometric dislocation measures behave in a genuinely different way to their nongeometric counterparts; we will discuss this difference further below. For technical reasons, we will also need to impose the condition that $\int_{\mathcal{S}_1} s_1^{-1-\rho} \nu(\mathrm{d}\mathbf{s}) < \infty$ for some $\rho > 0$. This assumption is not very restrictive: for example, it is always satisfied for fragmentations where blocks split into at most N sub-blocks (N being fixed) since then $s_1 + \cdots + s_N = 1$, and so the largest mass s_1 is bounded below by $1/N$ ν-a.s.

We consider the usual Skorokhod topology on the space of càdlàg functions $f : [0, \infty) \to \mathcal{S}$. By convention, we will set $F(t) = \mathbf{1}$ for $t < 0$. Our principal result is then the following theorem.

THEOREM 1.1. *Suppose that ν is nongeometric and that $\int_{\mathcal{S}_1} s_1^{-1-\rho} \nu(\mathrm{d}\mathbf{s}) < \infty$ for some $\rho > 0$. Then there exists C_∞, a càdlàg $\mathcal{S}$-valued self-similar process independent of ζ, such that*

$$\big(\varepsilon^{1/\alpha}\big(F\big((\zeta - \varepsilon t)-\big), t \geq 0\big), \zeta\big) \xrightarrow{law} \big(\big(C_\infty(t), t \geq 0\big), \zeta\big).$$

Moreover, $C_\infty(0) = \mathbf{0}$ and $\mathbb{P}(C_{\infty,i}(1) > 0) > 0$ for all $i \geq 1$.

In particular, as $\varepsilon \to 0$,

$$\varepsilon^{1/\alpha} F(\zeta - \varepsilon) \stackrel{\text{law}}{\to} C_\infty(1).$$

Since $\mathcal{S}$ is endowed with the ℓ_1-distance, this entails that the rescaled total mass $\varepsilon^{1/\alpha} M(\zeta - \varepsilon)$ has a nontrivial limit in distribution as $\varepsilon \to 0$.

The self-similarity of the limit process C_∞ takes the form

$$\left(a^{1/\alpha} C_\infty(at), t \geq 0\right) \stackrel{\text{law}}{=} \left(C_\infty(t), t \geq 0\right)$$

for all $a > 0$. We will specify the distribution of C_∞ more precisely below once we have established the necessary notation; see Definition 5.3. This process models the evolution of masses that coalesce, with a regular immigration of infinitesimally small masses, as illustrated in Figure 3. Reversing time, this gives a fragmentation process that starts from one infinitely large mass. A connection with a biased randomized version of F is made in Proposition 5.4.

In a first paper [18], we proved a result of the same form as Theorem 1.1 for a different subclass of self-similar fragmentations with negative index, the stable fragmentations. The stable fragmentations, which were introduced in [24], are qualitatively rather different in that they all have infinite dislocation measures. They can be represented in terms of stable Lévy trees (see [14, 15] for a definition), and the methods used in our earlier paper rely crucially on the excursion theory available for these trees. The methods used in the present work are quite different and are dependent on the finiteness of the dislocation measure. We conjecture, nonetheless, that Theorem 1.1 is true for generic nongeometric self-similar fragmentations with negative index.

The proof of Theorem 1.1 proceeds in two main steps. We begin by studying the *last fragment process* F_*, where $F_*(t)$ is the mass of the unique fragment present at time t that dies exactly at time ζ. We construct this process in Section 2, where we also discuss some properties of ζ. We are, of course, interested in the asymptotic behavior of F_* close to time ζ. A significant difficulty is that the evolution of the process F_* is not Markovian. To overcome this difficulty, we introduce the discrete-time process

$$Z_n = F_*(T_n)^\alpha (\zeta - T_n), \qquad n \geq 0,$$

where T_n denotes the nth jump time of the last fragment process F_*. The quantity Z_n can be thought of as an updated notion of the extinction time seen in the natural timescale of the last fragment at its nth jump time. It turns out that $(Z_n)_{n\geq 0}$ is a Markov chain which converges to a stationary distribution as $n \to \infty$. This is proved in Section 3 using standard Foster–Lyapunov criteria. Moreover, the Markov chain $(Z_n)_{n\geq 0}$ drives a bigger Markov chain which additionally tracks the relative sizes of the fragments produced by the split at time T_n. From this bigger Markov chain we derive a Markov renewal process in Section 4, and we then use a version of renewal theory, developed for such processes in [3, 4, 6, 21, 22, 25, 28], to obtain the behavior of F_* near ζ.

The second step of the proof consists of decomposing the fragmentation process along its *spine* F_*, in order to get the behavior of the whole process near ζ. This is the purpose of Sections 5 and 6, where we prove a detailed version of Theorem 1.1. Roughly speaking, the limiting process C_∞ is built from a spine, the limit process of F_* near ζ, by grafting onto it independent fragmentation processes conditioned to die before specific times. A significant technical difficulty in this proof is to deal with blocks which separated from the spine "a long time in the past" and have not yet become extinct, and for this we will need to establish a tightness criterion.

Spine methods are standard in the study of branching processes. In earlier work on fragmentation processes (e.g., in [9, 10]), the so-called *tagged fragment* has proved to be a very useful tool. This is again a sort of spine but of a rather different nature to ours (in particular, the tagged fragment is a Markov process). However, the tagged fragment vanishes at a time which is strictly smaller than ζ and, as a consequence, cannot help us to understand the behavior of the fragmentation near its extinction time ζ. We believe that the spine decomposition we develop in the present paper, based on the last fragment process, should not be particular to the finite dislocation measure case. However, our results do not immediately extend to the case of infinite dislocation measures.

As a direct consequence of Theorem 1.1, we are able to construct an invariant measure for the fragmentation process (since F is transient, this is necessarily an infinite measure).

THEOREM 1.2. *Under the conditions of Theorem 1.1, consider the occupation measure λ of C_∞, which is defined on $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ by*

$$\lambda(A) = \int_0^\infty \mathbb{P}\big(C_\infty(t) \in A\big)\,\mathrm{d}t$$

for all $A \in \mathcal{B}(\mathcal{S})$. Then λ is a σ-finite invariant measure for the transition kernel of the fragmentation process F; that is, for all $u > 0$ and all $A \in \mathcal{B}(\mathcal{S})$,

$$\lambda(A) = \int_{\mathcal{S}} \mathbb{P}_{\mathbf{s}}\big(F(u) \in A\big)\lambda(\mathrm{d}\mathbf{s}).$$

We can interpret λ heuristically as the "law" of C_∞ "sampled at a uniform time in $[0, \infty)$." To the best of our knowledge, this is the first time that invariant measures have been considered for self-similar fragmentation processes. Theorem 1.2 is proved in Section 7, where we will see that it is an easy consequence of the convergence in distribution of $\varepsilon^{1/\alpha}F(\zeta - \varepsilon)$ to $C_\infty(1)$. In particular, this invariance result also holds for the stable fragmentations and, more generally, for any fragmentation process such that $\varepsilon^{1/\alpha}F(\zeta - \varepsilon)$ has a nontrivial limit in distribution [in $(\mathcal{S}, d)$] as $\varepsilon \to 0$.

We conclude the main part of the paper in Section 8 by investigating the case of geometric fragmentations. These fragmentations should not be viewed simply as a degenerate special case: they can be interpreted in terms of various other models, in particular discounted branching random walks (introduced by Athreya [5]) and randomly growing k-ary trees (studied by Barlow, Pemantle and Perkins [7]). Theorem 1.1 is not valid for geometric fragmentations. Indeed, we will see in Proposition 8.1 that the rescaled sequence $\varepsilon^{1/\alpha}F(\zeta-\varepsilon)$ does not converge in distribution in this situation. However, we do obtain convergence along suitable subsequences, which entails the existence of a continuum set of distinct invariant measures, indexed by $x \in [0,1)$.

Appendix containing various technical lemmas. It is split into two sections. The first concerns criteria for convergence in the space $(\mathcal{S},d)$ and in the Skorokhod topology on càdlàg processes taking values in $(\mathcal{S},d)$. The second section contains the proofs of fine results about stationary and biased versions of the Markov chain $(Z_n)_{n\geq 0}$ which are necessary for the proof of Theorem 1.1 but which are not of much intrinsic interest.

2. The last fragment process. In this section, we gather together some results on the extinction time ζ and prove the existence of the last fragment process. We refer to [9, 11] for background on fragmentation processes. In particular, we will use the following *strong fragmentation property* on several occasions.

PROPOSITION 2.1 (Bertoin [9]). *Let T be a stopping time with respect to the filtration generated by F. Write, for $t \geq T$,*

$$F(t) = \left(F^{(1,T)}(t), F^{(2,T)}(t), \ldots\right),$$

where, for each $i \geq 1$, $F^{(i,T)}$ is the process evolving in $\mathcal{S}$ which has $F^{(i,T)}(T) = F_i(T)$ and, for $t > T$, tracks the evolution of the fragments coming from the ith block of $F(T)$. Then

$$F^{(i,T)}(T+t) = F_i(T)G^{(i)}\left(tF_i(T)^\alpha\right) \qquad \forall i \geq 1,$$

where the processes $G^{(i)}$ are independent and have the same distribution as F. They are also independent of T and $F(T)$.

2.1. *The extinction time.* We begin by establishing some properties of the extinction time ζ, which will be useful to us in the sequel. We will make use of Proposition 14 of [19], which states that

$$\mathbb{E}\left[\exp(a\zeta)\right] < \infty \qquad \text{for all positive } a \text{ sufficiently small.}$$

LEMMA 2.2. *The distribution of ζ is absolutely continuous with respect to Lebesgue measure on $(0,\infty)$, and there exists a continuous and strictly positive version of its density, which we denote f_ζ. Furthermore*:

(i) $f_\zeta(x) \le 1$ *for all* $x \in (0, \infty)$;
(ii) $f_\zeta(x) = o(\exp(-cx))$ *as* $x \to \infty$, *for some* $c > 0$;
(iii) $f_\zeta(x) = o(1)$ *as* $x \to 0$ *and*, *moreover*, *for each* $\beta > \alpha$ *such that* $\int_{\mathcal{S}_1} s_1^{-\beta} \nu(\mathrm{d}\mathbf{s}) < \infty$, $\mathbb{F}_\zeta(x) := \mathbb{P}(\zeta \le x) = \mathcal{O}(x^{1-\beta/\alpha})$.

PROOF. Let $T_1 := \inf\{t \ge 0 : F(t) \ne (1, 0, \ldots)\}$ be the first splitting time of F. Then T_1 is exponentially distributed with parameter 1, and $F(T_1)$ is distributed according to ν. Moreover, since T_1 is a stopping time with respect to the filtration generated by F, we get from Proposition 2.1 that

$$\zeta = T_1 + \sup_{i \ge 1}\{F_i(T_1)^{-\alpha} \zeta^{(i)}\},$$

where T_1, $F(T_1)$ and $(\zeta^{(i)}, i \ge 1)$ are independent, and $(\zeta^{(i)}, i \ge 1)$ is a collection of independent random variables with the same distribution as ζ. Since T_1 has an exponential distribution, this implies that ζ possesses a density, say f_ζ, which in turn implies that $\xi := \sup_{i \ge 1}\{F_i(T_1)^{-\alpha} \zeta^{(i)}\}$ possesses a density, given by

$$f_\xi(y) = \int_{\mathcal{S}_1} \sum_{i : s_i > 0} f_\zeta(s_i^\alpha y) s_i^\alpha \prod_{j \ne i} \mathbb{F}_\zeta(s_j^\alpha y) \nu(\mathrm{d}\mathbf{s}), \tag{2.1}$$

where $\mathbb{F}_\zeta$ is the cumulative distribution function corresponding to f_ζ. Note that if $\mathbb{F}_\zeta(s_j^\alpha y) > 0$, for all $j \ne i$, then necessarily $\prod_{j \ne i} \mathbb{F}_\zeta(s_j^\alpha y) > 0$. This is obvious when the set $\{j : s_j > 0\}$ is finite. When it is infinite, taking logarithms and using the fact that

$$\log(\mathbb{F}_\zeta(s_j^\alpha y)) \sim -\mathbb{P}(\zeta > s_j^\alpha y)$$

as $j \to \infty$, we see that the above product is null if and only if the sum $\sum_{j \ne i} \mathbb{P}(\zeta > s_j^\alpha y)$ is infinite. But this never happens when $\sum_{j \ne i} s_j \le 1$, since

$$\mathbb{P}(\zeta > s_j^\alpha y) \le \mathbb{E}[\zeta^{-1/\alpha}] s_j y^{-1/\alpha}$$

and ζ has exponential moments.

Now, choose f_ζ so that

$$f_\zeta(x) = \exp(-x) \int_0^x \exp(y) f_\xi(y)\, \mathrm{d}y \qquad \text{for all } x > 0. \tag{2.2}$$

Then, f_ζ is continuous and $f_\zeta(x) \le \mathbb{P}(\xi \le x) \to 0$ as $x \to 0$. In particular, we get (i) and the first assertion of (iii). Note also that if $f_\zeta(x) = 0$ for some $x > 0$, then f_ξ equals 0 a.e. on $[0, x]$. Hence, using (2.1) and the remark following it, we see that f_ζ equals 0 on $[0, x']$ for some $x' > x$. This easily entails that f_ζ equals 0 on $\mathbb{R}_+$, which is impossible. Hence, $f_\zeta(x) > 0$ for all $x > 0$.

Next, to prove (ii), note that for all $0 \le a \le 1$,

$$\exp(ax) f_\zeta(x) \le \int_0^x \exp(ay) f_\xi(y)\, \mathrm{d}y \le \mathbb{E}[\exp(a\xi)] \le \mathbb{E}[\exp(a\zeta)]$$

since $\zeta = T_1 + \xi$. The last expectation is finite for all positive a sufficiently small, and so $\exp(cx) f_\zeta(x) \to 0$ as $x \to \infty$ for all $c < a$.

It remains to prove the second assertion of (iii). Let $\Gamma := \{\gamma \geq 0 \text{ s.t. } \exists C_\gamma < \infty : \mathbb{F}_\zeta(x) \leq C_\gamma x^\gamma, \forall x \geq 0\}$. Since $\mathbb{F}_\zeta$ is smaller than 1, Γ is an interval whose left endpoint is 0. Moreover, since $f_\zeta(x) \leq 1$ for all $x > 0$, we have $[0, 1] \subseteq \Gamma$. In particular, we have checked the assertion for $\beta \leq 0$. Now consider $\gamma \in \Gamma$. We have

$$\begin{aligned} f_\zeta(x) \leq \mathbb{P}(\xi \leq x) &\leq \int_{\mathcal{S}_1} \mathbb{F}_\zeta(s_1^\alpha x)\nu(\mathrm{d}\mathbf{s}) \qquad \bigl[\text{since } \xi \geq F_1^{-\alpha}(T_1)\zeta^{(1)}\bigr] \\ &\leq C_\gamma x^\gamma \int_{\mathcal{S}_1} s_1^{\alpha\gamma}\nu(\mathrm{d}\mathbf{s}), \end{aligned}$$

which implies that $\gamma + 1$ is in Γ provided that $\int_{\mathcal{S}_1} s_1^{\alpha\gamma}\nu(\mathrm{d}\mathbf{s}) < \infty$. The second assertion of (iii) is then straightforward. $\square$

2.2. *Building the last fragment*. For all $t \geq 0$ and all $i \in \mathbb{N}$, denote by $F^{(i,t)}$ the fragmentation process starting from $(F_i(t), 0, \dots)$ which tracks the evolution of the masses emanating from $F_i(t)$. Let $Z^{(i,t)} := \inf\{s \geq 0 : F^{(i,t)}(s) = \mathbf{0}\}$ be the first time at which this process is reduced to dust.

LEMMA 2.3. *Almost surely, for all* $0 \leq t < \zeta$, *there exists a unique index* $i(t)$ *such that* $Z^{(i(t),t)} = \sup_{j\in\mathbb{N}} Z^{(j,t)} = \zeta - t$.

PROOF. Fix $t > 0$. By Proposition 2.1, $Z^{(i,t)} = F_i(t)^{-\alpha}\zeta^{(i,t)}$, where $(\zeta^{(i,t)}, i \geq 1)$ is a collection of i.i.d. random variables, with the same distribution as ζ, independent of $F(t)$. Hence

$$\mathbb{E}\biggl[\sum_{i\geq 1}\bigl(Z^{(i,t)}\bigr)^{-1/\alpha}\biggr] = \mathbb{E}\biggl[\sum_{i\geq 1} F_i(t)\bigl(\zeta^{(i,t)}\bigr)^{-1/\alpha}\biggr] \leq \mathbb{E}\bigl[\zeta^{-1/\alpha}\bigr] < \infty.$$

In particular, the sum $\sum_{i\geq 1}(Z^{(i,t)})^{-1/\alpha}$ is almost surely finite, which implies that $Z^{(i,t)} \to 0$ a.s. as $i \to \infty$. Hence, the supremum $\sup_{j\in\mathbb{N}} Z^{(j,t)}$ is attained for some $i \in \mathbb{N}$. Conditional on $t < \zeta$, this index i is necessarily a.s. unique, since

$$\begin{aligned} &\mathbb{P}\bigl(\exists k, j : F_k^{-\alpha}(t)\zeta^{(k,t)} = F_j^{-\alpha}(t)\zeta^{(j,t)}, F_k(t) \neq 0, F_j(t) \neq 0\bigr) = 0 \\ &\quad \Leftrightarrow \quad \forall k, j, \qquad \mathbb{P}\bigl(F_k^{-\alpha}(t)\zeta^{(k,t)} = F_j^{-\alpha}(t)\zeta^{(j,t)}, F_k(t) \neq 0, F_j(t) \neq 0\bigr) = 0, \end{aligned}$$

which is clearly satisfied, since $\zeta^{(k,t)}$ and $\zeta^{(j,t)}$ are absolutely continuous (by Lemma 2.2) and independent of $F(t)$. Hence, conditionally on $t < \zeta$, there almost surely exists a unique index $i(t)$ such that $Z^{(i(t),t)} = \sup_{j\in\mathbb{N}} Z^{(j,t)}$. To conclude, note that when $i(t)$ exists and is unique, then, for all $s \leq t$, $i(s)$ is automatically defined as the index of the ancestor at time s of $F_i(t)$. Therefore, with probability one, the indices $i(t)$ are well defined for all $0 \leq t < \zeta$. $\square$

Let $(\Omega, \mathcal{F})$ denote the measurable space on which we work.

DEFINITION 2.4. Let $E := \{\omega \in \Omega : \forall t < \zeta(\omega), \exists!\, i(t)(\omega) \text{ s.t. } Z^{(i(t)(\omega),t)}(\omega) = \sup_{j\in\mathbb{N}} Z^{(j,t)}(\omega)\}$, and define for all $t \geq 0$,

$$F_*(t)(\omega) = \begin{cases} F_{i(t)(\omega)}(t)(\omega), & \text{if } \omega \in E \text{ and } t < \zeta(\omega), \\ 0, & \text{otherwise.} \end{cases}$$

The process F_* is called the *last fragment process*. It is nonincreasing, càdlàg and $\zeta = \inf\{t \geq 0 : F_*(t) = 0\}$ a.s. (by Lemma 2.3).

REMARK 2.5. Almost surely, for all $t \geq 0$, $F_*(t) > 0$ implies that the number of jumps of F_* in $[0, t]$ is finite. This is obvious if $\nu(s_1 \leq a) = 1$ for some $a < 1$. Otherwise, it can be easily seen via the Poissonian construction of the fragmentation in [8, 9].

In the sequel, we will use the last fragment as a "spine" for the fragmentation process: when blocks separate from the last fragment, they evolve essentially as independent fragmentation processes which are conditioned to die before the last fragment. We emphasize that it is not measurable with respect to the natural filtration of the fragmentation process.

3. Asymptotics along a subsequence. We now derive a convergent Markov chain from the last fragment process F_*, which demonstrates that F_* restricted to its jump times behaves as expected near ζ. We prove the Markov property of the chain in Section 3.1 and show that it converges exponentially fast to its stationary distribution in Section 3.2. In Section 3.3, we consider an eternal stationary version of the Markov chain. We also introduce a biased version of this eternal chain, which is an essential building-block for the process C_∞.

3.1. *A Markov chain.* Let $T_1 < T_2 < \cdots < T_n < \cdots$ be the increasing sequence of times at which F_* splits, that is, $T_1 = \inf\{t \geq 0 : F_*(t) < 1\}$ and, for $n \geq 2$,

$$T_n = \inf\{t \geq T_{n-1} : F_*(t) < F_*(T_{n-1})\}.$$

For convenience, set $T_0 = 0$. We note that only T_0 and T_1 are stopping times with respect to the natural filtration of the fragmentation process. From Remark 2.5 and since $\zeta = \inf\{t \geq 0 : F_*(t) = 0\}$, we clearly have that

$$T_n \to \zeta \qquad \text{a.s. as } n \to \infty.$$

Define, for $n \geq 0$,

$$Z_n := \bigl(F_*(T_n)\bigr)^\alpha (\zeta - T_n), \tag{3.1}$$

and note that $Z_n^{1/\alpha}$ is the value of the process $\varepsilon^{1/\alpha} F_*(\zeta - \varepsilon)$ at $\varepsilon = \zeta - T_n$. Intuitively, Z_n is a version of the extinction time updated according to what we know about the last fragment at time T_n.

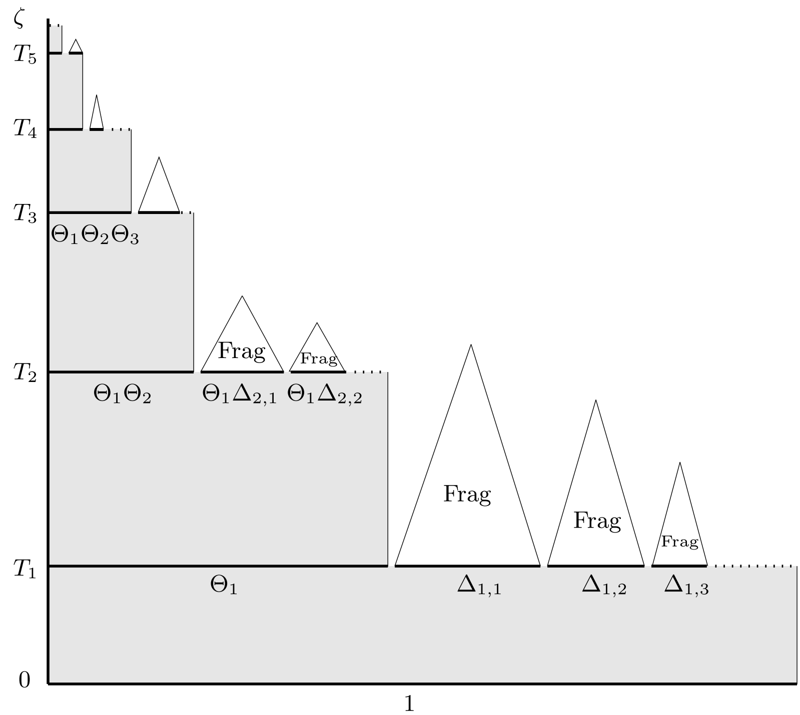

FIG. 1. *The spine decomposition. Time runs up the page. The size of the last fragment, F_*, which is constant on the intervals $[T_i, T_{i+1}-)$ is shaded. The blocks which split off from F_* start their own fragmentation processes, each conditioned to become extinct before ζ.*

Note also that $Z_0 = \zeta$, and set $\Theta_0 = 1$, $\mathbf{\Delta}_0 = (0, 0, \ldots)$. For $n \geq 1$, let $\Theta_n = F_*(T_n)/F_*(T_{n-1})$, and let $\mathbf{\Delta}_n = (\Delta_{n,1}, \Delta_{n,2}, \ldots)$ be the relative sizes of the other sub-blocks resulting from the split of F_* which occurs at time T_n, ordered so that $\Delta_{n,1} \geq \Delta_{n,2} \geq \cdots \geq 0$. Then

$$\big(F_*(T_{n-1})\Delta_{n,1}, F_*(T_{n-1})\Delta_{n,2}, \ldots\big)$$

are the sizes of the blocks which *split off* from the last fragment at time T_n. As a consequence of the fact that ν is conservative, we have $\Theta_n + \sum_{i=1}^{\infty} \Delta_{n,i} = 1$ almost surely. See Figure 1 for an illustration.

PROPOSITION 3.1. (a) *The process $(Z_n, \Theta_n, \mathbf{\Delta}_n)_{n\geq 0}$ is a time-homogeneous Markov chain. Moreover, conditional on $\sigma(Z_m, \Theta_m, \mathbf{\Delta}_m, m \leq n)$, the law of $(Z_{n+1}, \Theta_{n+1}, \mathbf{\Delta}_{n+1})$ depends only on the value of Z_n.*

(b) *The transition densities $P(x, \mathrm{d}y)$, $x > 0$, of $(Z_n)_{n\geq 0}$ are given by*

$$\begin{aligned} &P(x, \mathrm{d}y) \\ &\quad = \frac{e^{-x}}{f_\zeta(x)} f_\zeta(y) \left(\int_{\mathcal{S}_1} \sum_{i\,:\, s_i>0} e^{s_i^{-\alpha} y} \prod_{j\neq i} \mathbb{F}_\zeta\big(s_j^\alpha s_i^{-\alpha} y\big) \mathbb{1}_{\{0<y<s_i^\alpha x\}} \nu(\mathrm{d}\mathbf{s}) \right) \mathrm{d}y, \end{aligned} \tag{3.2}$$

where $\mathbb{F}_\zeta$ *is the cumulative distribution function of* ζ.

We refer to $(Z_n)_{n\geq 0}$ as the *driving chain* of $(Z_n, \Theta_n, \mathbf{\Delta}_n)_{n\geq 0}$.

REMARK 3.2. The density in (3.2) is strictly positive for all $x, y > 0$. This is a consequence of the positivity of f_ζ on $(0, \infty)$ (Lemma 2.2) and of the fact that $\prod_{j\neq i} \mathbb{F}_\zeta(s_j^\alpha s_i^{-\alpha} y) > 0$ when $s_i^{-\alpha} y > 0$ (as explained in the proof of Lemma 2.2).

Let $Y_0 := \zeta^{1/\alpha}$, and for $n \geq 1$, let

$$Y_n := \left(\frac{\zeta - T_n}{\zeta - T_{n-1}}\right)^{1/\alpha} = \frac{Z_n^{1/\alpha}}{Z_{n-1}^{1/\alpha}\Theta_n}.$$

Later on it will turn out to be convenient to work with Y_n, essentially because the times to extinction $\zeta - T_n$ can then be expressed in the multiplicative form $\zeta \prod_{i=1}^n Y_i^\alpha$. To this end, we need the following simple corollary of Proposition 3.1.

COROLLARY 3.3. *The process* $(Z_n, Y_n, \mathbf{\Delta}_n)_{n\geq 0}$ *is a time-homogeneous Markov chain with driving chain* $(Z_n, n \geq 0)$.

The rest of this section is devoted to the proof of Proposition 3.1. Recall from Proposition 2.1 that for $t \geq 0$, $F(T_1 + t)$ is the decreasing rearrangement of the terms of the sequences

$$F_1(T_1)G^{(1)}(tF_1(T_1)^\alpha), F_2(T_1)G^{(2)}(tF_2(T_1)^\alpha), \dots,$$

where the processes $G^{(i)}$ are independent fragmentations, all having the same distribution as F. They are also independent of T_1 and $F(T_1)$. Now let $\zeta^{(i)} = \inf\{t \geq 0 : G^{(i)}(t) = \mathbf{0}\}$, so that

$$\zeta = T_1 + \sup_{i\geq 1}\{F_i(T_1)^{-\alpha}\zeta^{(i)}\}. \tag{3.3}$$

By Lemma 2.3, this supremum is a maximum. Let $I := \operatorname{argmax}_{i\geq 1}\{F_i(T_1)^{-\alpha}\zeta^{(i)}\}$, and note that $F_*(T_1) = F_I(T_1)$ and $Z_1 = \zeta^{(I)}$. Let

$$H^{(i,j)} = G^{(j+\mathbb{1}_{\{j\geq i\}})} = \begin{cases} G^{(j)}, & \text{if } j < i, \\ G^{(j+1)}, & \text{if } j \geq i. \end{cases} \tag{3.4}$$

Finally, for $x > 0$ and suitable test functions ϕ and ψ, we write

$$A(\phi, x) = \mathbb{E}[\phi(F) | \zeta = x] \quad \text{and} \quad B(\psi, x) = \mathbb{E}[\psi(F) | \zeta < x].$$

REMARK 3.4. The function $A(\phi, \cdot)$ is well defined only up to a Borel set of Lebesgue measure 0, and is Borel-measurable. However, when applied to a positive and absolutely continuous random variable, say X, this is enough to define the random variable $A(\phi, X)$ properly up to a set of probability 0. This remark is also valid for any forthcoming functions defined as expectations conditional on $\zeta = x$.

The following lemma is the key result needed to prove the Markov property of $(Z_n, \Theta_n, \mathbf{\Delta}_n)_{n\geq 0}$.

LEMMA 3.5. *For all suitable test functions ϕ and ψ_j, $j \geq 1$,*

$$\mathbb{E}\left[\phi(G^{(I)}) \prod_{j=1}^{\infty} \psi_j(H^{(I,j)}) \middle| \zeta, \zeta^{(I)}, F_I(T_1), (F_k(T_1), k \neq I)\right]$$
$$= A(\phi, \zeta^{(I)}) \prod_{j=1}^{\infty} B(\psi_j, F_I^{-\alpha}(T_1) F^{\alpha}_{j+\mathbb{1}_{\{j\geq I\}}}(T_1)\zeta^{(I)}).$$

In particular, conditional on $\zeta^{(I)}$, $G^{(I)}$ is independent of ζ, $F(T_1)$ and $F_I(T_1)$, and is distributed as a fragmentation process conditioned to die at time $\zeta^{(I)}$.

PROOF OF LEMMA 3.5. We will, in fact, prove that

$$\mathbb{E}\left[\phi(G^{(I)}) \prod_{j=1}^{\infty} \psi_j(H^{(I,j)}) \middle| \zeta, \zeta^{(I)}, F(T_1), I\right]$$
$$= A(\phi, \zeta^{(I)}) \prod_{j=1}^{\infty} B(\psi_j, F_I^{-\alpha}(T_1) F^{\alpha}_{j+\mathbb{1}_{\{j\geq I\}}}(T_1)\zeta^{(I)}),$$

which implies the statement of the lemma. Let χ be another test function. For $i \neq j$, set $S_{i,j} = \{F_i^{-\alpha}(T_1)\zeta^{(i)} \geq F_j^{-\alpha}(T_1)^{-\alpha}\zeta^{(j)}\}$ and note that $\{I = i\} = \bigcap_{j\geq 1} S_{i,j+\mathbb{1}_{\{j\geq i\}}}$. We have

$$\mathbb{E}\left[\phi(G^{(I)}) \prod_{j=1}^{\infty} \psi_j(H^{(I,j)}) \chi(\zeta, \zeta^{(I)}, F(T_1)) \mathbb{1}_{\{I=i\}}\right]$$
$$= \mathbb{E}\left[\phi(G^{(i)}) \prod_{j=1}^{\infty} \psi_j(H^{(i,j)}) \chi(T_1 + F_i^{-\alpha}(T_1)\zeta^{(i)}, \zeta^{(i)}, F(T_1)) \mathbb{1}_{\{I=i\}}\right]$$
$$= \mathbb{E}\Bigg[\chi(T_1 + F_i^{-\alpha}(T_1)\zeta^{(i)}, \zeta^{(i)}, F(T_1))$$
$$\times \mathbb{E}\left[\phi(G^{(i)}) \prod_{j=1}^{\infty} \psi_j(G^{(j+\mathbb{1}_{\{j\geq i\}})}) \mathbb{1}_{S_{i,j+\mathbb{1}_{\{j\geq i\}}}} \middle| T_1, F(T_1), \zeta^{(i)}\right]\Bigg].$$

Since $G^{(j)}$, $j \geq 1$ are independent fragmentations, independent of T_1 and $F(T_1)$, we see that

$$\begin{aligned}
&\mathbb{E}\left[\phi(G^{(i)}) \prod_{j=1}^{\infty} \psi_j(G^{(j+\mathbb{1}_{\{j\geq i\}})}) \mathbb{1}_{S_{i,j+\mathbb{1}_{\{j\geq i\}}}} \middle| T_1, F(T_1), \zeta^{(i)}\right] \\
&\quad = \mathbb{E}[\phi(G^{(i)})|\zeta^{(i)}] \prod_{j=1}^{\infty} \mathbb{E}[\psi_j(G^{(j+\mathbb{1}_{\{j\geq i\}})}) \mathbb{1}_{S_{i,j+\mathbb{1}_{\{j\geq i\}}}} | F(T_1), \zeta^{(i)}] \\
&\quad = A(\phi, \zeta^{(i)}) \prod_{j=1}^{\infty} B(\psi_j, F_i^{-\alpha}(T_1) F_{j+\mathbb{1}_{\{j\geq i\}}}^{\alpha}(T_1) \zeta^{(i)}) \\
&\qquad\qquad \times \mathbb{P}(\zeta^{(j+\mathbb{1}_{\{j\geq i\}})} < F_i^{-\alpha}(T_1) F_{j+\mathbb{1}_{\{j\geq i\}}}^{\alpha}(T_1) \zeta^{(i)} | F(T_1), \zeta^{(i)}) \\
&\quad = A(\phi, \zeta^{(i)}) \left[\prod_{j=1}^{\infty} B(\psi_j, F_i^{-\alpha}(T_1) F_{j+\mathbb{1}_{\{j\geq i\}}}^{\alpha}(T_1) \zeta^{(i)})\right] \mathbb{P}(I = i | F(T_1), \zeta^{(i)}).
\end{aligned}$$

Then

$$\begin{aligned}
&\mathbb{E}\left[\phi(G^{(I)}) \prod_{j=1}^{\infty} \psi_j(H^{(I,j)}) \chi(\zeta, \zeta^{(I)}, F(T_1)) \mathbb{1}_{\{I=i\}}\right] \\
&\quad = \mathbb{E}\Bigg[A(\phi, \zeta^{(I)}) \prod_{j=1}^{\infty} B(\psi_j, F_I^{-\alpha}(T_1) F_{j+\mathbb{1}_{\{j\geq I\}}}^{\alpha}(T_1) \zeta^{(I)}) \\
&\qquad\qquad \times \chi(\zeta, \zeta^{(I)}, F(T_1)) \mathbb{1}_{\{I=i\}}\Bigg],
\end{aligned}$$

and the result follows. $\square$

PROOF OF PROPOSITION 3.1. (a) We start by proving that $(Z, \Theta, \mathbf{\Delta})$ is a time-homogeneous Markov chain with driving chain Z. To see this, we will show that for all suitable test functions f, g_i and all $n \geq 1$,

$$(\mathfrak{R}_n) \quad \begin{aligned}
&\mathbb{E}\left[f(Z_n, \Theta_n, \mathbf{\Delta}_n) \prod_{i=0}^{n-1} g_i(Z_i, \Theta_i, \mathbf{\Delta}_i)\right] \\
&\quad = \mathbb{E}\left[\mathsf{F}_f(Z_{n-1}) \prod_{i=0}^{n-1} g_i(Z_i, \Theta_i, \mathbf{\Delta}_i)\right],
\end{aligned}$$

where $\mathsf{F}_f(x) = \mathbb{E}[f(Z_1, \Theta_1, \mathbf{\Delta}_1)|Z_0 = x]$. Note that $\mathsf{F}_f(x)$ is well defined for Lebesgue a.e. $x > 0$, since $Z_0 = \zeta$ is absolutely continuous. We will prove by induction on n that $(\mathfrak{R}_n)$ is valid and that Z_n is absolutely continuous, so that

$\mathsf{F}_f(Z_{n-1})$ is almost surely well defined. In fact, once ($\mathfrak{R}_n$) is proved, the absolute continuity of Z_n is a direct consequence of the absolute continuity of Z_{n-1} and of ($\mathfrak{R}_n$), taking test functions f of the form $f = \mathbb{1}_A$ for Borel sets A with Lebesgue measure 0. So it is enough to focus in the following on the proof of ($\mathfrak{R}_n$) for $n \geq 1$.

($\mathfrak{R}_1$) is an immediate consequence of the fact that Θ_0 and $\mathbf{\Delta}_0$ are deterministic. Now assume that ($\mathfrak{R}_n$) holds, and recall that the last fragment process F_* can be written as

$$F_*(T_1 + t) = F_I(T_1)G^{(I)}\big(tF_I^{\alpha}(T_1)\big), \qquad t \geq 0. \tag{3.5}$$

As for the standard fragmentation process, the last fragment process of $G^{(I)}$ is well-defined since $G^{(I)}$ is a randomized version of the fragmentation. We denote it by $(G_*^{(I)}(t), t \geq 0)$. Then for $k \geq 1$, let $T_k^{(I)}$ be the kth time at which $G_*^{(I)}$ splits, let

$$\Theta_k^{(I)} := G_*^{(I)}\big(T_k^{(I)}\big)/G_*^{(I)}\big(T_{k-1}^{(I)}\big)$$

and let $\mathbf{\Delta}_k^{(I)}$ be the relative sizes of the other sub-blocks resulting from the split of $G_*^{(I)}$ at time $T_k^{(I)}$. From (3.5), we get that $T_{k+1} = T_1 + F_I^{-\alpha}(T_1)T_k^{(I)}$, $\Theta_{k+1} = \Theta_k^{(I)}$, $\mathbf{\Delta}_{k+1} = \mathbf{\Delta}_k^{(I)}$ and

$$Z_{k+1} = \big(G^{(I)}(T_k^{(I)})\big)^{\alpha}\big(Z_1 - T_k^{(I)}\big) := Z_k^{(I)}.$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\Bigg[f(Z_{n+1}, \Theta_{n+1}, \mathbf{\Delta}_{n+1}) \prod_{i=0}^{n} g_i(Z_i, \Theta_i, \mathbf{\Delta}_i)\Bigg] \\
&\quad = \mathbb{E}\Bigg[f\big(Z_n^{(I)}, \Theta_n^{(I)}, \mathbf{\Delta}_n^{(I)}\big) g_0(Z_0, \Theta_0, \mathbf{\Delta}_0) g_1(Z_1, \Theta_1, \mathbf{\Delta}_1) \\
&\qquad\qquad \times \prod_{i=1}^{n-1} g_{i+1}\big(Z_i^{(I)}, \Theta_i^{(I)}, \mathbf{\Delta}_i^{(I)}\big)\Bigg] \\
&\quad = \mathbb{E}\Bigg[g_0(Z_0, \Theta_0, \mathbf{\Delta}_0) g(Z_1, \Theta_1, \mathbf{\Delta}_1) \\
&\qquad \times \mathbb{E}\Bigg[f\big(Z_n^{(I)}, \Theta_n^{(I)}, \mathbf{\Delta}_n^{(I)}\big) \\
&\qquad\qquad \times \prod_{i=1}^{n-1} g_{i+1}\big(Z_i^{(I)}, \Theta_i^{(I)}, \mathbf{\Delta}_i^{(I)}\big) \Big| Z_0, Z_1, F(T_1), F_I(T_1)\Bigg]\Bigg].
\end{aligned}$$

Similarly,

$$\mathbb{E}\left[\mathsf{F}_f(Z_n)\prod_{i=0}^{n} g_i(Z_i,\Theta_i,\mathbf{\Delta}_i)\right]$$
$$= \mathbb{E}\Bigg[g_0(Z_0,\Theta_0,\mathbf{\Delta}_0)g(Z_1,\Theta_1,\mathbf{\Delta}_1)$$
$$\times \mathbb{E}\left[\mathsf{F}_f\big(Z_{n-1}^{(I)}\big)\prod_{i=1}^{n-1} g_{i+1}\big(Z_i^{(I)},\Theta_i^{(I)},\mathbf{\Delta}_i^{(I)}\big)\middle| Z_0, Z_1, F(T_1), F_I(T_1)\right]\Bigg].$$

Then by Lemma 3.5 (recall that $Z_0=\zeta$, $Z_1=\zeta^{(I)}$) applied to the functions $\psi_j\equiv 1, \forall j\in\mathbb{N}$ and $\phi(G^{(I)}) = f(Z_n^{(I)},\Theta_n^{(I)},\mathbf{\Delta}_n^{(I)})\prod_{i=1}^{n-1} g_{i+1}(Z_i^{(I)},\Theta_i^{(I)},\mathbf{\Delta}_i^{(I)})$,

$$\mathbb{E}\left[f\big(Z_n^{(I)},\Theta_n^{(I)},\mathbf{\Delta}_n^{(I)}\big)\prod_{i=1}^{n-1} g_{i+1}\big(Z_i^{(I)},\Theta_i^{(I)},\mathbf{\Delta}_i^{(I)}\big)\middle| Z_0, Z_1, F(T_1), F_I(T_1)\right]$$
$$= u(Z_1),$$

where

$$u(x) = \mathbb{E}\left[f(Z_n,\Theta_n,\mathbf{\Delta}_n)\prod_{i=1}^{n-1} g_{i+1}(Z_i,\Theta_i,\mathbf{\Delta}_i)\middle|\zeta = x\right],$$

and similarly

$$\mathbb{E}\left[\mathsf{F}_f\big(Z_{n-1}^{(I)}\big)\prod_{i=1}^{n-1} g_{i+1}\big(Z_i^{(I)},\Theta_i^{(I)},\mathbf{\Delta}_i^{(I)}\big)\middle| Z_0, Z_1, F(T_1), F_I(T_1)\right] = v(Z_1),$$

where

$$v(x) = \mathbb{E}\left[\mathsf{F}_f(Z_{n-1})\prod_{i=1}^{n-1} g_{i+1}(Z_i,\Theta_i,\mathbf{\Delta}_i)\middle|\zeta = x\right].$$

To get $(\mathfrak{R}_{n+1})$, it remains to prove that $u(x)=v(x)$ for Lebesgue-a.e. $x>0$. For this we use the induction hypothesis $(\mathfrak{R}_n)$ which implies that the random variables

$$f(Z_n,\Theta_n,\mathbf{\Delta}_n)\prod_{i=1}^{n-1} g_{i+1}(Z_i,\Theta_i,\mathbf{\Delta}_i) \quad \text{and} \quad \mathsf{F}_f(Z_{n-1})\prod_{i=1}^{n-1} g_{i+1}(Z_i,\Theta_i,\mathbf{\Delta}_i)$$

have the same expectation conditional on ζ since

$$\mathbb{E}\left[h(\zeta)f(Z_n,\Theta_n,\mathbf{\Delta}_n)\prod_{i=1}^{n-1} g_{i+1}(Z_i,\Theta_i,\mathbf{\Delta}_i)\right]$$
$$= \mathbb{E}\left[h(\zeta)\mathsf{F}_f(Z_{n-1})\prod_{i=1}^{n-1} g_{i+1}(Z_i,\Theta_i,\mathbf{\Delta}_i)\right]$$

for all bounded measurable functions h. The result follows by induction.

(b) It remains to prove that the transition densities of the chain $(Z_n)_{n\geq 0}$ are given by identity (3.2). To get this, we compute the joint density of (Z_0, Z_1). The first step is to use the independence of T_1, $F(T_1)$ and $(\zeta^{(j)}, j \geq 1)$ [defined in (3.3)] and the fact that $F(T_1)$ is distributed according to ν, to get that, for any test function χ,

$$
\begin{aligned}
&\mathbb{E}[\chi(F_I(T_1), \zeta, T_1)] \\
&\quad = \sum_{i=1}^{\infty} \mathbb{E}[\chi(F_i(T_1), T_1 + F_i(T_1)^{-\alpha}\zeta^{(i)}, T_1)\mathbb{1}_{\{I=i\}}] \\
&\quad = \int_{\mathcal{S}_1} \sum_{i\,:\,s_i>0} \int_0^{\infty} \mathbb{E}[\chi(s_i, t + s_i^{-\alpha}\zeta^{(i)}, t)\mathbb{1}_{\{s_i^{-\alpha}\zeta^{(i)} \geq \max_{j\neq i} s_j^{-\alpha}\zeta^{(j)}\}}]e^{-t}\,\mathrm{d}t\,\nu(\mathrm{d}\mathbf{s}) \\
&\quad = \int_{\mathcal{S}_1} \sum_{i\,:\,s_i>0} \int_0^{\infty}\int_0^{\infty} \chi(s_i, t + s_i^{-\alpha}z, t) f_\zeta(z) \prod_{j\neq i} \mathbb{F}_\zeta(s_j^\alpha s_i^{-\alpha} z) e^{-t}\,\mathrm{d}t\,\mathrm{d}z\,\nu(\mathrm{d}\mathbf{s}).
\end{aligned}
$$

In the inner integral, let $x = t + s_i^{-\alpha}z$ [then $z = s_i^\alpha(x-t)$] to get that this last is equal to

$$
\int_{\mathcal{S}_1} \sum_{i\,:\,s_i>0} \int_0^{\infty}\int_0^{x} s_i^\alpha \chi(s_i, x, t) f_\zeta(s_i^\alpha(x-t)) \prod_{j\neq i} \mathbb{F}_\zeta(s_j^\alpha(x-t))e^{-t}\,\mathrm{d}t\,\mathrm{d}x\,\nu(\mathrm{d}\mathbf{s}).
$$

Taking $\chi(F_I(T_1), \zeta, T_1) = \phi(\zeta, F_I(T_1)^\alpha(\zeta - T_1))$, we obtain

$$
\begin{aligned}
&\mathbb{E}[\phi(Z_0, Z_1)] \\
&\quad = \mathbb{E}[\phi(\zeta, F_I(T_1)^\alpha(\zeta - T_1))] \\
&\quad = \int_{\mathcal{S}_1} \sum_{i\,:\,s_i>0} \int_0^{\infty}\int_0^{x} s_i^\alpha \phi(x, s_i^\alpha(x-t)) f_\zeta(s_i^\alpha(x-t)) \\
&\qquad\qquad \times \prod_{j\neq i} \mathbb{F}_\zeta(s_j^\alpha(x-t))e^{-t}\,\mathrm{d}t\,\mathrm{d}x\,\nu(\mathrm{d}\mathbf{s}) \\
&\quad = \int_{\mathcal{S}_1} \sum_{i\,:\,s_i>0} \int_0^{\infty}\int_0^{s_i^\alpha x} e^{s_i^{-\alpha}y - x} \phi(x, y) f_\zeta(y) \prod_{j\neq i} \mathbb{F}_\zeta(s_j^\alpha s_i^{-\alpha} y)\,\mathrm{d}y\,\mathrm{d}x\,\nu(\mathrm{d}\mathbf{s}),
\end{aligned}
$$

where we have used the change of variable $y = s_i^\alpha(x-t)$ in the inner integral, so that $t = x - s_i^{-\alpha}y$. It follows that the joint density of (Z_0, Z_1) is given by

$$
f_{Z_0,Z_1}(x, y) = e^{-x} f_\zeta(y) \int_{\mathcal{S}_1} \sum_{i\,:\,s_i>0} e^{s_i^{-\alpha}y}\mathbb{1}_{\{y<s_i^\alpha x\}} \prod_{j\neq i} \mathbb{F}_\zeta(s_j^\alpha s_i^{-\alpha} y)\nu(\mathrm{d}\mathbf{s}),
\qquad x, y > 0.
$$

In particular, the density of Z_1 conditioned on $Z_0 = x$ is given by $f_{Z_0,Z_1}(x, y)/f_\zeta(x)$, as desired. $\square$

3.2. *Geometric ergodicity of the driving chain.* In view of the role of $(Z_n)_{n\geq 0}$ as driving chain, it will suffice to study its ergodic properties in order to deduce those of $(Z_n, \Theta_n, \mathbf{\Delta}_n)_{n\geq 0}$. This section is devoted to the proof of the following result.

THEOREM 3.6. *Suppose that* $\int_{\mathcal{S}_1} s_1^{-1}\nu(\mathrm{d}\mathbf{s}) < \infty$. *Then the Markov chain* $(Z_n)_{n\geq 0}$ *is positive Harris recurrent and possesses a unique stationary distribution on* $(0,\infty)$, π_{stat}. *This stationary distribution is absolutely continuous* (*with respect to Lebesgue measure*) *and its density, which* (*with a slight abuse of notation*) *we also denote by* π_{stat}, *is the unique solution to the equation*

$$\pi(x) = f_\zeta(x)\int_{\mathcal{S}_1}\left(\sum_{i=1}^{\infty} e^{s_i^{-\alpha}x}\prod_{j\neq i}\mathbb{F}_\zeta(s_j^\alpha s_i^{-\alpha}x)\left(\int_{s_i^{-\alpha}x}^{\infty}\frac{e^{-y}\pi(y)}{f_\zeta(y)}\,\mathrm{d}y\right)\right)\nu(\mathrm{d}\mathbf{s}). \tag{3.6}$$

Moreover, the distribution $\mathcal{L}(Z_n)$ *of* Z_n *converges to* π_{stat} *exponentially fast*; *more precisely, there exists a constant* $r > 1$ *such that*

$$\sum_{n\geq 1} r^n \|\mathcal{L}(Z_n) - \pi_{\text{stat}}\|_{\text{TV}} < \infty, \tag{3.7}$$

where $\|\cdot\|_{\text{TV}}$ *denotes the total variation norm.*

We have not been able to extract an explicit expression for π_{stat} from (3.6). (However, Lemmas A.7 and A.8 in the Appendix give some qualitative information about it.) Note also that (3.6) implies that $\pi_{\text{stat}}(x) > 0$ for $x > 0$.

To prove Theorem 3.6, we use the geometric ergodic theorem of Meyn and Tweedie [23], Theorem 15.0.1, which is based on a Foster–Lyapounov drift criterion; see (3.10) below. To understand the meaning of this criterion, we first need to introduce the concept of a small set. With this in hand, all we will require in order to obtain Theorem 3.6 from the geometric ergodic theorem are the forthcoming Lemmas 3.7 and 3.8. In the following, for each integer n, P^n denotes the n-step transition probability kernel of the chain $(Z_n)_{n\geq 0}$.

Following page 109 of Meyn and Tweedie [23], a *small set* C is a Borel subset of $\mathbb{R}_+^*$, for which there exist an integer $m_C > 0$ and a nontrivial measure μ_C such that

$$P^{m_C}(x, B) \geq \mu_C(B) \qquad \text{for all Borel sets } B \subseteq (0,\infty) \text{ and all } x \in C. \tag{3.8}$$

In our case, subsets of a compact subset of $(0,\infty)$ are clearly small sets. Indeed, let $C \subseteq [a,b]$, $0 < a < b$, and recall from Lemma 2.2 that $f_\zeta(x) \leq 1$ for all $x > 0$. It is then easy to see that for all Borel sets $B \subseteq (0,\infty)$ and all $x \in C$,

$$P(x, B) \geq e^{-b}\mu_C(B),$$

where the measure μ_C is defined for all B by

$$\mu_C(B) = \int_B f_\zeta(y)\left(\int_{\mathcal{S}_1}\sum_{i\,:\,s_i>0} e^{s_i^{-\alpha}y}\prod_{j\neq i}\mathbb{F}_\zeta(s_j^\alpha s_i^{-\alpha}y)\mathbf{1}_{\{0<y<s_i^\alpha a\}}\nu(\mathrm{d}\mathbf{s})\right)\mathrm{d}y. \tag{3.9}$$

The Markov chain $(Z_n, n \geq 0)$ is *Lebesgue-irreducible* if, for all Borel sets $B \subseteq (0, \infty)$ with strictly positive Lebesgue measure and all $x > 0$, there exists an integer n with $P^n(x, B) > 0$. It is said to be *strong aperiodic* if there exists a small set C with $m_C = 1$ and $\mu_C(C) > 0$.

LEMMA 3.7. *$(Z_n, n \geq 0)$ is both Lebesgue-irreducible and strong aperiodic.*

(In fact, the geometric ergodic theorem is valid if we replace strong aperiodicity by aperiodicity, but the definition of strong aperiodicity is easier to write down and easy to check in our context.)

PROOF OF LEMMA 3.7. By (3.2) and Remark 3.2 we have $P(x, B) > 0$ for all $x > 0$ and all Borel sets B with strictly positive Lebesgue measure; Lebesgue-irreducibility follows. Strong aperiodicity follows directly from the above proof that subsets of compact subsets of $(0, \infty)$ are small. $\square$

LEMMA 3.8 (Foster–Lyapounov drift criterion). *Assume that $\int_{\mathcal{S}_1} s_1^{-1} \nu(\mathrm{d}\mathbf{s}) < \infty$. Then there exists a small set C, a function $V : (0, \infty) \to [1, \infty)$ and constants $b < \infty$ and $\beta > 0$ satisfying*

$$\mathbb{P}V(x) - V(x) \leq -\beta V(x) + b\mathbb{1}_C(x) \qquad \forall x > 0, \tag{3.10}$$

where $\mathbb{P}V(x) := \int_0^\infty V(y) P(x, \mathrm{d}y)$. Moreover, $\int_0^\infty V(x) f_\zeta(x)\, \mathrm{d}x < \infty$.

Note that in Theorem 15.0.1 of [23], the words *small sets* are replaced by *petite sets*. However, small implies petite, and so we lose nothing here by using the former notion.

PROOF OF LEMMA 3.8. Let

$$V(x) := \frac{\exp(-cx)}{f_\zeta(x)}, \qquad x > 0,$$

where $c \in (0, 1/2)$ is such that $\exp(cx) f_\zeta(x) \to 0$ as $x \to \infty$; such a c exists by Lemma 2.2. Hence, $V(x) \to \infty$ as $x \to \infty$ and, still by Lemma 2.2, it is continuous and $V(x) \to \infty$ as $x \to 0$. In particular, it possesses a strictly positive minimum on $(0, \infty)$, which, up to normalization, may be supposed to be 1.

For the remainder of the proof, we proceed in three steps. The goal of the first two steps is to check that $\mathbb{P}V(x) < \infty$ for all $x > 0$ and that

$$\frac{\mathbb{P}V(x)}{V(x)} = f_\zeta(x) \exp(cx) \mathbb{P}V(x) \to 0 \qquad \text{as } x \to 0 \text{ or } x \to \infty.$$

To this end, write $\mathbb{P}V(x) = \mathbb{P}_1 V(x) + \mathbb{P}_2 V(x)$ where

$$\begin{aligned}\mathbb{P}_1 V(x) &\\ &:= \frac{e^{-x}}{f_\zeta(x)}\\ &\quad \times \int_0^\infty V(y) f_\zeta(y)\biggl(\int_{\mathcal{S}_1} \sum_{i\,:\,s_i>c_1} e^{s_i^{-\alpha} y} \prod_{j\neq i} \mathbb{F}_\zeta\bigl(s_j^\alpha s_i^{-\alpha} y\bigr) \mathbb{1}_{\{0<y<s_i^\alpha x\}} \nu(\mathrm{d}\mathbf{s})\biggr)\,\mathrm{d}y,\end{aligned}$$

with $c_1 \in (0, c^{-1/\alpha})$.

Step 1. We prove that the quantity $f_\zeta(x)\exp(cx)\mathbb{P}_1 V(x)$ is finite for all $x > 0$ and converges to 0 as x tends to 0 or ∞. To see this, note first that $s_i \le i^{-1}$, $\forall i \ge 1$, for ν-a.e. sequence $\mathbf{s}$, and, therefore, that the sum involved in $\mathbb{P}_1 V(x)$ only concerns indices $i < c_1^{-1}$. Since this set of indices is finite, it is sufficient to check that for all $i < c_1^{-1}$,

$$e^{(c-1)x} \int_{\mathcal{S}_1} \mathbb{1}_{\{s_i>c_1\}} \biggl(\int_0^{s_i^\alpha x} V(y) f_\zeta(y) e^{s_i^{-\alpha} y} \prod_{j\neq i} \mathbb{F}_\zeta\bigl(s_j^\alpha s_i^{-\alpha} y\bigr)\,\mathrm{d}y\biggr)\nu(\mathrm{d}\mathbf{s})$$

is finite and converges to 0 as x tends to 0 or to ∞. This term is bounded above by

$$e^{(c-1)x} \int_{\mathcal{S}_1} \mathbb{1}_{\{s_i>c_1\}} \biggl(\int_0^{s_i^\alpha x} e^{-cy+s_i^{-\alpha} y}\,\mathrm{d}y\biggr)\nu(\mathrm{d}\mathbf{s}) \tag{3.11}$$

which is clearly finite and converges to 0 as $x \to 0$. To get a similar result when $x \to \infty$, recall that $c < 1/2$, and note that

$$\begin{aligned} e^{(c-1)x} &\int_0^{s_i^\alpha x} e^{-cy+s_i^{-\alpha} y}\,\mathrm{d}y \\ &\le \begin{cases} e^{(c-1)x} s_i^\alpha x, & \text{if } c_1^{-\alpha} < s_i^{-\alpha} \le c,\\ e^{(1-s_i^\alpha)cx} s_i^\alpha x, & \text{if } c < s_i^{-\alpha} \le \frac{1}{2},\\ \bigl(e^{(1-s_i^\alpha)cx} - e^{(c-1)x}\bigr)\bigl(s_i^{-\alpha} - c\bigr)^{-1}, & \text{if } s_i^{-\alpha} > \frac{1}{2}.\end{cases}\end{aligned}$$

In all three cases, the upper bound converges to 0 (since $s_i < 1$) ν-a.e. as $x \to \infty$ and is bounded above by a finite constant independent both of $x \ge 1$ and of s_i in the interval under consideration. Hence, by dominated convergence, term (3.11) tends to 0 as $x \to \infty$.

Step 2. We now prove a similar result to the one proved in step 1, but for $\mathbb{P}_2 V$. Here we use the hypothesis $\int_{\mathcal{S}_1} s_1^{-1}\nu(\mathrm{d}\mathbf{s}) < \infty$. It will be sufficient to show that

$$\int_{\mathcal{S}_1} \sum_{i\,:\,s_i\le c_1} \biggl(\int_0^\infty e^{(s_i^{-\alpha}-c)y} \prod_{j\neq i} \mathbb{F}_\zeta\bigl(s_j^\alpha s_i^{-\alpha} y\bigr)\,\mathrm{d}y\biggr)\nu(\mathrm{d}\mathbf{s}) < \infty, \tag{3.12}$$

using the fact that $\exp(c-1)x \to 0$ as $x \to \infty$ and monotone convergence near 0. To get (3.12), we use the existence of some finite constant m [see Lemma 2.2, and note that $\int_{\mathcal{S}_1} s_1^{-\alpha-1}\nu(\mathrm{d}\mathbf{s}) \le \int_{\mathcal{S}_1} s_1^{-1}\nu(\mathrm{d}\mathbf{s}) < \infty$] such that $\mathbb{F}_\zeta(s_1^\alpha s_i^{-\alpha} y) \le m s_1^{-1} s_i y^{-1/\alpha}$, for all $y > 0$. Hence, the double integral in (3.12) is bounded above by

$$\begin{aligned}
&\int_{\mathcal{S}_1} \mathbb{1}_{\{s_1 \le c_1\}} \biggl(\int_0^\infty e^{(c_1^{-\alpha} - c)y} \prod_{j \ge 2} \mathbb{F}_\zeta\bigl(s_j^\alpha s_1^{-\alpha} y\bigr)\,\mathrm{d}y \biggr) \nu(\mathrm{d}\mathbf{s}) \\
&\quad + m \int_{\mathcal{S}_1} \sum_{i \ge 2 : s_i \le c_1} s_1^{-1} s_i \biggl(\int_0^\infty e^{(c_1^{-\alpha} - c)y} y^{-1/\alpha}\,\mathrm{d}y \biggr) \nu(\mathrm{d}\mathbf{s}) \\
&\quad \le \bigl(c - c_1^{-\alpha}\bigr)^{-1} \int_{\mathcal{S}_1} \mathbb{1}_{\{s_1 \le c_1\}} \nu(\mathrm{d}\mathbf{s}) + m' \int_{\mathcal{S}_1} s_1^{-1} \nu(\mathrm{d}\mathbf{s}) < \infty.
\end{aligned}$$

Step 3. From expression (3.2) for the transition density and from the fact that f_ζ is continuous, we see that the function $x \mapsto \mathbb{P}V(x)$ is continuous on $(0, \infty)$. Let $0 < \beta < 1$, and introduce the set $C := \{x > 0 : \mathbb{P}V(x) - (1 - \beta)V(x) \ge 0\}$. The continuity of $\mathbb{P}V/V$ on $(0, \infty)$, together with steps 1 and 2, imply that C is a compact subset of $(0, \infty)$, and so it is a small set. Moreover $b := \sup_{x \in C}(\mathbb{P}V(x) - (1 - \beta)V(x)) < \infty$, since $\mathbb{P}V - (1 - \beta)V$ is continuous on $(0, \infty)$. Finally, for all $x > 0$,

$$\mathbb{P}V(x) \le (1 - \beta)V(x) + b\mathbb{1}_C(x),$$

which is the required drift criterion.

Finally, note that $\int_0^\infty V(x) f_\zeta(x)\,\mathrm{d}x < \infty$ since $V(x) f_\zeta(x) = \exp(-cx)$, $x > 0$ for some $c > 0$. $\square$

Theorem 3.6 now follows from the geometric ergodic theorem.

3.3. *The stationary and biased Markov chains*. In order to construct the limit process C_∞ appearing in Theorem 1.1, we need to introduce an eternal stationary version of $(Z_n, Y_n, \boldsymbol{\Delta}_n)_{n \ge 1}$ and then a biased version of this stationary version; see the forthcoming Definition 5.3 of C_∞. This biased version will appear in the limit when using the techniques of Markov renewal theory to pass from the convergence of (Z_n) to the asymptotic behavior of the continuous-time processes F_* and F near their extinction time.

First, we can construct a stationary version of $(Z_n, Y_n, \boldsymbol{\Delta}_n)_{n \ge 1}$ from a fragmentation process conditioned to have an extinction time distributed according to π_{stat}. Formally, the Markov chain $((Z_n^{\mathrm{stat}}, Y_n^{\mathrm{stat}}, \boldsymbol{\Delta}_n^{\mathrm{stat}})_{n \ge 1}, Z_0^{\mathrm{stat}})$ is defined by

$$\begin{aligned}
&\mathbb{E}\bigl[f\bigl((Z_n^{\mathrm{stat}}, Y_n^{\mathrm{stat}}, \Delta_n^{\mathrm{stat}})_{n \ge 1}, Z_0^{\mathrm{stat}}\bigr)\bigr] \\
&\quad = \int_0^\infty \mathbb{E}\bigl[f\bigl((Z_n, Y_n, \boldsymbol{\Delta}_n)_{n \ge 1}, Z_0\bigr) | \zeta = x\bigr] \pi_{\mathrm{stat}}(\mathrm{d}x)
\end{aligned}$$

for suitable test functions f. Since $Z_0 = \zeta$, the chain is then stationary: Z_n^{stat} is distributed according to π_{stat} for all $n \geq 0$ and

$$\left(Z_n^{\mathrm{stat}}, Y_n^{\mathrm{stat}}, \boldsymbol{\Delta}_n^{\mathrm{stat}}\right) \stackrel{\mathrm{law}}{=} \left(Z_1^{\mathrm{stat}}, Y_1^{\mathrm{stat}}, \boldsymbol{\Delta}_1^{\mathrm{stat}}\right) \qquad \text{for } n \geq 1,$$

since $(Z_n^{\mathrm{stat}}, n \geq 0)$ is the driving chain of the Markov chain $(Z_n^{\mathrm{stat}}, Y_n^{\mathrm{stat}}, \boldsymbol{\Delta}_n^{\mathrm{stat}})_{n\geq 1}$.

Now let

$$\left(Z_n^{\mathrm{stat}}, Y_n^{\mathrm{stat}}, \boldsymbol{\Delta}_n^{\mathrm{stat}}\right)_{n\in\mathbb{Z}} \tag{3.13}$$

be an *eternal* stationary version of $(Z_n, Y_n, \boldsymbol{\Delta}_n)_{n\geq 1}$. Recall that such process always exists: for all positive integers k, the distribution of the chain $(Z_n^{\mathrm{stat}}, Y_n^{\mathrm{stat}}, \boldsymbol{\Delta}_n^{\mathrm{stat}})_{n\geq -k}$ is defined to be that of $(Z_n^{\mathrm{stat}}, Y_n^{\mathrm{stat}}, \boldsymbol{\Delta}_n^{\mathrm{stat}})_{n\geq 1}$ and so, by Kolmogorov's consistency theorem, the full process $(Z_n^{\mathrm{stat}}, Y_n^{\mathrm{stat}}, \boldsymbol{\Delta}_n^{\mathrm{stat}})_{n\in\mathbb{Z}}$ is well defined.

Observe that

$$\int_0^\infty \mathbb{P}(Y_1 \leq 1 | Z_0 = x) f_\zeta(x)\,\mathrm{d}x = \mathbb{P}(Y_1 \leq 1) = 0$$

and that, by Lemma 2.2, $f_\zeta(x) > 0$ for all x. It follows that $\mathbb{P}(Y_1 > 1 | Z_0 = x) = 1$ for Lebesgue-a.e. x, and so we also have $\mathbb{P}(Y_1^{\mathrm{stat}} > 1) = 1$. The following lemma is a consequence of Lemma A.9 in the Appendix.

LEMMA 3.9. *Suppose that* $\int_{\mathcal{S}_1} s_1^{-1} \nu(\mathrm{d}\mathbf{s}) < \infty$. *Let*

$$\mu = \mathbb{E}\left[\log\left(Y_1^{\mathrm{stat}}\right)\right].$$

Then $\mu \in (0, \infty)$.

The biased version $(Z^{\mathrm{bias}}, Y^{\mathrm{bias}}, \boldsymbol{\Delta}^{\mathrm{bias}})$ of the eternal stationary Markov chain constructed just above is then defined by

$$\mathbb{E}\left[g\left(\left(Z_n^{\mathrm{bias}}, Y_n^{\mathrm{bias}}, \boldsymbol{\Delta}_n^{\mathrm{bias}}\right)_{n\in\mathbb{Z}}\right)\right] = \frac{1}{\mu}\mathbb{E}\left[\log\left(Y_1^{\mathrm{stat}}\right) g\left(\left(Z_n^{\mathrm{stat}}, Y_n^{\mathrm{stat}}, \boldsymbol{\Delta}_n^{\mathrm{stat}}\right)_{n\in\mathbb{Z}}\right)\right].$$

Note that the eternal process $(Z_n^{\mathrm{bias}}, Y_n^{\mathrm{bias}}, \boldsymbol{\Delta}_n^{\mathrm{bias}})_{n\in\mathbb{Z}}$ is a time-inhomogeneous Markov chain. However, if we restrict to times $n \geq 1$, it is time-homogeneous, with the same transition kernel as the stationary and standard versions (although a different initial distribution). As in the standard case, we set

$$\Theta_n^{\mathrm{bias}} := \frac{(Z_n^{\mathrm{bias}})^{1/\alpha}}{(Z_{n-1}^{\mathrm{bias}})^{1/\alpha} Y_n^{\mathrm{bias}}} \qquad \text{for } n \in \mathbb{Z}.$$

In Appendix A.2 we will prove various technical results about the stationary and biased Markov chains, which will be used in the main body of the paper.

4. Asymptotics of the last fragment. We will now determine the asymptotics of $\varepsilon^{1/\alpha}F_*(\zeta-\varepsilon)$ as $\varepsilon\to 0$, and then of the whole process $t\in\mathbb{R}_+\mapsto \varepsilon^{1/\alpha}F_*(\zeta-\varepsilon t)$. The key point in our approach is the ergodicity of the driving chain proved in the previous section.

From the biased Markov chain introduced in Section 3.3, we can now define what will be the *limit process*, which is denoted by $(C_{\infty,*}(t),t\geq 0)$. Let U be uniformly distributed on $[0,1]$, independently of $(Z^{\mathrm{bias}},Y^{\mathrm{bias}},\mathbf{\Delta}^{\mathrm{bias}})$. Let

$$R(k)=\begin{cases}(Y_1^{\mathrm{bias}})^{-\alpha U}\displaystyle\prod_{i=1}^{k}(Y_i^{\mathrm{bias}})^{\alpha}, & \text{if } k\geq 1,\\ (Y_1^{\mathrm{bias}})^{-\alpha U}, & \text{if } k=0,\\ (Y_1^{\mathrm{bias}})^{-\alpha U}\displaystyle\prod_{i=k+1}^{0}(Y_i^{\mathrm{bias}})^{-\alpha}, & \text{if } k\leq -1,\end{cases}$$

so that $R(k)$ is a decreasing function of $k\in\mathbb{Z}$. Note the multiplicative relation $R(k+1)=R(k)(Y_{k+1}^{\mathrm{bias}})^{\alpha}$, $\forall k\in\mathbb{Z}$. The following result follows from Lemma A.11 in the Appendix.

LEMMA 4.1. *We have $R(k)\to 0$ as $k\to\infty$ and $R(k)\to\infty$ as $k\to-\infty$ almost surely.*

The process $C_{\infty,*}$ is then a nondecreasing piecewise constant right-continuous process, which is defined by $C_{\infty,*}(0)=0$ and, for $t>0$,

$$C_{\infty,*}(t)=(Z_k^{\mathrm{bias}})^{1/\alpha}(R(k))^{-1/\alpha}\qquad \text{if } t\in[R(k+1),R(k)).$$

See Figure 2 for an illustration. The monotonicity of $C_{\infty,*}$ comes from the identity

$$(Z_k^{\mathrm{bias}})^{1/\alpha}\prod_{i=1}^{k}(Y_i^{\mathrm{bias}})^{-1}=(Z_0^{\mathrm{bias}})^{1/\alpha}\prod_{i=1}^{k}\Theta_i^{\mathrm{bias}},\qquad k\geq 1$$

and from the fact that the random variables Θ_i^{bias} lie in $(0,1)$ a.s. A similar equality holds for negative k. Note that $R(1)<1<R(0)$ a.s. and so $C_{\infty,*}(1)=(Y_1^{\mathrm{bias}})^{U}(Z_0^{\mathrm{bias}})^{1/\alpha}$.

THEOREM 4.2. *Suppose that $\int_{\mathcal{S}_1}s_1^{-1}\nu(\mathrm{d}\mathbf{s})<\infty$ and that ν is nongeometric. Then, as $\varepsilon\to 0$,*

$$\big((\varepsilon^{1/\alpha}F_*((\zeta-\varepsilon t)-),t\geq 0),\zeta\big)\overset{law}{\to}\big((C_{\infty,*}(t),t\geq 0),\zeta\big),$$

where ζ and $C_{\infty,*}$ are independent in the limit. In particular,

$$\varepsilon^{1/\alpha}F_*(\zeta-\varepsilon)\overset{law}{\to}(Y_1^{\mathrm{bias}})^{U}(Z_0^{\mathrm{bias}})^{1/\alpha}.$$

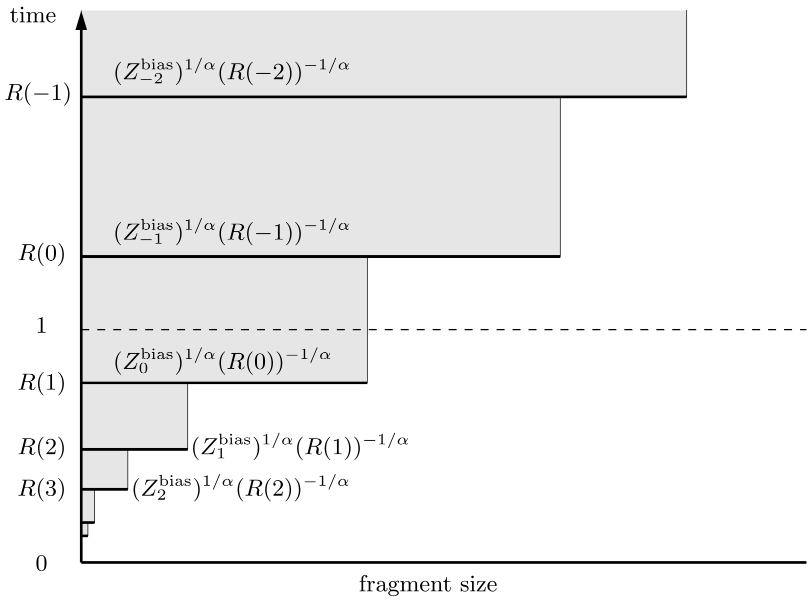

FIG. 2. *The limit* $(C_{\infty,*}(t), t \geq 0)$ *of the last fragment. The process is piecewise constant between the jumps which are indicated. Compare to Figure* 1: *here time has been reversed.*

The proof of this result is based on the convergence in distribution of the driving chain $(Z_n)_{n\geq 0}$, proved in the previous section, and uses results from Markov renewal theory, which are gathered in Section 4.1 below. In Section 4.2, we prove the convergence of the one-dimensional marginal distributions of the rescaled last fragment process. The full functional convergence is then proved in Section 4.3.

4.1. *Background on Markov renewal theory.* Let $S_0 = 0$, and for $n \geq 1$,

$$S_n := \sum_{i=1}^{n} \log Y_i .$$

As a consequence of Corollary 3.3, $(Z_n, S_n)_{n\geq 0}$ is a *Markov renewal process* in the terminology of [3, 4, 6, 21, 22, 25, 28]. We refer to Alsmeyer's paper [4] for background on this topic and results about asymptotic behaviors. As in standard renewal theory, these results depend on hypotheses of nonarithmeticity/arithmeticity for the support of the process. In our context, this is formulated as follows: the process is called *d-arithmetic* if $d \geq 0$ is the largest number for which there exists a measurable function $\gamma : (0, \infty) \to [0, d)$ such that

$$\mathbb{P}\big(\log Y_1 \in \gamma(Z_0) - \gamma(Z_1) + d\mathbb{Z}\big) = 1. \tag{4.1}$$

The process is *nonarithmetic* if no such d exists. The condition for nonarithmeticity in our setting is unsurprising.

LEMMA 4.3. *The process $(Z_n, S_n)_{n\geq 0}$ is nonarithmetic if and only if the dislocation measure ν is nongeometric.*

PROOF. Recall that $Y_1 = ((\zeta - T_1)/\zeta)^{1/\alpha}$ and $\zeta = T_1 + \Theta_1^{-\alpha} Z_1$, with T_1 independent of (Θ_1, Z_1), and $Z_0 = \zeta$. If ν is r-geometric for some $r \in (0,1)$, then $\Theta_1 \in r^{\mathbb{N}}$ a.s. and, consequently, $\log Y_1 \in \alpha^{-1}(\log Z_1 - \log Z_0) + (-\log r)\mathbb{N}$ a.s. The arithmeticity of $(Z_n, S_n)_{n\geq 0}$ follows.

Conversely, assume that (4.1) holds for some $d \geq 0$ and some measurable function γ. This is equivalent to

$$\mathbb{P}\big(\log \Theta_1 \in \overline{\gamma}(T_1 + \Theta_1^{-\alpha} Z_1) - \overline{\gamma}(Z_1) + d\mathbb{Z}\big) = 1$$

for some suitable function $\overline{\gamma}$. Since $\Theta_1^{-\alpha} Z_1$ has a strictly positive density on $(0, \infty)$ [see the discussion around (2.1)], and since T_1 is independent of (Θ_1, Z_1), this implies that for Lebesgue a.e. $a > 0$, there exists a real number b_a such that $\mathbb{P}(\overline{\gamma}(T_1 + a) \in b_a + d\mathbb{Z}) = 1$. But T_1 is exponentially distributed, and so $\overline{\gamma}(u + a) \in b_a + d\mathbb{Z}$ for Lebesgue-a.e. $u > 0$. This implies that

$$\mathbb{P}\big(\overline{\gamma}(Z_0) - \overline{\gamma}(Z_1) \in d\mathbb{Z} | Z_0 > a, Z_1 > a\big) = 1 \qquad \text{for Lebesgue a.e. } a > 0.$$

Hence, $\mathbb{P}(\overline{\gamma}(Z_0) - \overline{\gamma}(Z_1) \in d\mathbb{Z}) = 1$, and so $\mathbb{P}(\log \Theta_1 \in d\mathbb{Z}) = 1$. Note that this implies that $d > 0$. To conclude, assume that ν is nongeometric; that is, that for all $r \in (0,1)$, there exists some $i_r \in \mathbb{N}$ such that $\nu(s_{i_r} \notin r^{\mathbb{N}}, s_{i_r} > 0) > 0$. Then

$$\mathbb{P}\big(\log \Theta_1 \notin (\log r)\mathbb{N}\big) \geq \mathbb{P}\big(\Theta_1 = F_i(T_1), F_i(T_1) \notin r^{\mathbb{N}}\big) \qquad \forall i \in \mathbb{N}.$$

Since $\mathbb{P}(\Theta_1 = F_i(T_1) | F_i(T_1)) > 0$ when $F_i(T_1) > 0$ [this is due to the fact that $\prod_{j \neq i} \mathbb{F}_\zeta(s_j^\alpha x) > 0$, for $\mathbf{s} \in \mathcal{S}_1$, when $x > 0$, as explained in the proof of Lemma 2.2] and, since $\mathbb{P}(F_{i_r}(T_1) \notin r^{\mathbb{N}} \cup \{0\}) > 0$ by assumption, we have that $\mathbb{P}(\log \Theta_1 \notin (\log r)\mathbb{N}) > 0$ for all $r \in (0,1)$, which contradicts the fact that $\mathbb{P}(\log \Theta_1 \in d\mathbb{Z}) = 1$ for some $d > 0$. Hence, ν is geometric when (4.1) holds. $\square$

Theorem 1 of Alsmeyer [4] applied to $(Z_n, S_n)_{n\geq 0}$ yields the following result, with $\mu = \mathbb{E}[\log(Y_1^{\text{stat}})] \in (0, \infty)$; see Lemma 3.9.

THEOREM 4.4. *Suppose that the dislocation measure ν is nongeometric and such that $\int_{\mathcal{S}_1} s_1^{-1} \nu(\mathrm{d}\mathbf{s}) < \infty$. Suppose that $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a measurable function which is such that* (a) *$g(x, \cdot)$ is Lebesgue-almost everywhere continuous for Lebesgue-almost all $x \in \mathbb{R}_+$ and* (b) *$\int_0^\infty \sum_{n \in \mathbb{Z}_+} \sup_{n\rho \leq y < (n+1)\rho} |g(x, y)| \times \pi_{\text{stat}}(\mathrm{d}x) < \infty$ for some $\rho > 0$. Then as $r \to \infty$,*

$$\mathbb{E}\left[\sum_{n\geq 0} g(Z_n, r - S_n) \Big| Z_0 = z\right] \to \frac{1}{\mu} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} g(x, y) \, \mathrm{d}y \, \pi_{\text{stat}}(\mathrm{d}x),$$

for Lebesgue-almost all $z \in \mathbb{R}_+$.

In terms of the biased process introduced in Section 3.3, Corollary 1 of [4] reads as follows.

COROLLARY 4.5. *Suppose that the dislocation measure* ν *is nongeometric and such that* $\int_{\mathcal{S}_1} s_1^{-1}\nu(\mathrm{d}\mathbf{s}) < \infty$. *Let* $h:\mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *be a measurable function such that* $g:\mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *defined by* $g(x, y) = h(x, y)\mathbb{P}(\log(Y_1) > y|Z_0 = x)$ *satisfies conditions* (a) *and* (b) *of Theorem* 4.4. *Let*

$$J(r) = \sup\{n \geq 0 : S_n \leq r\},$$

and assume that $J(r) < \infty$ *for all* $r \in \mathbb{R}_+$. *Then for Lebesgue-almost all* $z \in \mathbb{R}_+$, *as* $r \to \infty$,

$$\mathbb{E}[h(Z_{J(r)}, r - S_{J(r)})|Z_0 = z] \to \mathbb{E}[h(Z_0^{\mathrm{bias}}, U \log(Y_1^{\mathrm{bias}}))],$$

where U *is uniformly distributed on* $[0, 1]$ *and independent of* $(Z_0^{\mathrm{bias}}, \log(Y_1^{\mathrm{bias}}))$.

REMARK 4.6. We have replaced all the "for π_{stat}-almost all x" in Alsmeyer's results by "for Lebesgue-almost all x" since π_{stat} is equivalent to Lebesgue measure on $\mathbb{R}_+$. Note also that a bounded measurable function $h:\mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ which is such that $h(x, \cdot)$ is Lebesgue-almost everywhere continuous for Lebesgue-almost all $x \in \mathbb{R}_+$, satisfies the conditions of Corollary 4.5. Indeed, the measurability and condition (a) are obvious. For condition (b), take $\rho = 1$, set $\|h\|_\infty = \sup_{x \geq 0} |h(x)|$ and note that

$$\begin{aligned}
&\int_{\mathbb{R}_+} \sum_{n \in \mathbb{Z}_+} \sup_{n \leq y < n+1} |h(x, y)| \mathbb{P}(\log(Y_1) > y|Z_0 = x)\pi_{\mathrm{stat}}(\mathrm{d}x) \\
&\quad \leq \|h\|_\infty \int_{\mathbb{R}_+} \sum_{n \in \mathbb{Z}_+} \mathbb{P}(\log(Y_1) > n|Z_0 = x)\pi_{\mathrm{stat}}(\mathrm{d}x) \\
&\quad \leq \|h\|_\infty (1 + \mu) < \infty,
\end{aligned}$$

since $\mathbb{E}[Z] + 1 \geq \sum_{n \in \mathbb{Z}_+} \mathbb{P}(Z > n)$ for any positive random variable Z.

4.2. *One-dimensional convergence*. We use Corollary 4.5 to obtain the convergence in distribution of the rescaled last fragment at time $\zeta - \varepsilon$ as $\varepsilon \to 0$,

$$(\varepsilon^{1/\alpha} F_*(\zeta - \varepsilon), \zeta) \overset{\mathrm{law}}{\to} ((Z_0^{\mathrm{bias}})^{1/\alpha} (Y_1^{\mathrm{bias}})^U, \zeta), \tag{4.2}$$

with ζ independent of $(Z_0^{\mathrm{bias}})^{1/\alpha}(Y_1^{\mathrm{bias}})^U$ in the limit. In fact, this result will be an immediate consequence of the proof of Theorem 4.2 in the next section. However, its proof is instructive and so, by way of a brief warm-up, we give the details here.

Let

$$
\begin{aligned}
N_\varepsilon &= \sup\{n \geq 0 : \zeta - \varepsilon \geq T_n\} = \sup\left\{n \geq 0 : \prod_{i=0}^{n} Y_i^\alpha \geq \varepsilon\right\} \\
&= \sup\left\{n \geq 0 : \sum_{i=0}^{n} \log Y_i \leq \frac{1}{\alpha} \log \varepsilon\right\},
\end{aligned} \tag{4.3}
$$

with the convention that $\sup \varnothing = -\infty$. Note that for all $\varepsilon > 0$, since $T_n \to \zeta$ almost surely, $N_\varepsilon < \infty$ almost surely. Also,

$$\mathbb{P}(N_\varepsilon \neq -\infty) = \mathbb{P}(\zeta \geq \varepsilon) \to 1 \qquad \text{as } \varepsilon \to 0.$$

Therefore,

$$F_*(\zeta - \varepsilon) = F_*(T_{N_\varepsilon}) \mathbb{1}_{\{N_\varepsilon \neq -\infty\}} + \mathbb{1}_{\{N_\varepsilon = -\infty\}} = \prod_{i=0}^{N_\varepsilon} \Theta_i \mathbb{1}_{\{N_\varepsilon \neq -\infty\}} + \mathbb{1}_{\{N_\varepsilon = -\infty\}}.$$

Hence, since $\prod_{i=0}^{N_\varepsilon} \Theta_i = Z_{N_\varepsilon}^{1/\alpha} \prod_{i=0}^{N_\varepsilon} Y_i^{-1}$,

$$
\begin{aligned}
&\varepsilon^{1/\alpha} F_*(\zeta - \varepsilon) \\
&\quad = Z_{N_\varepsilon}^{1/\alpha} \exp\left(\frac{1}{\alpha} \log \varepsilon - S_{N_\varepsilon} - \frac{1}{\alpha} \log \zeta\right) \mathbb{1}_{\{N_\varepsilon \neq -\infty\}} + \varepsilon^{1/\alpha} \mathbb{1}_{\{N_\varepsilon = -\infty\}}.
\end{aligned}
$$

Next, let $f : \mathbb{R}_+ \to \mathbb{R}$ be a bounded continuous test function. To obtain (4.2), it is sufficient to prove that for Lebesgue-almost all $z > 0$,

$$\mathbb{E}[f(\varepsilon^{1/\alpha} F_*(\zeta - \varepsilon)) | \zeta = z] \to \mathbb{E}[f((Z_0^{\text{bias}})^{1/\alpha} (Y_1^{\text{bias}})^U)].$$

So let $z > 0$ and note that, conditional on $\zeta = z$, $N_\varepsilon \neq -\infty$ for all $\varepsilon \leq z$. Hence, for $\varepsilon \leq z$, since $Z_0 = \zeta$,

$$
\begin{aligned}
&\mathbb{E}[f(\varepsilon^{1/\alpha} F_*(\zeta - \varepsilon)) | \zeta = z] \\
&\quad = \mathbb{E}\left[f\left(Z_{N_\varepsilon}^{1/\alpha} \exp\left(\frac{1}{\alpha} \log \varepsilon - S_{N_\varepsilon} - \frac{1}{\alpha} \log z\right)\right) \Big| Z_0 = z\right] \\
&\quad = \mathbb{E}\left[f\left(Z_{J(\alpha^{-1} \log(\varepsilon/z))}^{1/\alpha} \exp\left(\frac{1}{\alpha} \log(\varepsilon/z) - S_{J(\alpha^{-1} \log(\varepsilon/z))}\right)\right) \Big| Z_0 = z\right],
\end{aligned}
$$

where J is defined in Corollary 4.5. The last expectation converges to $\mathbb{E}[f((Z_0^{\text{bias}})^{1/\alpha} (Y_1^{\text{bias}})^U)]$ as $\varepsilon \to 0$, by Corollary 4.5, since the function h defined on $(0, \infty) \times [0, \infty)$ by

$$h(x, y) = f(x^{1/\alpha} \exp(y))$$

and by, say, $h(0, y) = 0$ for $y \in \mathbb{R}_+$, satisfies the conditions of Corollary 4.5; see Remark 4.6.

4.3. *Functional convergence.* We take as a convention (for the standard version of our Markov chain, started from $Z_0 = \zeta$) that $Z_i = Y_i = 0$ and $\mathbf{\Delta}_i = \mathbf{0}$ for $i < 0$.

LEMMA 4.7. *Endow $(\mathbb{R}_+^2 \times \mathcal{S}_1)^{\mathbb{Z}} \times \mathbb{R}_+$ with the product topology. Then for Lebesgue a.e. $z > 0$, conditional on $\zeta = z$, we have*

$$\left((Z_{N_\varepsilon+n}, Y_{N_\varepsilon+n}, \mathbf{\Delta}_{N_\varepsilon+n})_{n\in\mathbb{Z}}, \frac{1}{\alpha}\log(\varepsilon/\zeta) - S_{N_\varepsilon}\right) \xrightarrow{law} \big((Z_n^{\mathrm{bias}}, Y_n^{\mathrm{bias}}, \mathbf{\Delta}_n^{\mathrm{bias}})_{n\in\mathbb{Z}}, U\log\big(Y_1^{\mathrm{bias}}\big)\big)$$

as $\varepsilon \to 0$, where U is independent of the process $(Z_n^{\mathrm{bias}}, Y_n^{\mathrm{bias}}, \mathbf{\Delta}_n^{\mathrm{bias}})_{n\in\mathbb{Z}}$.

PROOF. It is sufficient to prove that for all $k \geq 1$ and Lebesgue a.e. $z > 0$, conditional on $\zeta = z$,

$$\left((Z_{N_\varepsilon+n}, Y_{N_\varepsilon+n}, \mathbf{\Delta}_{N_\varepsilon+n})_{n\geq -k}, \frac{1}{\alpha}\log(\varepsilon/\zeta) - S_{N_\varepsilon}\right) \xrightarrow{\mathrm{law}} \big((Z_n^{\mathrm{bias}}, Y_n^{\mathrm{bias}}, \mathbf{\Delta}_n^{\mathrm{bias}})_{n\geq -k}, U\log\big(Y_1^{\mathrm{bias}}\big)\big).$$

So, in the following, we fix $k \geq 1$.

Recall that, conditionally on $\zeta = Z_0 = z$, $N_\varepsilon \neq -\infty$ for all $\varepsilon \leq z$. Moreover, $N_\varepsilon \to \infty$ as $\varepsilon \to 0$ almost surely. It is therefore sufficient to show that for Lebesgue a.e. $z > 0$ and all bounded continuous functions $f : (\mathbb{R}_+^2 \times \mathcal{S}_1)^{\mathbb{Z}} \times \mathbb{R}_+ \to \mathbb{R}$,

$$\begin{aligned}\mathbb{E}\left[f\left((Z_{N_\varepsilon+n}, Y_{N_\varepsilon+n}, \mathbf{\Delta}_{N_\varepsilon+n})_{n\geq -k}, \frac{1}{\alpha}\log(\varepsilon/z) - S_{N_\varepsilon}\right)\mathbb{1}_{\{N_\varepsilon\geq k+1\}}\middle| Z_0 = z\right]\\ \to \mathbb{E}\big[f\big((Z_n^{\mathrm{bias}}, Y_n^{\mathrm{bias}}, \mathbf{\Delta}_n^{\mathrm{bias}})_{n\geq -k}, U\log\big(Y_1^{\mathrm{bias}}\big)\big)\big].\end{aligned}$$

To show this, note that for $\varepsilon \leq z$,

$$\begin{aligned}&\mathbb{E}\left[f\left((Z_{N_\varepsilon+n}, Y_{N_\varepsilon+n}, \mathbf{\Delta}_{N_\varepsilon+n})_{n\geq -k}, \frac{1}{\alpha}\log(\varepsilon/z) - S_{N_\varepsilon}\right)\mathbb{1}_{\{N_\varepsilon\geq k+1\}}\middle| Z_0 = z\right]\\ &\quad= \sum_{i=1}^{\infty}\mathbb{E}\left[f\left((Z_{i+k+n}, Y_{i+k+n}, \mathbf{\Delta}_{i+k+n})_{n\geq -k}, \frac{1}{\alpha}\log(\varepsilon/z) - S_{i+k}\right)\right.\\ &\qquad\qquad\left.\times\, \mathbb{1}_{\{S_{i+k}\leq 1/\alpha\log(\varepsilon/z)<S_{i+k+1}\}}\middle| Z_0 = z\right]\\ &\quad= \sum_{i=0}^{\infty}\mathbb{E}\left[g\left(Z_i, \frac{1}{\alpha}\log(\varepsilon/z) - S_i\right)\middle| Z_0 = z\right],\end{aligned}$$

where

$$g(x,y)=\mathbb{E}\Bigg[f\Bigg((Z_{k+n+1},Y_{k+n+1},\mathbf{\Delta}_{k+n+1})_{n\geq -k},y-\sum_{j=1}^{k+1}\log Y_j\Bigg)\\ \times \mathbb{1}_{\{\sum_{j=1}^{k+1}\log Y_j\leq y<\sum_{j=1}^{k+2}\log Y_j\}}\Bigg|Z_0=x\Bigg],$$

the first equality being a consequence of the definition of N_ε and the second of the Markov property of the process. Note that g satisfies the assumptions of Theorem 4.4; see Remark 4.6. Consequently, as $\varepsilon\to 0$, for Lebesgue a.e. $z>0$,

$$\mathbb{E}\Bigg[f\Bigg((Z_{N_\varepsilon+n},Y_{N_\varepsilon+n},\mathbf{\Delta}_{N_\varepsilon+n})_{n\geq -k},\frac{1}{\alpha}\log(\varepsilon/z)-S_{N_\varepsilon}\Bigg)\mathbb{1}_{\{N_\varepsilon\geq k+1\}}\Bigg|Z_0=z\Bigg]\\ \to\frac{1}{\mu}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}g(x,y)\,\mathrm{d}y\,\pi_{\mathrm{stat}}(\mathrm{d}x).$$

Using the change of variables $u=(y-\sum_{j=1}^{k+1}\log Y_j)/\log(Y_{k+2})$, we get, for U uniform on $[0,1]$ and independent of the process $(X,Y,\mathbf{\Delta})$, that this limit can be written as

$$\frac{1}{\mu}\int_{\mathbb{R}_+}\mathbb{E}\big[\log(Y_{k+2})f\big((Z_{k+n+1},Y_{k+n+1},\mathbf{\Delta}_{k+n+1})_{n\geq -k},U\log Y_{k+2}\big)|Z_0=x\big]\\ \times\pi_{\mathrm{stat}}(\mathrm{d}x)\\ =\frac{1}{\mu}\mathbb{E}\big[\log\big(Y_{k+2}^{\mathrm{stat}}\big)f\big(\big(Z_{k+n+1}^{\mathrm{stat}},Y_{k+n+1}^{\mathrm{stat}},\mathbf{\Delta}_{k+n+1}^{\mathrm{stat}}\big)_{n\geq -k},U\log Y_{k+2}^{\mathrm{stat}}\big)\big]\\ =\frac{1}{\mu}\mathbb{E}\big[\log\big(Y_1^{\mathrm{stat}}\big)f\big(\big(Z_n^{\mathrm{stat}},Y_n^{\mathrm{stat}},\mathbf{\Delta}_n^{\mathrm{stat}}\big)_{n\geq -k},U\log Y_1^{\mathrm{stat}}\big)\big],$$

by stationarity of the process $(Z^{\mathrm{stat}},Y^{\mathrm{stat}},\mathbf{\Delta}^{\mathrm{stat}})$. □

PROOF OF THEOREM 4.2. Let $\varepsilon\leq\zeta$. Recall that for $0<t\leq\zeta/\varepsilon$,

$$N_{\varepsilon t}=\sup\Bigg\{n\geq 0:\prod_{i=0}^{n}Y_i^\alpha\geq\varepsilon t\Bigg\}\neq -\infty$$

and

$$\varepsilon^{1/\alpha}F_*(\zeta-\varepsilon t)=\varepsilon^{1/\alpha}Z_{N_{\varepsilon t}}^{1/\alpha}\prod_{i=0}^{N_{\varepsilon t}}Y_i^{-1}.$$

We will want to re-center all times around N_ε (which is $\neq -\infty$ since $\varepsilon\leq\zeta$). To this end, let

$$R_\varepsilon(k)=\varepsilon^{-1}\prod_{i=0}^{N_\varepsilon+k}Y_i^\alpha,\qquad k\geq -N_\varepsilon$$

so that $R_\varepsilon(k)$ is strictly decreasing in $k \geq -N_\varepsilon$ and

$$N_{\varepsilon t} = N_\varepsilon + \sup\{k \geq -N_\varepsilon : R_\varepsilon(k) \geq t\}.$$

Note that $R_\varepsilon(k) = \varepsilon^{-1}(\zeta - T_{N_\varepsilon+k})$ and therefore that $(R_\varepsilon(k), k \geq -N_\varepsilon + 1)$ is the (decreasing) sequence of jump times of the process $(\varepsilon^{1/\alpha} F_*(\zeta - \varepsilon t), t \geq 0)$. Re-centering times around N_ε, we obtain that $R_\varepsilon(k)$ may be written as

$$R_\varepsilon(k) = \begin{cases} \exp\big(\alpha S_{N_\varepsilon} - \log(\varepsilon/\zeta)\big) \displaystyle\prod_{i=1}^{k} Y^{\alpha}_{N_\varepsilon+i}, & \text{if } k \geq 1, \\ \exp\big(\alpha S_{N_\varepsilon} - \log(\varepsilon/\zeta)\big), & \text{if } k = 0, \\ \exp\big(\alpha S_{N_\varepsilon} - \log(\varepsilon/\zeta)\big) \displaystyle\prod_{i=k+1}^{0} Y^{-\alpha}_{N_\varepsilon+i}, & \text{if } -N_\varepsilon \leq k \leq -1. \end{cases} \tag{4.4}$$

Similar to the construction of $C_{\infty,*}$, the process $(\varepsilon^{1/\alpha} F_*(\zeta - \varepsilon t), t \geq 0)$ is piecewise constant and may be constructed from $(Z_n, Y_n)_{n\geq 0}$ as follows: for $0 < t \leq \zeta/\varepsilon$,

$$\varepsilon^{1/\alpha} F_*(\zeta - \varepsilon t) = (Z_{N_\varepsilon+k})^{1/\alpha}\big(R_\varepsilon(k)\big)^{-1/\alpha}$$
$$\text{when } t \in \big(R_\varepsilon(k+1), R_\varepsilon(k)\big]\ \forall k \geq -N_\varepsilon.$$

Next, by Lemma 4.7 and the Skorokhod representation theorem [the space $(\mathbb{R}_+^2 \times \mathcal{S}_1)^{\mathbb{Z}} \times \mathbb{R}_+$ is Polish], for Lebesgue a.e. $z > 0$, there exists for all $\varepsilon > 0$ a version of

$$\left((Z_{N_\varepsilon+n}, Y_{N_\varepsilon+n}, \mathbf{\Delta}_{N_\varepsilon+n})_{n\in\mathbb{Z}}, \frac{1}{\alpha}\log(\varepsilon/\zeta) - S_{N_\varepsilon}\right)\Big| \zeta = z$$

that converges almost surely as $\varepsilon \to 0$ to a version of $((Z^{\text{bias}}, Y^{\text{bias}}, \mathbf{\Delta}^{\text{bias}}), U\log(Y_1^{\text{bias}}))$. Then for all $t > 0$ and all $\varepsilon \leq z$, construct from this new version a process $\varepsilon^{1/\alpha}(\tilde{F}_*(\tilde{\zeta} - \varepsilon t), t \geq 0)$ (with $\tilde{\zeta} = z$), exactly as $\varepsilon^{1/\alpha}(F_*(\zeta - \varepsilon t), t \geq 0)$ is constructed above from

$$\left((Z_{N_\varepsilon+n}, Y_{N_\varepsilon+n}, \mathbf{\Delta}_{N_\varepsilon+n})_{n\in\mathbb{Z}}, \frac{1}{\alpha}\log(\varepsilon/\zeta) - S_{N_\varepsilon}\right).$$

By Lemma A.4 in the Appendix, the càdlàg process $\varepsilon^{1/\alpha}(\tilde{F}_*((\tilde{\zeta} - \varepsilon t)-), t \geq 0)$ then converges almost surely as $\varepsilon \to 0$ to a process which is distributed as $C_{\infty,*}$. □

5. The spine decomposition for the fragmentation. We are now ready to introduce our spine decomposition for a fragmentation process. It may help the reader to refer to Figure 1. We need a little notation. Write $\bar{F}^{(x)}$ to denote the (left-continuous) time-reversal of a fragmentation process F conditioned to become extinct before time x, that is, $\bar{F}^{(x)}(0) = \mathbf{0}$, $\bar{F}^{(x)}(x) = \mathbf{1}$, $\bar{F}^{(x)}$ is làdcàg on $\mathbb{R}_+$, and for any suitable test function f,

$$\mathbb{E}\big[f\big(\bar{F}^{(x)}(t), t \geq 0\big)\big] = \mathbb{E}\big[f\big(F(x-t)\big)\mathbb{1}_{\{0\leq t\leq x\}} + f(\mathbf{0})\mathbb{1}_{\{t>x\}}|\zeta < x\big].$$

[We emphasize that $\bar{F}^{(x)}(t) = \mathbf{0}$ for $t > x$.] Note that since $\bar{F}^{(x)}$ is làdcàg, the process $(\bar{F}^{(x)}(t+), t \geq 0)$ is càdlàg. Moreover, the probability that a fragmentation process jumps at a fixed deterministic time t is 0. (This can be seen as a consequence of its Poissonian construction [8, 9]. Equivalently, and in a more elementary way since the dislocation measure is finite here, this can be seen using the genealogical description of the fragmentation developed in Chapter 1.2 of [11].) It is clear from its definition that $\bar{F}^{(x)}$ inherits this property on $(0, x)$.

Recall the definitions of N_ε and $R_\varepsilon(k)$ from (4.3) and (4.4), respectively.

PROPOSITION 5.1 (Spine decomposition). *On the event $\{N_\varepsilon \neq -\infty\} = \{\varepsilon \leq \zeta\}$, the process $(F(\zeta - \varepsilon t), 0 < t \leq \zeta/\varepsilon)$ can be rewritten in the form*

$$\Bigg(\Bigg\{\prod_{j=0}^{N_\varepsilon+K_\varepsilon(t)} \Theta_j,\quad \Bigg(\prod_{j=0}^{N_\varepsilon+i-1} \Theta_j\Bigg)\Delta_{N_\varepsilon+i,m}\bar{F}_{i,m}^{(\Delta^\alpha_{N_\varepsilon+i,m}\Theta^{-\alpha}_{N_\varepsilon+i}Z_{N_\varepsilon+i})}\Bigg(\varepsilon t\Bigg(\prod_{j=0}^{N_\varepsilon+i-1}\Theta_j\Bigg)^\alpha \Delta^\alpha_{N_\varepsilon+i,m}\Bigg):$$
$$m \geq 1, -N_\varepsilon + 1 \leq i \leq K_\varepsilon(t)\Bigg\}^{\downarrow}, 0 < t \leq \zeta/\varepsilon\Bigg),$$

where $K_\varepsilon(t)$ is the unique integer $k \geq -N_\varepsilon$ such that $R_\varepsilon(k) \geq t > R_\varepsilon(k+1)$, and $\bar{F}_{i,m}^{(\Delta^\alpha_{N_\varepsilon+i,m}\Theta^{-\alpha}_{N_\varepsilon+i}Z_{N_\varepsilon+i})}, i \in \mathbb{Z}, m \geq 1$ is a collection of conditioned fragmentation processes which are independent for distinct i and m, conditionally on $(Z_n, \Theta_n, \mathbf{\Delta}_n)_{n\geq 0}$.

Although this expression may seem a little intimidating, the idea behind it is simple: the decreasing sequence $F(\zeta - \varepsilon t)$ is composed of $F_*(\zeta - \varepsilon t)$ (the spine term) and the masses of fragments coming from the fragmentation of all blocks that detached from the spine F_* before time $\zeta - \varepsilon t$.

PROOF OF PROPOSITION 5.1. Consider the state of the fragmentation at some time $\zeta - \varepsilon t$. Each block present is either the last fragment, or descends from a block which split off from the last fragment at time T_n for some $1 \leq n \leq N_{\varepsilon t}$ (this ensures that $T_n \leq \zeta - \varepsilon t$). In other words, the current state may be written as the decreasing rearrangement of the blocks of

$$\big(F_*(\zeta - \varepsilon t), \tilde{F}_{n,m}(\zeta - \varepsilon t - T_n), m \geq 1, 1 \leq n \leq N_{\varepsilon t}\big), \tag{5.1}$$

where $\tilde{F}_{n,m}(s)$ represents the collection of blocks present at time $T_n + s$ which are descended from the mth-largest block to split off from F_* at time T_n. Note that the process $\tilde{F}_{n,m}$ must itself have extinction time at most $\zeta - T_n$ (since it must die before the last fragment), that is, $\tilde{F}_{n,m}(s) = \mathbf{0}$ for some $s < \zeta - T_n$.

By construction,

$$F_*(T_n) = \prod_{j=0}^{n} \Theta_j.$$

For $K_\varepsilon(t)$ defined to be the unique integer $k \geq -N_\varepsilon$ such that $R_\varepsilon(k) \geq t > R_\varepsilon(k+1)$, we have $N_{\varepsilon t} = N_\varepsilon + K_\varepsilon(t)$ and so

$$F_*(\zeta - \varepsilon t) = F_*(T_{N_{\varepsilon t}}) = \prod_{j=0}^{N_{\varepsilon t}} \Theta_j = \prod_{j=0}^{N_\varepsilon + K_\varepsilon(t)} \Theta_j.$$

For $1 \leq n \leq N_\varepsilon + K_\varepsilon(t)$, the blocks descending from the last fragment at time T_{n-1} which split off from the last fragment at time T_n have sizes $\{F_*(T_{n-1})\Delta_{n,m}, m \geq 1\}$, that is, $\tilde{F}_{n,m}(0) = F_*(T_{n-1})\Delta_{n,m}$. Note that

$$F_*(T_{n-1})\Delta_{n,m} = \left(\prod_{j=0}^{n-1} \Theta_j\right)\Delta_{n,m}.$$

Let us write $H_{n,m}(s) = (\tilde{F}_{n,m}(0))^{-1}\tilde{F}_{n,m}(\tilde{F}_{n,m}(0)^{-\alpha}s)$ for $\tilde{F}_{n,m}$ with its natural time- and space-rescaling, in order that we may later exploit the scaling property. We can then rewrite (5.1) as

$$\left(\prod_{j=0}^{N_\varepsilon + K_\varepsilon(t)} \Theta_j, \left(\prod_{j=0}^{n-1} \Theta_j\right)\Delta_{n,m} H_{n,m}\left((\zeta - \varepsilon t - T_n)\left(\prod_{j=0}^{n-1} \Theta_j \Delta_{n,m}\right)^\alpha\right),\right.$$
$$\left. m \geq 1, 1 \leq n \leq N_\varepsilon + K_\varepsilon(t)\right).$$

Now observe that

$$(\zeta - T_n)\left(\prod_{j=0}^{n-1} \Theta_j \Delta_{n,m}\right)^\alpha = Z_n \Theta_n^{-\alpha} \Delta_{n,m}^\alpha,$$

so that we in fact have

$$\left(\prod_{j=0}^{N_\varepsilon + K_\varepsilon(t)} \Theta_j, \left(\prod_{j=0}^{n-1} \Theta_j\right)\Delta_{n,m} H_{n,m}\left(Z_n \Theta_n^{-\alpha} \Delta_{n,m}^\alpha - \varepsilon t \left(\prod_{j=0}^{n-1} \Theta_j\right)^\alpha \Delta_{n,m}^\alpha\right),\right.$$
$$\left. m \geq 1, 1 \leq n \leq N_\varepsilon + K_\varepsilon(t)\right). \tag{5.2}$$

So far, we know that $H_{n,m}$ is some sort of fragmentation process which is started from $H_{n,m}(0) = \mathbf{1}$ and becomes extinct before time $Z_n \Theta_n^{-\alpha} \Delta_{n,m}^\alpha$.

Suppose, temporarily, that we are on the event $\{N_\varepsilon + K_\varepsilon(t) = 1\}$; in other words, by time $\zeta - \varepsilon t$, the last fragment has split exactly once. Then, in the notation introduced just before Lemma 3.5, $H_{1,m} = H^{(I,m)}$, and Lemma 3.5 entails that, conditionally on $(\Theta_0, \Theta_1, Z_1, \Delta_{1,m}, m \geq 1)$, $H_{1,m}$ is distributed as a standard fragmentation process conditioned to become extinct before time $\Delta_{1,m}^\alpha \Theta_1^{-\alpha} Z_1$, independently for different $m \geq 1$. It follows that, in this case, (5.2) is distributed as

$$\left(\Theta_0\Theta_1, \Theta_0\Delta_{1,m}\bar{F}_{1,m}^{(\Delta_{1,m}^\alpha \Theta_1^{-\alpha} Z_1)}\left(\varepsilon t \Theta_0^\alpha \Delta_{1,m}^\alpha\right)\right).$$

To get to the result for general N_ε and $K_\varepsilon(t)$, note that Lemma 3.5 also tells us that, conditionally on $(\Theta_0, \Theta_1, Z_1, \Delta_{1,m}, m \geq 1)$, the evolution of the last fragment after its first split (suitably rescaled) is independent of the evolution of $H_{1,m}$ for $m \geq 1$. So we may apply Lemma 3.5 inductively, just as we did in the proof of Proposition 3.1, to obtain that (5.2) has the same distribution as

$$\begin{aligned}\Bigg(\prod_{j=0}^{N_\varepsilon+K_\varepsilon(t)} \Theta_j, \left(\prod_{j=0}^{n-1}\Theta_j\right)\Delta_{n,m}\bar{F}_{n,m}^{(\Delta_{n,m}^\alpha \Theta_n^{-\alpha} Z_n)}\left(\varepsilon t\left(\prod_{j=0}^{n-1}\Theta_j\right)^\alpha \Delta_{n,m}^\alpha\right), \\ m \geq 1, 1 \leq n \leq N_\varepsilon + K_\varepsilon(t)\Bigg).\end{aligned}$$

Finally, we will find it convenient to index the split times in such a way that index N_ε becomes 0. So we simply shift the indices down by N_ε (i.e., set $n = N_\varepsilon + i$). Now notice that everything we have done here is consistent as we vary t in $\mathbb{R}_+$, and so we obtain the desired result. $\square$

So far, we have mainly thought of the spine decomposition in terms of the forward direction of time for the fragmentation $(F(t), 0 \leq t \leq \zeta)$, with blocks gradually detaching from the spine and then further fragmenting until such a time as they are reduced to dust. We now adopt the opposite perspective and view $\varepsilon^{1/\alpha}F(\zeta - \varepsilon\cdot)$ as being composed of a spine plus other blocks which immigrate into the system and gradually coalesce with one another, before eventually coalescing with the spine. We group the nonspine blocks together into sub-collections formed of those which will attach to the spine at the same time. To this end, for $i \geq -N_\varepsilon + 1$, $m \geq 1$ and $t \geq 0$, define

$$\begin{aligned}H_{i,m}^\varepsilon(t) = {} & \frac{\Delta_{N_\varepsilon+i,m} Z_{N_{\varepsilon n}+i-1}^{1/\alpha}}{(R_\varepsilon(i-1))^{1/\alpha}} \\ & \times \bar{F}_{i,m}^{(Z_{N_\varepsilon+i-1} Y_{N_\varepsilon+i}^\alpha \Delta_{N_\varepsilon+i,m}^\alpha)}\left(t\Delta_{N_\varepsilon+i,m}^\alpha Z_{N_\varepsilon+i-1}\left(R_\varepsilon(i-1)\right)^{-1}+\right),\end{aligned}$$

where $\bar{F}_{i,m}^{(Z_{N_\varepsilon+i-1} Y_{N_\varepsilon+i}^\alpha \Delta_{N_\varepsilon+i,m}^\alpha)}$, $i \in \mathbb{Z}, m \geq 1$ is a collection of conditioned time-reversed fragmentation processes which are independent for distinct i and m, conditionally on $(Z_n, Y_n, \mathbf{\Delta}_n)_{n\geq 0}$. Let $H_i^{\varepsilon,\downarrow}(t)$ be the decreasing rearrangement of all terms involved in the sequences $H^{\varepsilon}_{i,m}(t)$, $m \geq 1$. [Note that $H^{\varepsilon,\downarrow}_i(t) \in \mathcal{S}$ since $\sum_{m\geq 1} \Delta_{N_\varepsilon+i,m} \leq 1$.] Thus, $H^{\varepsilon,\downarrow}_i$ tracks the evolution of the collection of blocks which attach to the spine at time $R_\varepsilon(i)$. The spine coalesces with other blocks only at times $R_\varepsilon(k), k \geq -N_\varepsilon + 1$.

Using this new notation, we can rewrite the expression for the spine decomposition in Proposition 5.1 in a form more adapted to our purposes.

COROLLARY 5.2. *Suppose that $t \in [R_\varepsilon(k+1), R_\varepsilon(k))$ for some $k \geq -N_\varepsilon$. Then $\varepsilon^{1/\alpha} F((\zeta - \varepsilon t)-)$ is the decreasing rearrangement of the masses which make up*:

- $Z^{1/\alpha}_{N_\varepsilon+k}(R_\varepsilon(k))^{-1/\alpha}$;
- $H^{\varepsilon,\downarrow}_i(t)$, $-N_\varepsilon + 1 \leq i \leq k$.

By Lemma 4.7, it is then, more or less, clear what the limit process should be. Recall that $(Z^{\text{bias}}_n, \Theta^{\text{bias}}_n, \mathbf{\Delta}^{\text{bias}}_n)_{n\in\mathbb{Z}}$ is the biased Markov chain introduced in Section 3.3. Let

$$H_{i,m}(t) = \frac{\Delta^{\text{bias}}_{i,m}(Z^{\text{bias}}_{i-1})^{1/\alpha}}{(R(i-1))^{1/\alpha}} \bar{F}^{(Z^{\text{bias}}_{i-1}(Y^{\text{bias}}_i)^\alpha(\Delta^{\text{bias}}_{i,m})^\alpha)}_{i,m}\big(t(\Delta^{\text{bias}}_{i,m})^\alpha Z^{\text{bias}}_{i-1}(R(i-1))^{-1}+\big),$$

where $\bar{F}^{(Z^{\text{bias}}_{i-1}(Y^{\text{bias}}_i)^\alpha(\Delta^{\text{bias}}_{i,m})^\alpha)}_{i,m}$, $i \in \mathbb{Z}, m \geq 1$ is a collection of conditioned time-reversed fragmentation processes which are independent for distinct i and m, conditionally on the chain $(Z^{\text{bias}}_n, Y^{\text{bias}}_n, \mathbf{\Delta}^{\text{bias}}_n)_{n\in\mathbb{Z}}$. Let $H^{\downarrow}_i(t)$ be the decreasing rearrangement of all terms involved in the sequences $H_{i,m}(t)$, $m \geq 1$.

DEFINITION 5.3. Let $C_\infty(0) = \mathbf{0}$. For all $k \in \mathbb{Z}$ and all $t \in [R(k+1), R(k))$, let $C_\infty(t)$ be the decreasing rearrangement of the masses which make up:

- $(Z^{\text{bias}}_k)^{1/\alpha}(R(k))^{-1/\alpha}$;
- $H^{\downarrow}_i(t)$, $i \leq k$.

See Figure 3 for an illustration. In a rough sense, the process C_∞ models the evolution of masses that coalesce, with a regular immigration of infinitesimally small masses. It turns out that reversing time, the distribution of C_∞ can be related to the distribution of a transformed biased fragmentation process in the following way. For all a, recall that $C_{\infty,*}(a)$ denotes the mass at time a of the spine. For $0 \leq t \leq a$, let $C^a_\infty(t)$ denote the subsequence of $C_\infty(t)$ composed of all of the blocks which will contribute to the mass $C_{\infty,*}(a)$ at time a. In other words, we are looking at the coagulation history of $C_{\infty,*}(a)$. Note that, for a fixed time t, each block of $C_\infty(t)$ belongs to a sequence $C^a_\infty(t)$ for some a sufficiently large. We are interested in the distribution of the $(C^a_\infty(t), 0 \leq t \leq a)$ process. By self-similarity it has the same distribution as $(a^{1/\alpha} C^1_\infty(at), 0 \leq t \leq 1)$, so we can focus on the C^1_∞

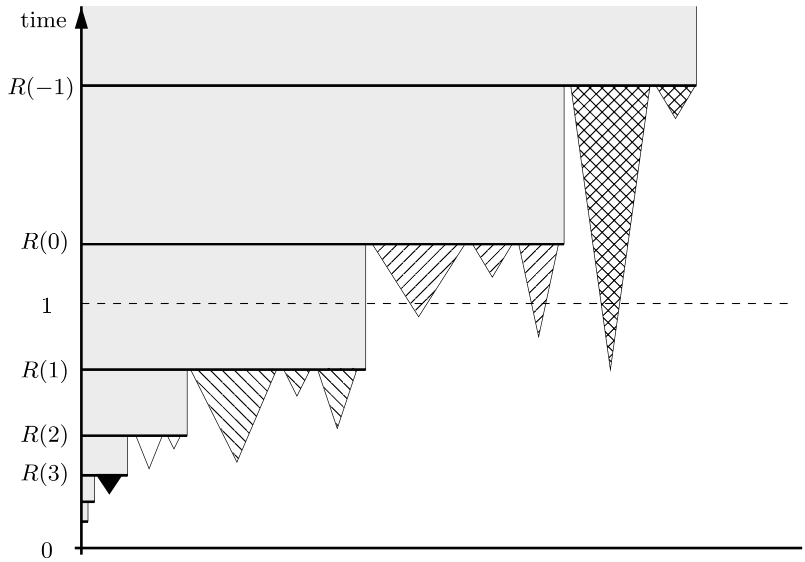

FIG. 3. *The process C_∞. The mass of the spine, which is piecewise constant, is shaded in gray. For each k, the mass of the spine at time $R(k)$ is the coagulation of the mass of the spine at time $R(k)-$ together with the masses $H_k^{\downarrow}(R(k)-)$ of some other fragments present at time $R(k)-$. The collections of patterned triangles represent, from left to right, the processes $H_3^{\downarrow}$, $H_2^{\downarrow}$, $H_1^{\downarrow}$, $H_0^{\downarrow}$ and $H_{-1}^{\downarrow}$ respectively.*

process. The proposition below connects the distribution of this process to that of a biased fragmentation process. We need the following elements:

- Let Z_0^{stat} be distributed according to π_{stat}, and independently, let F be a fragmentation process.
- Let F_{stat} be distributed as the process $(Z_0^{\mathrm{stat}})^{1/\alpha}F(Z_0^{\mathrm{stat}}\cdot)$ conditioned to die at time 1. Let $T_{\mathrm{stat},1}$ be the first jump time of F_{stat}.
- Independently, let U be uniformly distributed on $[0, 1]$.

PROPOSITION 5.4. *For all test functions ϕ,*

$$\begin{aligned}
&\mathbb{E}\big[\phi\big(C_\infty^1(t), 0 \le t \le 1\big)\big] \\
&\quad = \mathbb{E}\big[\log(1 - T_{\mathrm{stat},1}) \\
&\qquad\quad \times \phi\big((1 - T_{\mathrm{stat},1})^{U/\alpha} F_{\mathrm{stat}}\big(1 - (1 - T_{\mathrm{stat},1})^{U} t\big), 0 \le t \le 1\big)\big] \\
&\qquad \times \big(\mathbb{E}\big[\log(1 - T_{\mathrm{stat},1})\big]\big)^{-1}.
\end{aligned} \tag{5.3}$$

PROOF. First note that

$$\mathbb{E}\big[\phi(F_{\mathrm{stat}})\big] = \int_{\mathbb{R}_+} \mathbb{E}\big[\phi\big(x^{1/\alpha}F(x\cdot)\big) | \zeta = x\big]\pi_{\mathrm{stat}}(\mathrm{d}x). \tag{5.4}$$

Recall that the fragmentation F can be constructed from the Markov chain $(Z_n, Y_n, \Delta_n)_{n\geq 0}$ and a collection of conditioned fragmentation processes $\bar{F}_{i,m}$: roughly, F is then composed of a spine $(F_*(T_n), n \geq 1)$, where for $n \geq 1$

$$T_n = Z_0 - Z_0 \prod_{i=1}^n Y_i^\alpha, \qquad F_*(T_n) = \prod_{i=1}^n \Theta_i = \frac{Z_n^{1/\alpha}}{Z_0^{1/\alpha} \prod_{i=1}^n Y_i},$$

from which, at each time T_{n+1}, blocks split off to give rise to conditioned fragmentation processes

$$\frac{Z_n^{1/\alpha} \Delta_{n+1,m}}{Z_0^{1/\alpha} \prod_{i=1}^n Y_i} \bar{F}_{n+1,m}^{(\Delta_{n+1,m}^\alpha Z_n Y_{n+1}^\alpha)}\left(\Delta_{n+1,m}^\alpha Z_n \left(Y_{n+1}^\alpha - \frac{(\cdot - T_{n+1})}{Z_0 \prod_{i=1}^n Y_i^\alpha}\right)\right).$$

These conditioned processes are independent given $(Z_n, Y_n, \Delta_n)_{n\geq 0}$. From (5.4), we see that F_{stat} is constructed similarly from $(Z_n^{\mathrm{stat}}, Y_n^{\mathrm{stat}}, \Delta_n^{\mathrm{stat}})_{n\geq 0}$, a stationary version of $(Z_n, Y_n, \Delta_n)_{n\geq 0}$, and a collection of conditioned fragmentation processes as follows: F_{stat} is composed of a spine $(F_{\mathrm{stat},*}(T_{\mathrm{stat},n}), n \geq 1)$, where for $n \geq 1$

$$T_{\mathrm{stat},n} = 1 - \prod_{i=1}^n (Y_i^{\mathrm{stat}})^\alpha, \qquad F_*(T_{\mathrm{stat},n}) = \frac{(Z_n^{\mathrm{stat}})^{1/\alpha}}{\prod_{i=1}^n Y_i^{\mathrm{stat}}},$$

and from this spine, blocks split off at times $T_{\mathrm{stat},n+1}$ to give rise to conditioned fragmentation processes

$$\frac{(Z_n^{\mathrm{stat}})^{1/\alpha} \Delta_{n+1,m}^{\mathrm{stat}}}{\prod_{i=1}^n Y_i^{\mathrm{stat}}} \times \bar{F}_{n+1,m}^{((\Delta_{n+1,m}^{\mathrm{stat}})^\alpha Z_n^{\mathrm{stat}} (Y_{n+1}^{\mathrm{stat}})^\alpha)}\left((\Delta_{n+1,m}^{\mathrm{stat}})^\alpha Z_n^{\mathrm{stat}}\left((Y_{n+1}^{\mathrm{stat}})^\alpha - \frac{(\cdot - T_{\mathrm{stat},n+1})}{\prod_{i=1}^n (Y_i^{\mathrm{stat}})^\alpha}\right)\right).$$

To finish, multiply F_{stat} by $(1 - T_{\mathrm{stat},1})^{U/\alpha}$, perform the time change $t \mapsto 1 - (1 - T_{\mathrm{stat},1})^U t$ and note that $1 - T_{\mathrm{stat},1} = (Y_1^{\mathrm{stat}})^\alpha$. In order to obtain the expression in (5.3), we must now take a biased version of this stationary construction. It suffices to compare this biased, scaled and time-changed version of F_{stat} with Definition 5.3 to conclude the argument. $\square$

6. Convergence of the full fragmentation.

The aim of this section is to prove Theorem 1.1. Throughout, we will assume that ν is nongeometric and that $\int_{\mathcal{S}} s_1^{-1-\rho} \nu(\mathrm{d}\mathbf{s}) < \infty$ for some $\rho > 0$. We start by establishing several preliminary lemmas.

6.1. *Preliminary lemmas.*

We first deal with an important redundancy in our expression for $(C_\infty(t), t \geq 0)$: for each time t, most of the $H_{i,m}(t)$ do not contribute.

LEMMA 6.1. *Consider the expression for* $(C_\infty(t), t \geq 0)$ *given in Definition* 5.3. *Then almost surely for all* $t > 0$, $t \notin \{R(k), k \in \mathbb{Z}\}$, *only finitely many indices* i *and* m *contribute nonzero blocks to* $C_\infty(t)$.

PROOF. We start by proving that only finitely many indices i and m contribute nonzero blocks to the state a.s. for a *fixed* $t > 0$. By the self-similarity of C_∞, it suffices to prove that this holds for $t = 1$. Recall, moreover, that $R(1) < 1 < R(0)$ a.s. By the first Borel–Cantelli lemma, it suffices to check that the following sum is almost surely finite:

$$\sum_{i \leq 0} \sum_{m=1}^{\infty} \mathbb{P}\big(H_{i,m}(1) \neq \mathbf{0} | Z^{\mathrm{bias}}, Y^{\mathrm{bias}}, \Delta^{\mathrm{bias}}\big)$$

$$= \sum_{i \leq 0} \sum_{m=1}^{\infty} \mathbb{P}\big(\bar{F}_{i,m}^{(Z_{i-1}^{\mathrm{bias}}(Y_i^{\mathrm{bias}})^\alpha (\Delta_{i,m}^{\mathrm{bias}})^\alpha)}\big((\Delta_{i,m}^{\mathrm{bias}})^\alpha Z_{i-1}^{\mathrm{bias}}\big(R(i-1)\big)^{-1}+\big) \neq \mathbf{0} | Z^{\mathrm{bias}}, Y^{\mathrm{bias}}, \Delta^{\mathrm{bias}}\big). \tag{6.1}$$

For any $x > 0$ and any $0 \leq u \leq x$,

$$\mathbb{P}\big(\bar{F}^{(x)}(u) \neq \mathbf{0}\big) = \mathbb{P}(\zeta > x - u | \zeta < x) = \frac{\mathbb{F}_\zeta(x) - \mathbb{F}_\zeta(x-u)}{\mathbb{F}_\zeta(x)}.$$

Using Lemma 2.2(ii), we see that

$$\mathbb{P}\big(\bar{F}^{(x)}(u) \neq \mathbf{0}\big) \leq d \frac{u \exp(-c(x-u))}{\mathbb{F}_\zeta(x)}$$

for some constants $c, d > 0$. Hence (6.1) is bounded above by

$$d \sum_{i \leq 0} \sum_{m=1}^{\infty} Z_{i-1}^{\mathrm{bias}} (\Delta_{i,m}^{\mathrm{bias}})^\alpha \big(R(i-1)\big)^{-1} \frac{\exp(-c Z_{i-1}^{\mathrm{bias}} (Y_i^{\mathrm{bias}})^\alpha (\Delta_{i,m}^{\mathrm{bias}})^\alpha (1 - (R(i))^{-1}))}{\mathbb{F}_\zeta(Z_{i-1}^{\mathrm{bias}} (Y_i^{\mathrm{bias}})^\alpha (\Delta_{i,m}^{\mathrm{bias}})^\alpha)}$$

[note that $R(i) > 1$ for all $i \leq 0$]. Using the monotonicity of $\mathbb{F}_\zeta$, we see that we only require the finiteness of

$$\sum_{i \leq 0} \frac{(R(i))^{-1}}{\mathbb{F}_\zeta(Z_{i-1}^{\mathrm{bias}} (Y_i^{\mathrm{bias}})^\alpha)} \sum_{m=1}^{\infty} Z_{i-1}^{\mathrm{bias}} \big(Y_i^{\mathrm{bias}}\big)^\alpha \big(\Delta_{i,m}^{\mathrm{bias}}\big)^\alpha \times \exp\big(-c Z_{i-1}^{\mathrm{bias}} \big(Y_i^{\mathrm{bias}}\big)^\alpha \big(\Delta_{i,m}^{\mathrm{bias}}\big)^\alpha \big(1 - \big(R(i)\big)^{-1}\big)\big). \tag{6.2}$$

Since $\sum_{m=1}^{\infty} \Delta_{i,m}^{\mathrm{bias}} < 1$, we have $\Delta_{i,m}^{\mathrm{bias}} \leq m^{-1}$, and so the inner sum in m is bounded above by

$$C\big(1 - \big(R(i)\big)^{-1}\big)^{-1} \sum_{m=1}^{\infty} \exp\big(-c' Z_{i-1}^{\mathrm{bias}} \big(Y_i^{\mathrm{bias}}\big)^\alpha m^{-\alpha} \big(1 - \big(R(i)\big)^{-1}\big)\big), \tag{6.3}$$

for some constants $C > 0$ and $0 < c' < c$. Now observe that for $\theta > 0$,

$$\sum_{m=1}^{\infty} \exp(-\theta m^{-\alpha}) \le \int_0^{\infty} \exp(-\theta x^{-\alpha})\,\mathrm{d}x = \Gamma\left(1 - \frac{1}{\alpha}\right)\theta^{1/\alpha},$$

and so (6.3) is bounded above by

$$C'(Z_{i-1}^{\mathrm{bias}})^{1/\alpha} Y_i^{\mathrm{bias}} (1 - (R(i))^{-1})^{-1+1/\alpha}$$

for some constant $C' > 0$. Since by Lemma 4.1 we have that $R(i) \to \infty$ as $i \to -\infty$ almost surely, there exists $i_0 < 0$ such that for all $i \le i_0$, $(1-(R(i))^{-1})^{-1+1/\alpha}$ is bounded above, say by 2. For $i \le i_0$, let

$$B_i = \frac{2(R(i))^{-1}}{\mathbb{F}_\zeta(Z_{i-1}^{\mathrm{bias}}(Y_i^{\mathrm{bias}})^\alpha)} (Z_{i-1}^{\mathrm{bias}})^{1/\alpha} Y_i^{\mathrm{bias}}.$$

Then by Lemma A.11,

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \log(B_{-n}) &= \lim_{n\to\infty} \frac{\alpha}{n} \sum_{j=-n+1}^{0} \log(Y_j^{\mathrm{bias}}) - \lim_{n\to\infty} \frac{1}{n} \log(\mathbb{F}_\zeta(Z_{-n-1}^{\mathrm{bias}}(Y_{-n}^{\mathrm{bias}})^\alpha)) \\
&\quad + \lim_{n\to\infty} \frac{1}{\alpha n} \log(Z_{-n-1}^{\mathrm{bias}}) + \lim_{n\to\infty} \frac{1}{n} \log(Y_{-n}^{\mathrm{bias}}) \\
&= \alpha\mu < 0.
\end{aligned}$$

Hence, by Cauchy's root test, (6.2) is almost surely finite.

The statement of the lemma now follows easily: we know that almost surely for all rational numbers $q \in \mathbb{Q} \cap (0, \infty)$, only finitely many indices i and m contribute nonzero blocks to the state $C_\infty(q)$. On this event of probability one, for each positive time $t \notin \{R(k), k \in \mathbb{Z}\}$, say $t \in (R(k+1), R(k))$, consider a rational number $q \in (t, R(k))$. Since all indices i, m that contribute to the state $C_\infty(t)$ also contribute to the state $C_\infty(q)$, the statement follows. $\square$

LEMMA 6.2. *C_∞ is almost surely a càdlàg process taking values in $(\mathcal{S}, d)$.*

PROOF. We first prove that, with probability one, $C_\infty(t) \in \mathcal{S}$ for all $t \ge 0$. By Lemma 6.1, with probability one, for all $t \notin \{R(k), k \in \mathbb{Z}\}$, $t > 0$, $\|C_\infty(t)\|_1 < \infty$. If $t = 0$, $C_\infty(t) = \mathbf{0}$. Finally, for $t = R(k)$ for some k, we can argue via monotonicity. Let $u \in (R(k), R(k-1))$. Then $\|C_\infty(t)\|_1 \le \|C_\infty(u)\|_1 < \infty$ on the event of probability one we just considered.

We now turn to the continuity properties. We first show that $\|C_\infty(t)\|_1 \to 0$ as $t \downarrow 0$. First, recall that $C_{\infty,*}(t) \to 0$ as $t \downarrow 0$ (this was noted at the beginning of Section 4, as a consequence of Lemma 4.1). Now fix $\varepsilon > 0$. Then we can find $t_\varepsilon > 0$ such that $C_{\infty,*}(t_\varepsilon) < \varepsilon/2$. Moreover, we can always assume that t_ε is not one of the $R(k)$ and, therefore, that there are only finitely many indices i and

m which contribute to the state of $C_\infty(t_\varepsilon)$. Since the total mass in each of these fragmentations is decreasing to 0, it follows that there exists some time $t'_\varepsilon \in (0, t_\varepsilon)$ such that $\|C_\infty(t'_\varepsilon)\|_1 < \varepsilon$.

Now consider a fixed time $t \in (0, \infty)$, and suppose that $t \in [R(k+1), R(k))$ for some $k \in \mathbb{Z}$. Take $(t_n)_{n\in\mathbb{N}}$ to be such that $t_1 < R(k)$ and $t_n \downarrow t$. Then $C_\infty(t_n)$ is the decreasing rearrangement of $C_{\infty,*}(R(k+1))$ together with the blocks of $H_{i,m}(t_n)$ for $m \geq 1$, $i \leq k$. There are only finitely many indices i and m which contribute to the nonzero blocks of $C_\infty(t_1)$, and blocks can only disappear as t_n decreases in $(R(k+1), R(k))$. Hence

$$\sum_{i=-\infty}^{k} \sum_{m=1}^{\infty} \|H_{i,m}(t_n) - H_{i,m}(t_\infty)\|_1$$

is a sum with only finitely many nonzero terms. Since $\bar{F}_{i,m}^{(Z_{i-1}^{\mathrm{bias}}(Y_i^{\mathrm{bias}})^\alpha(\Delta_{i,m}^{\mathrm{bias}})^\alpha)}(\cdot+)$ is càdlàg for each i, m, each term converges to 0, and so the whole sum converges to 0. Using Lemma A.1, we deduce that $\|C_\infty(t_n) - C_\infty(t)\|_1 \to 0$.

The existence of a left limit at time $t \in (0, \infty)$ such that $t \in (R(k+1), R(k))$ follows similarly, because again the same finite collection of indices i, m are involved for all $t' \in (t - \varepsilon, t)$ for sufficiently small $\varepsilon > 0$. Finally, for times t such that $t = R(k)$ for some $k \in \mathbb{Z}$, there is a slight difference since the number of indices in the set $\{(k, m), m \geq 1\}$ that are involved may be infinite. However, the result still holds by Lemma A.1, since

$$\sum_{m\geq m_\eta} \Delta_{k,m}^{\mathrm{bias}} \left(Z_{k-1}^{\mathrm{bias}}\right)^{1/\alpha} \left(R(k-1)\right)^{-1/\alpha} \leq \eta$$

for some finite m_η and all $\eta > 0$. □

We now turn to an important tightness result, which will allow us to ignore, in the proof of Theorem 1.1, the possibility that there exist blocks in the system at time $R_\varepsilon(k)$ which persist for a very long time before coalescing with the spine. From now on, we use its spine decomposition, as discussed in the previous section. For each $\varepsilon > 0$ and each $k \in \mathbb{Z}$, let $I_\varepsilon(k)$ be the largest positive integer i such that at least one nonspine block present at time $R_\varepsilon(k)$ attaches to the spine at time $R_\varepsilon(k - i)$. Formally, when $k \geq -N_\varepsilon + 1$,

$$I_\varepsilon(k) = \sup\{1 \leq i \leq k + N_\varepsilon - 1 : H_{k-i}^{\varepsilon,\downarrow}(R_\varepsilon(k)) \neq \mathbf{0}\},$$

with the convention that $I_\varepsilon(k) = 0$ if this is the supremum of an empty set. We also set $I_\varepsilon(k) = 0$ when $k < -N_\varepsilon + 1$. Our goal is prove that with a high probability $I_\varepsilon(k)$ is not too large, simultaneously for all ε small enough.

LEMMA 6.3 (Tightness). *Let $z > 0$ be fixed and such that the convergence in distribution of Lemma 4.7 holds. Consider a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of strictly positive* *real numbers converging to* 0. *Then there exists a family of positive integers* $(j_\eta(k))$ *indexed by* $k \in \mathbb{Z}$, $\eta > 0$ *such that*

$$\mathbb{P}\big(I_{\varepsilon_n}(k) \geq j_\eta(k) | Z_0 = z\big) \leq \eta \qquad \forall n \in \mathbb{N}.$$

Consequently, $\forall n \in \mathbb{N}$,

$$\mathbb{P}\big(\{I_{\varepsilon_n}(0) \geq j_\eta(0)\} \cup \{\exists k \in \mathbb{Z} \setminus \{0\} : I_{\varepsilon_n}(k) \geq j_{\eta/k^2}(k)\} | Z_0 = z\big) \\ \leq (1 + 2\pi^2/6)\eta.$$

Having in mind the construction of C_∞, we define similarly $I(k)$, $k \in \mathbb{Z}$ to be the largest integer $i \geq 1$ such at least one nonspine block present at time $R(k)$ attaches to the spine at time $R(k-i)$ [and $I(k) = 0$ if no such $i \geq 1$ exists]. As a direct consequence of Lemmas 6.1 and 6.3, we have the following result, which is in the form we will use later for the proof of Theorem 1.1.

LEMMA 6.4. *Let* $z > 0$ *be fixed and such that the convergence of Lemma* 4.7 *holds*. *Consider a sequence* $(\varepsilon_n)_{n\in\mathbb{N}}$ *of strictly positive real numbers converging to* 0. *Then there exists a family of positive integers* $(i_\eta(k))$ *indexed by* $k \in \mathbb{Z}$, $\eta > 0$ *such that*

$$\mathbb{P}\big(\exists k \in \mathbb{Z} : I_{\varepsilon_n}(k) \geq i_\eta(k) | Z_0 = z\big) \leq \eta \qquad \forall n \in \mathbb{N}$$

and

$$\mathbb{P}\big(\exists k \in \mathbb{Z} : I(k) \geq i_\eta(k)\big) \leq \eta.$$

In order to prove Lemma 6.3, we gather together some technical results in the following lemma. They follow from Lemmas A.8, A.9, A.10 and A.12 in the Appendix.

LEMMA 6.5. *We have that for* $p > 0$ *and* $\delta > 0$ *sufficiently small*,

$$\mathbb{E}\big[\big|\log\big(Z_0^{\mathrm{stat}}\big)\big|^p\big] < \infty, \qquad \mathbb{E}\big[\big(\log\big(Y_1^{\mathrm{stat}}\big)\big)^p\big] < \infty$$

and

$$\mathbb{E}\big[\big|\log\big(\mathbb{F}_\zeta\big(Z_0^{\mathrm{stat}}\big(Y_1^{\mathrm{stat}}\big)^\alpha\big)\big)\big|^{1+\delta}\big] < \infty.$$

Moreover, *there exist constants* $A < \infty$ *and* $c_Y \in (0, 1)$ *such that*

$$\mathbb{E}\left[\prod_{i=2}^{n} \big(Y_i^{\mathrm{stat}}\big)^\alpha\right] \leq A c_Y^n.$$

PROOF OF LEMMA 6.3. The proof is similar for all $k \in \mathbb{Z}$, and so, in order to ease the notation, we will only write it out in the case where $k = 0$. In the following lines, $\eta > 0$ is fixed, and C denotes a finite positive constant that may vary from line to line.

Our main goal is to prove the existence of $N_\eta \in \mathbb{Z}$ and $\varepsilon_\eta > 0$ such that

$$\mathbb{P}\big(I_\varepsilon(0) \geq N_\eta | Z_0 = z\big) \leq \eta \qquad \forall 0 \leq \varepsilon \leq \varepsilon_\eta. \tag{6.4}$$

Since $(\varepsilon_n)_{n \in \mathbb{N}}$ is a sequence of strictly positive real numbers converging to 0, this will imply the existence of a positive integer $j_\eta(0)$ such that

$$\mathbb{P}\big(I_{\varepsilon_n}(0) \geq j_\eta(0) | Z_0 = z\big) \leq \eta \qquad \forall n \in \mathbb{N},$$

as expected.

Now, in order prove (6.4), note that for all integers $N \geq 1$, following the main lines of the proof of Lemma 6.1, we obtain that

$$\begin{aligned}
&\mathbb{P}\big(I_\varepsilon(0) \geq N | (Z, Y, \Delta)\big) \\
&\quad \leq C \sum_{i=N}^{N_\varepsilon - 1} \frac{R_\varepsilon(0) R_\varepsilon(-i)^{-1}}{(1 - R_\varepsilon(0) R_\varepsilon(-i)^{-1})^{1-1/\alpha}} \frac{Z_{N_\varepsilon - i - 1}^{1/\alpha} Y_{N_\varepsilon - i}}{\mathbb{F}_\zeta(Z_{N_\varepsilon - i - 1} Y_{N_\varepsilon - i}^\alpha)} \\
&\quad \leq C A_\varepsilon B_\varepsilon(N),
\end{aligned}$$

where

$$\begin{aligned}
A_\varepsilon &= \frac{R_\varepsilon(0) \exp(-\alpha S_{N_\varepsilon} + \log(\varepsilon / Z_0))}{(1 - R_\varepsilon(0) R_\varepsilon(-1)^{-1})^{1-1/\alpha}}, \\
B_\varepsilon(N) &= \sum_{i=N}^{N_\varepsilon - 1} \left(\prod_{k=-i+1}^{0} Y_{N_\varepsilon + k}^\alpha \right) \frac{Z_{N_\varepsilon - i - 1}^{1/\alpha} Y_{N_\varepsilon - i}}{\mathbb{F}_\zeta(Z_{N_\varepsilon - i - 1} Y_{N_\varepsilon - i}^\alpha)}.
\end{aligned}$$

Consequently, for every $A > 0$,

$$\begin{aligned}
\mathbb{P}\big(I_\varepsilon(0) \geq N | Z_0 = z\big) \leq \frac{\eta}{3} \mathbb{P}\big(A_\varepsilon B_\varepsilon(N) \leq \eta / 3C | Z_0 = z\big) \\
+ \mathbb{P}(A_\varepsilon \geq A / 3C | Z_0 = z) + \mathbb{P}\big(B_\varepsilon(N) \geq \eta / A | Z_0 = z\big).
\end{aligned}$$

But we know from Lemma 4.7 that when $\varepsilon \to 0$, conditional on $Z_0 = z$,

$$A_\varepsilon \overset{\text{law}}{\to} \frac{R(0) (Y_1^{\text{bias}})^{\alpha U}}{(1 - R(0) R(-1)^{-1})^{1-1/\alpha}},$$

and the limit is almost surely finite. Hence if we fix A sufficiently large, then for all ε sufficiently small, say $\varepsilon \leq \varepsilon_0$, and all $N \geq 1$,

$$\mathbb{P}\big(I_\varepsilon(0) \geq N | Z_0 = z\big) \leq \frac{2\eta}{3} + \mathbb{P}\big(B_\varepsilon(N) \geq \eta / A | Z_0 = z\big).$$

Let us now deal with this last probability (A is now fixed). We have

$$\begin{aligned}
&\mathbb{P}\big(B_\varepsilon(N) \geq \eta/A | Z_0 = z\big) \\
&\quad \leq \sum_{i=N}^{\infty} \mathbb{P}\left(\left(\prod_{k=-i+1}^{0} Y_{N_\varepsilon+k}^{\alpha}\right) \frac{Z_{N_\varepsilon-i-1}^{1/\alpha} Y_{N_\varepsilon-i}}{\mathbb{F}_\zeta(Z_{N_\varepsilon-i-1} Y_{N_\varepsilon-i}^{\alpha})} \mathbb{1}_{\{i \leq N_\varepsilon - 1\}} \geq \frac{6\eta}{A\pi^2 i^2} \middle| Z_0 = z\right)
\end{aligned}$$

(since $\sum_{i \geq N} i^{-2} \leq \pi^2/6$). Recall that, on the event $\{Z_0 = z\}$, we have $\{N_\varepsilon = j\} = \{S_j \leq \alpha^{-1} \log(\varepsilon/z) < S_{j+1}\}$. Hence

$$\begin{aligned}
&\mathbb{P}\left(\left(\prod_{k=-i+1}^{0} Y_{N_\varepsilon+k}^{\alpha}\right) \frac{Z_{N_\varepsilon-i-1}^{1/\alpha} Y_{N_\varepsilon-i}}{\mathbb{F}_\zeta(Z_{N_\varepsilon-i-1} Y_{N_\varepsilon-i}^{\alpha})} \mathbb{1}_{\{i \leq N_\varepsilon - 1\}} \geq \frac{6\eta}{A\pi^2 i^2} \middle| Z_0 = z\right) \\
&\qquad = \sum_{j=i+1}^{\infty} \mathbb{P}\left(\left(\prod_{k=-i+1}^{0} Y_{j+k}^{\alpha}\right) \frac{Z_{j-i-1}^{1/\alpha} Y_{j-i}}{\mathbb{F}_\zeta(Z_{j-i-1} Y_{j-i}^{\alpha})} \geq \frac{6\eta}{A\pi^2 i^2}, \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. S_j \leq \frac{\log(\varepsilon/z)}{\alpha} < S_{j+1} \middle| Z_0 = z\right) \\
&\qquad = \sum_{j=0}^{\infty} \mathbb{P}\left(\left(\prod_{k=-i+1}^{0} Y_{j+i+1+k}^{\alpha}\right) \frac{Z_j^{1/\alpha} Y_{j+1}}{\mathbb{F}_\zeta(Z_j Y_{j+1}^{\alpha})} \geq \frac{6\eta}{A\pi^2 i^2}, \right. \\
&\qquad\qquad\qquad\qquad\qquad \left. S_{j+i+1} \leq \frac{\log(\varepsilon/z)}{\alpha} < S_{j+i+2} \middle| Z_0 = z\right) \\
&\qquad = \sum_{j=0}^{\infty} \mathbb{E}\left[g_i\left(Z_j, \frac{\log(\varepsilon/z)}{\alpha} - S_j\right) \middle| Z_0 = z\right],
\end{aligned}$$

where for $x > 0$, $y \in \mathbb{R}$ and $i \geq 1$,

$$g_i(x, y) = \mathbb{1}_{\{y \geq 0\}} \mathbb{P}\left(\left(\prod_{k=2}^{i+1} Y_k^{\alpha}\right) \frac{x^{1/\alpha} Y_1}{\mathbb{F}_\zeta(x Y_1^{\alpha})} \geq \frac{6\eta}{A\pi^2 i^2}, S_{i+1} \leq y < S_{i+2} \middle| \zeta = x\right).$$

So, finally,

$$\mathbb{P}\big(B_\varepsilon(N) \geq \eta/A | Z_0 = z\big) \leq \sum_{j=0}^{\infty} \mathbb{E}\big[g\big(Z_j, \alpha^{-1} \log(\varepsilon/z) - S_j\big) | Z_0 = z\big],$$

where $g(x, y) = \sum_{i \geq N} g_i(x, y)$. Assume for the moment that this function g satisfies conditions (a) and (b) of Theorem 4.4. Then, as a consequence of that theorem,

$$\limsup_{\varepsilon \to 0} \mathbb{P}\big(B_\varepsilon(N) \geq \eta/A | Z_0 = z\big) \leq \frac{1}{\mu} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \sum_{i \geq N} g_i(x, y) \pi_{\mathrm{stat}}(\mathrm{d}x) < \infty.$$

We then use the monotone convergence theorem to conclude that there exists some N_η and then some ε_η ($\le \varepsilon_0$) such that

$$\mathbb{P}\big(B_\varepsilon(N_\eta) \ge \eta/A | Z_0 = z\big) \le \frac{\eta}{3} \qquad \forall \varepsilon \le \varepsilon_\eta,$$

which was the missing piece we needed to get (6.4).

It remains to check that g satisfies conditions (a) and (b) of Theorem 4.4. Note that we do not even know yet that $g(x, y) < \infty$ for Lebesgue a.e. x, y. We start with (b). For this, note that if $y \in [n, n+1)$ and $y \in [S_{i+1}, S_{i+2})$, then $S_{i+2} > n$ and $S_{i+1} < n+1$. Moreover, the number of integers $n \in (S_{i+1} - 1, S_{i+2})$ is smaller than $S_{i+2} - (S_{i+1} - 1) + 1 = \log(Y_{i+2}) + 2$. Thus

$$\begin{aligned}\int_0^\infty \sum_{n \in \mathbb{Z}_+} \sup_{y \in [n,n+1)} g(x, y)\pi_{\text{stat}}(\mathrm{d}x) &\\ \le \sum_{i \ge N} \mathbb{E}\big[\big(\log\big(Y_{i+2}^{\text{stat}}\big) + 2\big) &\\ \times \mathbb{1}_{\{(\prod_{k=2}^{i+1}(Y_k^{\text{stat}})^\alpha)(Z_0^{\text{stat}})^{1/\alpha} Y_1^{\text{stat}} / (\mathbb{F}_\zeta(Z_0^{\text{stat}}(Y_1^{\text{stat}})^\alpha)) \ge 6\eta/(A\pi^2 i^2)\}}\big].&\end{aligned}$$

Fix $\delta \in (0, 1)$. By Hölder's inequality, and for any $c \in (0, 1)$,

$$\begin{aligned}&\mathbb{E}\big[\big(\log\big(Y_{i+2}^{\text{stat}}\big) + 2\big)\mathbb{1}_{\{(\prod_{k=2}^{i+1}(Y_k^{\text{stat}})^\alpha)(Z_0^{\text{stat}})^{1/\alpha} Y_1^{\text{stat}} / (\mathbb{F}_\zeta(Z_0^{\text{stat}}(Y_1^{\text{stat}})^\alpha)) \ge 6\eta/(A\pi^2 i^2)\}}\big] \\ &\quad \le \mathbb{E}\big[\big(\log\big(Y_1^{\text{stat}}\big) + 2\big)^{1/\delta}\big]^\delta \mathbb{P}\left(\left(\prod_{k=2}^{i+1}\big(Y_k^{\text{stat}}\big)^\alpha\right) \frac{(Z_0^{\text{stat}})^{1/\alpha} Y_1^{\text{stat}}}{\mathbb{F}_\zeta(Z_0^{\text{stat}}(Y_1^{\text{stat}})^\alpha)} \ge \frac{6\eta}{A\pi^2 i^2}\right)^{1-\delta} \\ &\quad \le C\left(\mathbb{P}\left(c^{-i}\left(\prod_{k=2}^{i+1}\big(Y_k^{\text{stat}}\big)^\alpha\right) \ge \frac{6\eta}{A\pi^2 i^2}\right)^{1-\delta}\right. \\ &\qquad\qquad \left. + \mathbb{P}\left(c^i \frac{(Z_0^{\text{stat}})^{1/\alpha} Y_1^{\text{stat}}}{\mathbb{F}_\zeta(Z_0^{\text{stat}}(Y_1^{\text{stat}})^\alpha)} \ge 1\right)^{1-\delta}\right),\end{aligned}$$

where, in the last inequality, we have used the finiteness of the expectation $\mathbb{E}[(\log(Y_1^{\text{stat}}))^{1/\delta}]$; see Lemma 6.5. By Markov's inequality, the first probability on the right-hand side above is smaller than $Ci^2(c_Y/c)^i$, where c_Y is defined in Lemma 6.5. For the second term on the right-hand side, first take the logarithm inside the probability, and then use Markov's inequality to bound it from above by

$$\begin{aligned}Ci^{-(1+2\delta)}\big(\mathbb{E}\big[\big|\log\big(Z_0^{\text{stat}}\big)\big|^{1+2\delta}\big] + \mathbb{E}\big[\big|\log\big(Y_1^{\text{stat}}\big)\big|^{1+2\delta}\big] &\\ + \mathbb{E}\big[\big|\log\big(\mathbb{F}_\zeta\big(Z_0^{\text{stat}}\big(Y_1^{\text{stat}}\big)^\alpha\big)\big)\big|^{1+2\delta}\big]\big).&\end{aligned}$$

By Lemma 6.5, this sum of three expectations is finite for $\delta > 0$ small enough. Consequently, for $c \in (c_Y, 1)$,

$$\int_0^\infty \sum_{n \in \mathbb{Z}_+} \sup_{y \in [n,n+1)} g(x, y)\pi_{\text{stat}}(\mathrm{d}x) \le \sum_{i \ge N} C\left(i^2\left(\frac{c_Y}{c}\right)^{i(1-\delta)} + i^{-(1+2\delta)(1-\delta)}\right),$$

and this sum on $i \geq N$ is finite as soon as $(1+2\delta)(1-\delta) > 1$, that is, as soon as $\delta < 1/2$. Hence, condition (b) of Theorem 4.4 is satisfied.

To get condition (a), note that we have shown in the last paragraph that for all $c \in (0,1)$ and all $\delta > 0$ small enough,

$$\mathbb{P}\left(\left(\prod_{k=2}^{i+1}(Y_k^{\mathrm{stat}})^\alpha\right)\frac{(Z_0^{\mathrm{stat}})^{1/\alpha}Y_1^{\mathrm{stat}}}{\mathbb{F}_\zeta(Z_0^{\mathrm{stat}}(Y_1^{\mathrm{stat}})^\alpha)} \geq \frac{6\eta}{A\pi^2 i^2}\right) \leq C\big(i^2(c_Y/c)^i + i^{-1-2\delta}\big),$$

where C depends both on c and δ, but not on $i \geq N$. Consequently, considering $c \in (c_Y, 1)$, we get

$$\int_0^\infty \sum_{i\geq N}\mathbb{P}\left(\left(\prod_{k=2}^{i+1}Y_k^\alpha\right)\frac{x^{1/\alpha}Y_1}{\mathbb{F}_\zeta(xY_1^\alpha)} \geq \frac{6\eta}{A\pi^2 i^2}\middle|\zeta = x\right)\pi_{\mathrm{stat}}(\mathrm{d}x) < \infty,$$

hence for Lebesgue a.e. $x > 0$, $\sum_{i\geq N}\mathbb{P}((\prod_{k=2}^{i+1}Y_k^\alpha)\frac{x^{1/\alpha}Y_1}{\mathbb{F}_\zeta(xY_1^\alpha)} \geq \frac{6\eta}{A\pi^2 i^2}|\zeta = x)$ is finite. For those x, $g(x,y) < \infty$ for all $y \geq 0$ and we can apply the dominated convergence theorem to deduce that $g(x,\cdot)$ is continuous at each point which is not an atom of one of the S_i, $i \geq 1$. The Lebesgue measure of this set of atoms is 0; hence condition (a) is also satisfied. □

6.2. *Proof of Theorem* 1.1. Consider a sequence (ε_n) of strictly positive real numbers converging to 0, and recall from Corollary 5.2 the spine construction of

$$\big(\varepsilon_n^{1/\alpha}F\big((\zeta - \varepsilon_n t)-\big), 0 \leq t \leq \zeta/\varepsilon_n\big)$$

in terms of the Markov chain $(Z_k, Y_k, \mathbf{\Delta}_k)_{k\geq 0}$ and the time-reversed fragmentations

$$\bar{F}_{i,m}^{(Z_{N_{\varepsilon_n}+i-1}Y^\alpha_{N_{\varepsilon_n}+i}\Delta^\alpha_{N_{\varepsilon_n}+i,m})}, \qquad i \in \mathbb{Z}, m \geq 1,$$

where these fragmentations are conditionally independent given $(Z_k, Y_k, \mathbf{\Delta}_k)_{k\geq 0}$. For the rest of this proof, we fix $z > 0$ such that the conditional convergence of Lemma 4.7 holds.

Step 1. As we have already mentioned, an important technical issue is the possibility that, among the blocks present at time t, there are some which will persist in the system for a very long time before coalescing with spine. In other words, we would like to be able to say that $H_i^{\varepsilon_n,\downarrow}$ does not contribute to the state for large negative i (uniformly in n). For this reason, we introduce, for all $\eta > 0$ and $n \in \mathbb{N}$, the modified process

$$\big(\varepsilon_n^{1/\alpha}F^{(\eta)}\big((\zeta - \varepsilon_n t)-\big), 0 \leq t \leq \zeta/\varepsilon_n\big)$$

whose spine decomposition is constructed from $(Z_k, Y_k, \mathbf{\Delta}_k)_{k\geq 0}$ in a way very similar to $(\varepsilon_n^{1/\alpha}F((\zeta - \varepsilon_n\cdot)-))$ except that some terms are omitted: for $t \in [R_{\varepsilon_n}(k+1), R_{\varepsilon_n}(k))$, $k \geq -N_{\varepsilon_n}$, we take $\varepsilon_n^{1/\alpha}F^{(\eta)}((\zeta - \varepsilon_n t)-)$ to be the decreasing rearrangement of the terms involved in:

- $Z^{1/\alpha}_{N_{\varepsilon_n}+k}(R_{\varepsilon_n}(k))^{-1/\alpha}$,
- $H_i^{\varepsilon_n,\downarrow}(t)$, $k - i_\eta(k) \le i \le k$,

where the (deterministic) integers $i_\eta(k)$ are those introduced in Lemma 6.4. If $t > \zeta/\varepsilon_n$, we set $F^{(\eta)}((\zeta - \varepsilon_n t)-) = \mathbf{1}$. By Lemma 6.4, the processes $F^{(\eta)}((\zeta - \varepsilon_n \cdot)-)$ and $F((\zeta - \varepsilon_n \cdot)-)$ are identical with a high probability independently of n, namely

$$\mathbb{P}\big((\varepsilon_n^{1/\alpha} F^{(\eta)}((\zeta - \varepsilon_n t)-), t \ge 0) \ne (\varepsilon_n^{1/\alpha} F((\zeta - \varepsilon_n t)-), t \ge 0) | \zeta = z\big) \le \eta.$$

Consequently, for every bounded continuous test function $f : \mathcal{S} \to \mathbb{R}$,

$$\big|\mathbb{E}[f(\varepsilon_n^{1/\alpha} F((\zeta - \varepsilon_n \cdot)-)) | \zeta = z] - \mathbb{E}[f(\varepsilon_n^{1/\alpha} F^{(\eta)}((\zeta - \varepsilon_n \cdot)-)) | \zeta = z]\big| \le C\eta,$$

where C is independent of n and η. Similarly, again by Lemma 6.4, $|\mathbb{E}[f(C_\infty)] - \mathbb{E}[f(C_\infty^{(\eta)})]| \le C\eta$, where $C_\infty^{(\eta)}(t)$ is defined for $t \in [R(k+1), R(k))$ to be the decreasing rearrangement of the terms involved in:

- $(Z_k^{\mathrm{bias}})^{1/\alpha}(R(k))^{-1/\alpha}$,
- $H_i^{\downarrow}(t)$, $k - i_\eta(k) \le i \le k$.

Therefore, the expected convergence in distribution will be proved if we show that the process $(\varepsilon_n^{1/\alpha} F^{(\eta)}((\zeta - \varepsilon_n \cdot)-))$ converges in distribution (conditional on $\zeta = z$) to $C_\infty^{(\eta)}$, for each $\eta > 0$.

Step 2. Fix $\eta > 0$. Our goal is to prove that conditionally on $\zeta = z$, there exist versions of $(\varepsilon_n^{1/\alpha} F^{(\eta)}((\zeta - \varepsilon_n t)-), 0 \le t \le \zeta/\varepsilon_n)$, $n \in \mathbb{N}$, that converge to a version of $C_\infty^{(\eta)}$, almost surely as $\varepsilon_n \to 0$. With step 1 above, this will clearly entail Theorem 1.1.

By Lemma 4.7 and the Skorokhod representation theorem, conditionally on $\zeta = z$, there exist versions of

$$\left((Z_{N_{\varepsilon_n}+k}, Y_{N_{\varepsilon_n}+k}, \mathbf{\Delta}_{N_{\varepsilon_n}+k})_{k\in\mathbb{Z}}, \frac{1}{\alpha}\log(\varepsilon_n/\zeta) - S_{N_{\varepsilon_n}}\right) \tag{6.5}$$

that converge almost surely as $\varepsilon_n \to 0$ to a version of $((Z^{\mathrm{bias}}, Y^{\mathrm{bias}}, \mathbf{\Delta}^{\mathrm{bias}}), U\log(Y_1^{\mathrm{bias}}))$. From now on, we always consider these versions. Using Lemma A.5, we get the joint Skorokhod convergence in distribution, conditional on $\zeta = z$, of the càdlàg processes

$$H_{i,m}^{\varepsilon_n} \to H_{i,m} \qquad \text{as } \varepsilon_n \to 0, i \in \mathbb{Z}, m \ge 1.$$

By the Skorokhod representation theorem, we may again assume that these convergences hold almost surely. Without changing notation, we work with these versions for the rest of this proof. In fact, we will implicitly work on the event of probability one where the convergence of (6.5) to $((Z^{\mathrm{bias}}, Y^{\mathrm{bias}}, \mathbf{\Delta}^{\mathrm{bias}}), U\log(Y_1^{\mathrm{bias}}))$ holds, as well as all convergences of processes $H_{i,m}^{\varepsilon_n}$ to $H_{i,m}$, $i \in \mathbb{Z}$, $m \ge 1$.

Step 2(a). We then claim that for each $i \in \mathbb{Z}$,

$$H_i^{\varepsilon_n,\downarrow} \to H_i^{\downarrow} \qquad \text{as } \varepsilon_n \to 0,$$

in the Skorokhod sense (for the distance d on $\mathcal{S}$). To see this, we use Proposition A.3 and Lemma A.6 from the Appendix. For this, fix a time $t \geq 0$ and a sequence (t_{ε_n}) converging to t. The integer i being fixed, our goal is to check that the functions $H_i^{\varepsilon_n,\downarrow}$ and $H_i^{\downarrow}$ satisfy assertions (a), (b) and (c) of Proposition A.3 for the sequence of times (t_{ε_n}). In order to do this, we distinguish three cases: $t \in [0,\infty) \setminus \{R(i), 0\}$, $t = 0$ and $t = R(i)$.

First assume that $t \neq R(i)$ and $t > 0$. Since a reversed fragmentation process $\bar{F}^{(x)}$ almost surely does not jump at any given fixed time except x, the processes $H_{i,m}, m \geq 1$ cannot jump simultaneously on $\mathbb{R}_+ \setminus \{R(i)\}$. So at most one process among $H_{i,m}, m \geq 1$ jumps at time t (almost surely). Let m_t be the index of this process if it exists. For $m \neq m_t$, $H_{i,m}^{\varepsilon_n}(t_{\varepsilon_n}) \to H_{i,m}(t)$ and this leads to the convergence in $\mathcal{S}$ of the decreasing rearrangement of all terms involved in at least one sequence $H_{i,m}^{\varepsilon_n}(t_{\varepsilon_n})$ for some $m \neq m_t$, to the decreasing rearrangement of all terms involved in at least one sequence $H_{i,m}(t)$ for some $m \neq m_t$, although the number of m involved may be infinite. Indeed, this is due to the continuity property for finite decreasing rearrangements (Lemma A.2) and to the fact that

$$\begin{aligned}\sum_{m \geq M} \|H_{i,m}^{\varepsilon_n}(t_{\varepsilon_n})\|_1 &\leq Z_{N_{\varepsilon_n}+i-1}^{1/\alpha}\big(R_{\varepsilon_n}(i-1)\big)^{-1/\alpha} \sum_{m \geq M} \Delta_{N_{\varepsilon_n}+i,m} \\ &\underset{\varepsilon_n \to 0}{\to} \big(Z_{i-1}^{\text{bias}}\big)^{1/\alpha}\big(R(i-1)\big)^{-1/\alpha} \sum_{m \geq M} \Delta_{i,m}^{\text{bias}},\end{aligned}$$

which implies that for all $\delta > 0$ there exists $M_\delta \in \mathbb{N}$ such that for all ε_n small enough,

$$\sum_{m \geq M_\delta} \|H_{i,m}^{\varepsilon_n}(t_{\varepsilon_n})\|_1 \leq \delta \quad \text{and} \quad \sum_{m \geq M_\delta} \|H_{i,m}(t)\|_1 \leq \delta. \tag{6.6}$$

Hence, $\sum_{m \geq 1, m \neq m_t} d(H_{i,m}^{\varepsilon_n}(t_{\varepsilon_n}), H_{i,m}(t)) \to 0$, and so, by Lemma A.1, the decreasing rearrangement $\{H_{i,m}^{\varepsilon_n}(t_{\varepsilon_n}), m \neq m_t\}^{\downarrow}$ converges in $\mathcal{S}$ to $\{H_{i,m}(t), m \neq m_t\}^{\downarrow}$. Now, we also have that $H_{i,m_t}^{\varepsilon_n}$ converges in the Skorokhod sense to H_{i,m_t}. It follows, using Lemma A.6(i), that $H_i^{\varepsilon_n,\downarrow}$ and $H_i^{\downarrow}$ satisfy assertions (a), (b) and (c) of Proposition A.3 for the sequence of times (t_{ε_n}).

Next assume that $t = 0$. Let $(s_k)_{k \in \mathbb{N}}$ be a decreasing sequence of strictly positive times that are not jump times of $H_i^{\downarrow}$, and that converge to 0. Then, since $s_k \neq R(i)$ and $s_k > 0$, as we have just seen,

$$\|H_i^{\varepsilon_n,\downarrow}(s_k)\|_1 \underset{n \to \infty}{\to} \|H_i^{\downarrow}(s_k)\|_1 \qquad \forall k \in \mathbb{N}.$$

We conclude by using a monotonicity argument: for all k and then all ε_n sufficiently small, we have $t_{\varepsilon_n} \leq s_k$, and so

$$\|H_i^{\varepsilon_n,\downarrow}(t_{\varepsilon_n})\|_1 \leq \|H_i^{\varepsilon_n,\downarrow}(s_k)\|_1,$$

and then

$$\limsup_{\varepsilon_n\to 0}\left\|H_i^{\varepsilon_n,\downarrow}(t_{\varepsilon_n})\right\|_1 \le \left\|H_i^{\downarrow}(s_k)\right\|_1 \le \left\|C_\infty(s_k)\right\|_1 \qquad \forall k\in\mathbb{N}.$$

Now let $k\to\infty$, so that $\|C_\infty(s_k)\|_1\to 0$, by the right-continuity of C_∞ at 0. Hence, $H_i^{\varepsilon_n,\downarrow}(t_{\varepsilon_n})\to \mathbf{0}=H_i^{\downarrow}(0)$ as $\varepsilon_n\to 0$.

Finally, for $t=R(i)$, consider the subsequences $(t_{\varepsilon_{\phi(n)}})$ and $(t_{\varepsilon_{\psi(n)}})$ of (t_{ε_n}) characterized by

$$R_{\varepsilon_{\phi(n)}}(i)\le t_{\varepsilon_{\phi(n)}} < R_{\varepsilon_{\phi(n)}}(i-1),$$
$$R_{\varepsilon_{\psi(n)}}(i+1)\le t_{\varepsilon_{\psi(n)}} < R_{\varepsilon_{\psi(n)}}(i).$$

For N large enough, there always exists a n such that either $N=\phi(n)$ or $N=\psi(n)$. Since $H_i^{\varepsilon_n,\downarrow}(s)=\mathbf{0}$ for all $s\ge R_{\varepsilon_n}(i)$, we clearly have that

$$H_i^{\varepsilon_n,\downarrow}(t_{\varepsilon_{\phi(n)}})\to \mathbf{0}=H_i^{\downarrow}(t).$$

Next, note that $H_{i,m}^{\varepsilon_n}(t_{\varepsilon_{\psi(n)}})\to H_{i,m}(t-)$ for all $m\ge 1$. Moreover, similar to (6.6), for all $\delta>0$, there exists an integer M_δ such that for all ε_n small enough,

$$\sum_{m\ge M_\delta}\left\|H_{i,m}^{\varepsilon_n}(t_{\varepsilon_{\psi(n)}})\right\|\le\delta \quad\text{and}\quad \sum_{m\ge M_\delta}\left\|H_{i,m}(t-)\right\|\le\delta.$$

From this and Lemma A.1 we deduce that

$$H_i^{\varepsilon_n,\downarrow}(t_{\varepsilon_{\psi(n)}})\to H_i^{\downarrow}(t-).$$

Assertion (a) of Proposition A.3 follows. To get assertion (b), note that if

$$H_i^{\varepsilon_n,\downarrow}(t_{\varepsilon_n})\to \mathbf{0}=H_i^{\downarrow}(t),$$

then necessarily $R_{\varepsilon_n}(i)\le t_{\varepsilon_n}<R_{\varepsilon_n}(i-1)$ for n large enough [since $H_i^{\downarrow}(t-)\ne\mathbf{0}$]. Hence if (s_{ε_n}) is a sequence converging to t with $s_{\varepsilon_n}\ge t_{\varepsilon_n}$, one has $R_{\varepsilon_n}(i)\le s_{\varepsilon_n}<R_{\varepsilon_n}(i-1)$ for n large enough and then $H_i^{\varepsilon_n,\downarrow}(s_{\varepsilon_n})=\mathbf{0}=H_i^{\downarrow}(t)$. We obtain assertion (c) similarly.

Step 2(b). Conditionally on $\zeta=z$, we consider for all n the version of

$$\left(\varepsilon_n^{1/\alpha}F^{(\eta)}\left((\zeta-\varepsilon_n t)-\right),0\le t\le\zeta/\varepsilon_n\right) \tag{6.7}$$

built from the chain $(Z_{N_{\varepsilon_n}+k},Y_{N_{\varepsilon_n}+k},\mathbf{\Delta}_{N_{\varepsilon_n}+k})_{k\in\mathbb{Z}}$, the real number $\frac{1}{\alpha}\log(\varepsilon_n/\zeta)-S_{N_{\varepsilon_n}}$ and the processes $H_i^{\varepsilon_n,\downarrow}, i\in\mathbb{Z}$. We know that (almost surely) these quantities converge, respectively, to $(Z^{\text{bias}},Y^{\text{bias}},\mathbf{\Delta}^{\text{bias}})$, $U\log(Y_1^{\text{bias}})$ and $H_i^{\downarrow}, i\in\mathbb{Z}$. To prove that this version of (6.7) converges for the Skorokhod topology as $\varepsilon_n\to 0$ to a version of $C_\infty^{(\eta)}$ [indeed, the version constructed from $(Z^{\text{bias}},Y^{\text{bias}},\mathbf{\Delta}^{\text{bias}})$, $U\log(Y_1^{\text{bias}})$ and $H_i^{\downarrow}, i\in\mathbb{Z}$], we will again use Proposition A.3 and Lemma A.6.

We start by proving the Skorokhod convergence on any compact set $[a,b]\subseteq(0,\infty)$. Let $R(k_a)$ be the largest $R(k)$ strictly smaller than a and similarly $R(k_b)$

be the smallest $R(k)$ strictly larger than b. For all ε_n small enough, $R_{\varepsilon_n}(k_a) < a$ and $R_{\varepsilon_n}(k_b) > b$. This implies that the processes $(\varepsilon_n^{1/\alpha} F^{(\eta)}((\zeta - \varepsilon_n t)-), t \in [a, b])$ and $(C_\infty^{(\eta)}(t), t \in [a, b])$ are constructed from the sequences $H_i^{\varepsilon_n,\downarrow}$ and $H_i^{\downarrow}$, respectively, with $k_b - i_\eta(k_b) \le i \le k_a - 1$ [together with the terms $Z^{1/\alpha}_{N_{\varepsilon_n}+k}(R_{\varepsilon_n}(k))^{-1/\alpha}$, $(Z_k^{\mathrm{bias}})^{1/\alpha}(R(k))^{-1/\alpha}$, for $k_b \le k \le k_a - 1$]. Crucially, the number of processes $H_i^{\varepsilon_n,\downarrow}$, $H_i^{\downarrow}$ involved in these constructions is finite, independently of n. Moreover, the processes $H_i^{\downarrow}, i \in \mathbb{Z}$ do not jump simultaneously (almost surely). We can therefore apply Lemma A.6(ii) to obtain the Skorokhod convergence of $\varepsilon_n^{1/\alpha} F^{(\eta)}((\zeta - \varepsilon_n \cdot)-)$ to $C_\infty^{(\eta)}$ on any compact set $[a, b] \subseteq (0, \infty)$.

It remains to check that for any sequence (t_{ε_n}) converging to 0, $\varepsilon_n^{1/\alpha} F^{(\eta)}((\zeta - \varepsilon_n t_{\varepsilon_n})-)$ converges to $\mathbf{0} = C_\infty^{(\eta)}(0)$. This can be done via a monotonicity argument, exactly as in the case $t = 0$ of step 2(a).

7. An invariant measure for the fragmentation process. This section is devoted to the proof of Theorem 1.2. Throughout, we will assume that the assumptions of Theorem 1.1 are satisfied. Recall that the occupation measure λ on $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ is defined by

$$\lambda(A) = \int_0^\infty \mathbb{P}\big(C_\infty(t) \in A\big)\,\mathrm{d}t \quad \text{for all } A \in \mathcal{B}(\mathcal{S}).$$

By definition of the process C_∞, it is clear that $\lambda(\{\mathbf{0}\}) = 0$ and also, using its self-similarity, that

$$\lambda\big(\{\mathbf{s} \in \mathcal{S} : s_i \le a_i x, \forall i \ge 1\}\big) = x^{-\alpha} \lambda\big(\{\mathbf{s} \in \mathcal{S} : s_i \le a_i, \forall i \ge 1\}\big)$$

for all $a_i \ge 0$ and all $x > 0$.

Recall the notation $\|\mathbf{s}\|_1 = \sum_{i \ge 1} s_i$ for $\mathbf{s} \in \mathcal{S}$. Our goal in this section is to prove first that

$$\lambda\big(\{\mathbf{s} \in \mathcal{S} : \|\mathbf{s}\|_1 \le x\}\big) < \infty \qquad \forall x > 0$$

(which implies that λ is σ-finite) and second that

$$\int_{\mathcal{S}} f(\mathbf{s}) \lambda(\mathrm{d}\mathbf{s}) = \int_{\mathcal{S}} \mathbb{E}_{\mathbf{s}}\big[f\big(F(u)\big)\big] \lambda(\mathrm{d}\mathbf{s})$$

for all $u > 0$ and all continuous functions $f : \mathcal{S} \to \mathbb{R}_+$ such that $f(\mathbf{s}) \le \mathbb{1}_{\{0 < \|\mathbf{s}\|_1 \le c\}}$ for some $c > 0$.

LEMMA 7.1. *For all continuous functions $f : \mathcal{S} \to \mathbb{R}_+$ such that $f(\mathbf{s}) \le \mathbb{1}_{\{\|\mathbf{s}\|_1 \le c\}}$ for some $c > 0$,*

$$\int_0^\infty \mathbb{E}\big[f\big(\varepsilon^{1/\alpha} F(\zeta - \varepsilon t)\big)\big]\,\mathrm{d}t \underset{\varepsilon \to 0}{\to} \int_{\mathcal{S}} f(\mathbf{s}) \lambda(\mathrm{d}\mathbf{s}) \in [0, \infty).$$

PROOF. To simplify the notation, we assume that $c = 1$; a similar argument works for a general $c > 0$. By Theorem 1.1, for all $t > 0$, $\mathbb{E}[f(\varepsilon^{1/\alpha} F(\zeta - \varepsilon t))] \to \mathbb{E}[f(C_\infty(t))]$. It remains to check that we can apply the dominated convergence theorem. For this, we introduce for every $a > 0$ the stopping time $\tau_a = \inf\{u \geq 0 : \|F(u)\|_1 \leq a\}$. By Proposition 2.1, we may write

$$\zeta - \tau_a = \sup_{i \geq 1}\{F_i(\tau_a)^{-\alpha} \zeta^{(i)}\},$$

where the $\zeta^{(i)}$s are i.i.d. distributed as ζ and independent of $F(\tau_a)$. Hence, for all $\beta \geq 1$,

$$\begin{aligned}
\mathbb{E}\big[f\big(\varepsilon^{1/\alpha} F(\zeta - \varepsilon t)\big)\big] &\leq \mathbb{P}(\zeta - \varepsilon t \geq \tau_{\varepsilon^{-1/\alpha}}) \\
&\leq \mathbb{P}\big((\zeta - \tau_{\varepsilon^{-1/\alpha}})^{-\beta/\alpha} \geq (\varepsilon t)^{-\beta/\alpha}\big) \\
&\leq \mathbb{P}\bigg(\sum_{i \geq 1} F_i(\tau_{\varepsilon^{-1/\alpha}})^{\beta} \big(\zeta^{(i)}\big)^{-\beta/\alpha} \geq (\varepsilon t)^{-\beta/\alpha}\bigg) \\
&\leq \frac{\mathbb{E}[\zeta^{-\beta/\alpha}]\mathbb{E}[\sum_{i \geq 1} F_i(\tau_{\varepsilon^{-1/\alpha}})^{\beta}]}{(\varepsilon t)^{-\beta/\alpha}} \\
&\leq \frac{\mathbb{E}[\zeta^{-\beta/\alpha}]}{t^{-\beta/\alpha}},
\end{aligned}$$

by definition of $\tau_{\varepsilon^{-1/\alpha}}$ and the fact that $\beta \geq 1$. Taking β larger if necessary so that $-\beta/\alpha > 1$ and recalling that $\mathbb{E}[\zeta^{-\beta/\alpha}] < \infty$, we obtain

$$\mathbb{E}\big[f\big(\varepsilon^{1/\alpha} F(\zeta - \varepsilon t)\big)\big] \leq \min\big(1, Ct^{\beta/\alpha}\big) \qquad \forall t \geq 0$$

for some finite constant C, independently of ε. The result follows. □

PROOF OF THEOREM 1.2. Consider the potential measure

$$\lambda_\varepsilon(A) := \int_0^\infty \mathbb{P}_{\varepsilon^{1/\alpha}\mathbf{1}}\big(F(t) \in A\big)\,\mathrm{d}t.$$

Equivalently, $\lambda_\varepsilon(A)$ is the expected time spent in A by a fragmentation process started from $\varepsilon^{1/\alpha}\mathbf{1}$. Suppose now that $f : \mathcal{S} \to \mathbb{R}_+$ is continuous and such that $f(\mathbf{s}) \leq \mathbb{1}_{\{0 < \|\mathbf{s}\|_1 \leq c\}}$ for some $c > 0$. By the self-similarity of the fragmentation process,

$$\begin{aligned}
\int_{\mathcal{S}} f(\mathbf{s})\lambda_\varepsilon(\mathrm{d}\mathbf{s}) &= \int_0^\infty \mathbb{E}\big[f\big(\varepsilon^{1/\alpha} F(\varepsilon t)\big)\big]\,\mathrm{d}t \\
&\underset{(\text{if } \varepsilon^{1/\alpha} > c)}{=} \int_0^\infty \mathbb{E}\big[f\big(\varepsilon^{1/\alpha} F(\zeta - \varepsilon t)\big)\big]\,\mathrm{d}t \to \int_{\mathcal{S}} f(\mathbf{s})\lambda(\mathrm{d}\mathbf{s})
\end{aligned} \tag{7.1}$$

as $\varepsilon \to 0$, by Lemma 7.1.

From now on, fix $u > 0$, $c > 0$ and a continuous function $f : \mathcal{S} \to \mathbb{R}_+$ such that $f(\mathbf{s}) \le \mathbb{1}_{\{0<\|\mathbf{s}\|_1 \le c\}}$. Our goal is to check, on the one hand, that

$$\int_{\mathcal{S}} \mathbb{E}_{\mathbf{s}}\big[f\big(F(u)\big)\big]\lambda_\varepsilon(\mathrm{d}\mathbf{s}) \to \int_{\mathcal{S}} f(\mathbf{s})\lambda(\mathrm{d}\mathbf{s}) \tag{7.2}$$

and, on the other, that

$$\int_{\mathcal{S}} \mathbb{E}_{\mathbf{s}}\big[f\big(F(u)\big)\big]\lambda_\varepsilon(\mathrm{d}\mathbf{s}) \to \int_{\mathcal{S}} \mathbb{E}_{\mathbf{s}}\big[f\big(F(u)\big)\big]\lambda(\mathrm{d}\mathbf{s}). \tag{7.3}$$

Together, these will yield the invariance of λ.

We start with (7.2). By the definition of λ_ε,

$$\begin{aligned}
&\int_{\mathcal{S}} \mathbb{E}_{\mathbf{s}}\big[f\big(F(u)\big)\big]\lambda_\varepsilon(\mathrm{d}\mathbf{s}) \\
&\quad = \int_0^\infty \mathbb{E}\big[f\big(\varepsilon^{1/\alpha} F\big(\varepsilon(u+t)\big)\big)\big]\,\mathrm{d}t \\
&\quad = \int_0^\infty \mathbb{E}\big[f\big(\varepsilon^{1/\alpha} F(\varepsilon t)\big)\big]\,\mathrm{d}t - \int_0^u \mathbb{E}\big[f\big(\varepsilon^{1/\alpha} F(\varepsilon t)\big)\big]\,\mathrm{d}t.
\end{aligned}$$

The first integral in the last line converges to $\int_{\mathcal{S}} f(\mathbf{s})\lambda(\mathrm{d}\mathbf{s})$, by (7.1). The second converges to 0, since $\mathbb{E}[f(\varepsilon^{1/\alpha} F(\varepsilon t))] \to 0$ for all $t > 0$ [as $\varepsilon^{1/\alpha}\|F(\varepsilon t)\|_1 > c$ for ε small enough, a.s.]. The convergence in (7.2) follows.

To get (7.3), set $g(\mathbf{s}) = \mathbb{E}_{\mathbf{s}}[f(F(u))]$. The function g is continuous, bounded and $\mathbb{R}_+$-valued, but is not supported by a set of the form $0 < \|\mathbf{s}\|_1 \le c'$ for some c', so we cannot conclude the desired result directly from the convergence of λ_ε to λ. Note that, for all $c' > 0$,

$$\begin{aligned}
&\int_{\mathcal{S}} g(\mathbf{s})\mathbb{1}_{\{\|\mathbf{s}\|_1 > c'\}}\lambda_\varepsilon(\mathrm{d}\mathbf{s}) \\
&\quad = \int_0^\infty \mathbb{E}\big[f\big(\varepsilon^{1/\alpha} F\big(\varepsilon(u+t)\big)\big)\mathbb{1}_{\{\|\varepsilon^{1/\alpha} F(\varepsilon t)\|_1 > c'\}}\big]\,\mathrm{d}t \\
&\quad \le \int_0^\infty \mathbb{P}\big(\big\|\varepsilon^{1/\alpha} F(\varepsilon t)\big\|_1 > c', 0 < \big\|\varepsilon^{1/\alpha} F\big(\varepsilon(t+u)\big)\big\|_1 \le c\big)\,\mathrm{d}t \\
&\quad \le \int_0^\infty \mathbb{P}\big(\big\|\varepsilon^{1/\alpha} F(\zeta - \varepsilon t - \varepsilon u)\big\|_1 > c', 0 < \big\|\varepsilon^{1/\alpha} F(\zeta - \varepsilon t)\big\|_1 \le c\big)\,\mathrm{d}t.
\end{aligned}$$

Using the dominated convergence theorem (and the same argument as in the proof of Lemma 7.1), we see that the right-hand side converges to $\int_0^\infty \mathbb{P}(\|C_\infty(t+u)\|_1 > c', \|C_\infty(t)\|_1 \le c)\,\mathrm{d}t$, which is finite. Hence, for all $\eta > 0$ and then all $c' > 0$ large enough, say $c' \ge c'_\eta$,

$$\limsup_{\varepsilon \to 0} \int_{\mathcal{S}} g(\mathbf{s})\mathbb{1}_{\{\|\mathbf{s}\|_1 > c'\}}\lambda_\varepsilon(\mathrm{d}\mathbf{s}) \le \eta.$$

Now,

$$\begin{aligned}\int_{\mathcal{S}} g(\mathbf{s})\mathbb{1}_{\{\|\mathbf{s}\|_1>c'\}}\lambda_\varepsilon(\mathrm{d}\mathbf{s})&\\&=\int_0^\infty \mathbb{E}\big[g\big(\varepsilon^{1/\alpha}F(\varepsilon t)\big)\mathbb{1}_{\{\|\varepsilon^{1/\alpha}F(\varepsilon t)\|_1>c'\}}\big]\,\mathrm{d}t\\&=\int_0^\infty \mathbb{E}\big[g\big(\varepsilon^{1/\alpha}F(\zeta-\varepsilon t)\big)\mathbb{1}_{\{\|\varepsilon^{1/\alpha}F(\zeta-\varepsilon t)\|_1>c'\}}\mathbb{1}_{\{\zeta\ge\varepsilon t\}}\big]\,\mathrm{d}t.\end{aligned}$$

Since the function $\mathbf{s}\mapsto g(\mathbf{s})\mathbb{1}_{\{\|\mathbf{s}\|_1>c'\}}$ is lower semi-continuous, by the Portmanteau theorem,

$$\begin{aligned}\liminf_{\varepsilon\to 0}\mathbb{E}\big[g\big(\varepsilon^{1/\alpha}F(\zeta-\varepsilon t)\big)\mathbb{1}_{\{\|\varepsilon^{1/\alpha}F(\zeta-\varepsilon t)\|_1>c'\}}\mathbb{1}_{\{\zeta\ge\varepsilon t\}}\big]&\\\ge\mathbb{E}\big[g\big(C_\infty(t)\big)\mathbb{1}_{\{\|C_\infty(t)\|_1>c'\}}\big].&\end{aligned}$$

Hence, by Fatou's lemma,

$$\int_0^\infty g(\mathbf{s})\mathbb{1}_{\{\|\mathbf{s}\|_1>c'\}}\lambda(\mathrm{d}\mathbf{s})\le\liminf_{\varepsilon\to 0}\int_{\mathcal{S}} g(\mathbf{s})\mathbb{1}_{\{\|\mathbf{s}\|_1>c'\}}\lambda_\varepsilon(\mathrm{d}\mathbf{s})\le\eta$$

for all $c'\ge c'_\eta$. Finally, fix $\eta>0$ and then $c'\ge c'_\eta$. Consider then $c''\in(c',\infty)$, and let $h:\mathcal{S}\to[0,1]$ be a continuous function such that $h(\mathbf{s})=1$ when $\|\mathbf{s}\|_1\le c'$ and $h(\mathbf{s})=0$ when $\|\mathbf{s}\|_1\ge c''$. Then

$$\begin{aligned}\left|\int_{\mathcal{S}} g(\mathbf{s})(\lambda_\varepsilon-\lambda)(\mathrm{d}\mathbf{s})\right|&\\\le\left|\int_{\mathcal{S}} g(\mathbf{s})h(\mathbf{s})(\lambda_\varepsilon-\lambda)(\mathrm{d}\mathbf{s})\right|&+\left|\int_{\mathcal{S}} g(\mathbf{s})\big(1-h(\mathbf{s})\big)\lambda_\varepsilon(\mathrm{d}\mathbf{s})\right|\\&+\left|\int_{\mathcal{S}} g(\mathbf{s})\big(1-h(\mathbf{s})\big)\lambda(\mathrm{d}\mathbf{s})\right|.\end{aligned}$$

We have chosen c' and h so that the second and third terms are each smaller than η for small enough ε. By (7.1), the first term converges to 0 as $\varepsilon\to 0$. The convergence in (7.3) follows. □

8. Discussion of geometric fragmentations. In this section, we consider geometric fragmentations; that is, we assume that the set of $r\in(0,1)$ such that

$$\nu\big(s_i\in r^{\mathbb{N}}\cup\{0\}, i\ge 1\big)=1 \tag{8.1}$$

is nonempty, and we let $r_{\min}$ denote its unique minimal element. It is easy to see that $r_{\min}$ exists and is characterized by the fact that ν-a.e. $s_i=r_{\min}^{n_i}$, $\forall i$ where the nonzero integers n_i have 1 as highest common factor. Moreover, for every $r\in(0,1)$ satisfying (8.1), there is a $q\in\mathbb{N}$ such that $r_{\min}=r^q$.

This case has some interesting connections to other parts of the probability literature, which we will briefly describe below. We will then see that $\varepsilon^{1/\alpha}F(\zeta-\varepsilon)$

cannot converge in distribution in this case. However, it does converge along appropriate subsequences. Finally, we will restrict attention to the simple case of k-ary fragmentations, when each fragmentation of a block produces k blocks with identical masses, and describe all possible limit distributions of the rescaled last fragment $\varepsilon^{1/\alpha} F_*(\zeta - \varepsilon)$ in these simple k-ary fragmentations.

8.1. *Related models.* Specialize, for the moment, to the case where the fragmentation has dislocation measure

$$\nu(d\mathbf{s}) = \delta_{(1/k,1/k,\dots,1/k,0,\dots)}(d\mathbf{s}), \qquad \mathbf{s} \in \mathcal{S}_1.$$

This fragmentation process has been studied in various different guises in the probability literature.

In [5], Athreya considers a model which he calls the *discounted branching random walk*. Start with a single particle situated at a distance to the right of the origin which is distributed as Exp(1). At each epoch, every particle present gives birth to two particles. At epoch n, these new particles have a displacement rightwards from the parent with distribution $\mathrm{Exp}(2^{-n\alpha})$, independently for different particles. It is easy to see that the positions of the 2^n particles at generation n correspond to the times at which the blocks of size 2^{-n} appear in the simple binary fragmentation (when $k = 2$). Athreya concerns himself particularly with a recursive equation for the distribution of the right-hand end of the support of the particle distribution at time ∞. This, of course, has the same distribution as ζ, and the recursive distributional equation is $\zeta = T_1 + 2^{n\alpha} \max\{\zeta^{(1)}, \zeta^{(2)}\}$ in our notation. This equation and others like it are discussed in more detail in Aldous and Bandyopadhyay [2]. The convergence of the last fragment in Theorem 3.6 (which is valid for geometric fragmentations) entails that the distance between the ancestor of generation n of the winning particle and the winning particle itself, rescaled by $2^{-n\alpha}$ converges in distribution as $n \to \infty$. Of course, this construction is easily extended to the case where each individual gives birth to k offspring.

Barlow, Pemantle and Perkins [7] consider a model of randomly-growing k-ary trees which has also been studied, in various versions, in [1, 12, 13, 26]. Suppose we grow the complete k-ary tree as follows. [For definiteness, label vertices in the tree by k-ary strings, so that the root is $\varnothing$, its neighbors are $0, 1, \dots, k-1$ and, in general, the descendants of a vertex labeled x are $x0, x1, \dots, x(k-1)$.] We start with the empty tree and wait an Exp(1) amount of time; then the root gets filled in. Let $A(0) = \{\varnothing\}$. In general, let $A(t)$ be the set of vertices in the k-ary tree which have not yet been filled in themselves, but whose parents in the tree have been filled in. A vertex in $A(t)$ at height n (where the root has height 0) becomes filled in at a rate $k^{-\alpha n}$. The vertices in $A(t)$ correspond exactly to blocks in our fragmentation at time t. In particular, a vertex at height n corresponds to a block of size k^{-n}. This model can be thought of as a sort of *first-passage percolation* or as *diffusion-limited aggregation* on a tree. In particular, Barlow, Pemantle and Perkins study the structure of the cluster at the first time that it hits the boundary of the tree. This corresponds to the time at which mass first disappears in the fragmentation. They show that at that time the cluster consists of a unique infinite backbone with small finite trees hanging off it. We are instead interested in what happens near the time at which the *last* point on the boundary of the tree is reached. Theorem 3.6 tells us that the time taken to reach this last point on the boundary from its ancestor in generation n, suitably rescaled, has a limit in distribution as $n \to \infty$.

We now turn to a more general context and prove some results which apply to these special cases.

8.2. *Absence of limit in distribution.* We return to the general case of a geometric fragmentation F, assuming solely that $\int_{\mathcal{S}_1} s_1^{-1} \nu(\mathrm{d}\mathbf{s})$ is finite. Recall that T_n is the nth jump time of the last fragment process F_*. From Theorem 3.6, we know that

$$Z_n^{1/\alpha} = (\zeta - T_n)^{1/\alpha} F_*(T_n)$$

converges in distribution to a law which is fully supported by $(0, \infty)$. However, we do not have convergence in distribution of the rescaled sequence $\varepsilon^{1/\alpha} F(\zeta - \varepsilon)$ as $\varepsilon \to 0$.

PROPOSITION 8.1. *In the geometric cases, $\varepsilon^{1/\alpha} F(\zeta - \varepsilon)$ and $\varepsilon^{1/\alpha} F_*(\zeta - \varepsilon)$ do not converge in distribution as $\varepsilon \to 0$. However, for each $x \in [0, 1)$, the sequence $r_{\min}^{-n-x} F_*(\zeta - r_{\min}^{-\alpha(n+x)})$ has a nonzero limit in distribution as $n \to \infty$, which depends on x.*

In the next section, we specify this limit and its dependence on x for the simple k-ary fragmentations.

PROOF OF PROPOSITION 8.1. Suppose (for a contradiction) that $\varepsilon^{1/\alpha} F(\zeta - \varepsilon)$ converges in distribution in $\mathcal{S}$. Then $\varepsilon^{1/\alpha} F_1(\zeta - \varepsilon)$ has a limit in distribution, say $L \in [0, \infty)$. Consider the sequence $\varepsilon_n = a r_{\min}^{-\alpha n}$, $n \geq 1$, where $a \in (0, \infty)$ is fixed. Then the random variables $\varepsilon_n^{1/\alpha} F_1(\zeta - \varepsilon_n)$ almost surely all belong to the set $a^{1/\alpha} r_{\min}^{\mathbb{Z}}$, and so $L \in a^{1/\alpha} r_{\min}^{\mathbb{Z}} \cup \{0\}$ a.s. But this assertion holds for all $a \in (0, \infty)$, hence $L = 0$ a.s. In particular, this implies that $\varepsilon^{1/\alpha} F_*(\zeta - \varepsilon)$ converges in distribution to 0. Similarly, supposing first that $\varepsilon^{1/\alpha} F_*(\zeta - \varepsilon)$ has a limit in distribution, we conclude that this limit is necessarily 0.

But a zero limit is not possible, because $r_{\min}^{-n} F_*(\zeta - r_{\min}^{-\alpha n})$ has a nonzero limit in distribution as $n \to \infty$, provided that $\int_{\mathcal{S}_1} s_1^{-1} \nu(\mathrm{d}\mathbf{s}) < \infty$. To see this, we use Corollary 2.2(b) of Alsmeyer [3], on Markov renewal theory in the geometric cases. Given this corollary, it is possible to check that the rescaled sequence $r_{\min}^{-n} F_*(\zeta - r_{\min}^{-\alpha n})$ has a nontrivial limit in distribution as $n \to \infty$, in exactly the same way as we proved the one-dimensional convergence in Section 4.2. Using arguments from Section 4.3 giving an expression for $N_{\varepsilon t}$ in terms of N_ε, it is then easy to deduce the convergence in distribution of $r_{\min}^{-n-x} F_*(\zeta - r_{\min}^{-\alpha n+\alpha x})$ to a nontrivial limit. We leave these extensions to the reader. $\square$

REMARK 8.2. This result then certainly leads to the convergence of $r_{\min}^{-n-x} F(\zeta - r_{\min}^{-\alpha(n+x)})$ to a nontrivial limit and more generally of the whole process $r_{\min}^{-n-x}(F((\zeta - r_{\min}^{-\alpha(n+x)} t)-), t \geq 0)$, at least when $\int_{\mathcal{S}_1} s_1^{-1-\rho} \nu(d\mathbf{s}) < \infty$ for some $\rho > 0$. In order to see this, one should mimic the proofs of Sections 4 and 6. However, for ease and brevity of exposition, we omit this part and leave it to the motivated reader. We emphasize that the limit process depends on x and cannot be self-similar. Moreover, the proofs of Lemma 7.1 and Theorem 1.2 in Section 7 are still valid when replacing ε by $\varepsilon_n(x) = r_{\min}^{-\alpha(n+x)}$ and letting $n \to 0$. Hence, we may deduce the existence of invariant measures for these geometric fragmentations. Note that the invariant measure constructed from the sequence $(\varepsilon_n(x))_{n\geq 0}$ is supported by elements $\mathbf{s}$ of $\mathcal{S}$ such that $s_i \in r_{\min}^{-x+\mathbb{Z}}$ for all i. We have, therefore, a continuum set of distinct invariant measures, indexed by $x \in [0, 1)$.

8.3. *Simple k-ary fragmentations.* From now on, we assume that the fragmentation has dislocation measure

$$\nu(d\mathbf{s}) = \delta_{(1/k,1/k,\dots,1/k,0,\dots)}(d\mathbf{s}), \qquad \mathbf{s} \in \mathcal{S}_1.$$

By adapting the method of proof of Theorems 5.1 and 5.2 of [7], we can obtain a stronger version of Theorem 3.6. Note that here $T_n = \inf\{t \geq 0 : F_*(t) = k^{-n}\}$ and $Z_n = k^{-n\alpha}(\zeta - T_n)$.

PROPOSITION 8.3. *The sequence $(Z_n)_{n\geq 0}$ is stochastically increasing. As a consequence,*

$$Z_n \overset{law}{\to} Z_\infty$$

as $n \to \infty$, where $Z_\infty \sim \pi_{\text{stat}}$ and $Z_\infty \geq_{\text{st}} \zeta$.

PROOF. We argue by induction, using the notation of Section 3. Recall that $Z_0 = \zeta$ and that $Z_1 = \zeta^{(I)} = \max_{1\leq i\leq k} \zeta^{(i)}$. It follows that $Z_0 \leq_{\text{st}} Z_1$. Let next

$$p(t, x) = \mathbb{P}\big(\zeta^{(I)} \geq t | \zeta = x\big)$$

in the sense of a regular conditional probability. Since $(Z_n)_{n\geq 0}$ is a Markov chain,

$$\mathbb{P}(Z_{n+1} \geq t) = \mathbb{E}\big[p(t, Z_n)\big].$$

Suppose for the moment that, for fixed t, $p(t, x)$ is increasing in x. Our induction hypothesis is that $Z_{n-1} \leq_{\text{st}} Z_n$. Then

$$\mathbb{P}(Z_{n+1} \geq t) = \mathbb{E}\big[p(t, Z_n)\big] \geq \mathbb{E}\big[p(t, Z_{n-1})\big] = \mathbb{P}(Z_n \geq t).$$

So it remains to show that $p(t,x)$ is increasing in x.

We have $\zeta = T_1 + k^\alpha \max_{1\le i\le k} \zeta^{(i)} = T_1 + k^\alpha \zeta^{(I)}$ with T_1 independent of $\zeta^{(I)}$. From this, it is easy to see that $(\zeta, \zeta^{(I)})$ has a density which may be written as

$$(x,y) \in \mathbb{R}_+^2 \mapsto f_{\zeta^{(I)}}(y) e^{k^\alpha y - x} \mathbb{1}_{\{x \ge k^\alpha y\}}.$$

Then for $t \le k^{-\alpha} x$,

$$p(t,x) = \frac{\int_t^{k^{-\alpha}x} f_{\zeta^{(I)}}(y) e^{k^\alpha y - x}\,\mathrm{d}y}{\int_0^{k^{-\alpha}x} f_{\zeta^{(I)}}(y) e^{k^\alpha y - x}\,\mathrm{d}y} = 1 - \frac{\int_0^t f_{\zeta^{(I)}}(y) e^{k^\alpha y}\,\mathrm{d}y}{\int_0^{k^{-\alpha}x} f_{\zeta^{(I)}}(y) e^{k^\alpha y}\,\mathrm{d}y}$$

and so $p(t,x)$ is, indeed, increasing in x. $\square$

Now, for $t \ge 0$, let

$$x(t) = \frac{1}{\alpha}\log_k t - \left[\frac{1}{\alpha}\log_k t\right].$$

We will now specify the asymptotics of the last fragment $F_*(\zeta - \varepsilon_n)$, according to the behavior of the sequence (ε_n) under the action of the function x.

PROPOSITION 8.4. *Let $(\varepsilon_n)_{n\ge 0}$ be any sequence of times converging to 0 such that $x(\varepsilon_n) \to x$ for some fixed $x \in [0,1)$. Then we have as $n \to \infty$*

$$\varepsilon_n^{1/\alpha} F_*(\zeta - \varepsilon_n) \stackrel{\text{law}}{\to} k^{x - N(x)},$$

where $N(x) = \sup\{n \in \mathbb{Z} : Z_n^{\text{stat}} \ge k^{(x-n)\alpha}\}$.

We note that $N(x) > -\infty$ almost surely, a statement which we will justify during the course of the proof. It is also the case that $N(x) < \infty$. As an example of an application of this proposition, for all $x \in [0,1)$, we have

$$k^{x+n} F_*\big(\zeta - k^{\alpha(x+n)}\big) \stackrel{law}{\to} k^{x-N(x)} \qquad \text{as } n \to \infty.$$

PROOF OF PROPOSITION 8.4. For any $\varepsilon \ge 0$, let $N_\varepsilon = \sup\{n \ge 0 : \zeta - \varepsilon \ge T_n\} = \sup\{n \ge 0 : \varepsilon \le \zeta - T_n\}$. Then,

$$\varepsilon^{1/\alpha} F_*(\zeta - \varepsilon) = \varepsilon^{1/\alpha} F_*(T_{N_\varepsilon}) = \varepsilon^{1/\alpha} k^{-N_\varepsilon}.$$

Using $Z_n = k^{-n\alpha}(\zeta - T_n)$, we have

$$N_\varepsilon = \sup\big\{n \ge 0 : Z_n \ge k^{-n\alpha}\varepsilon\big\}.$$

Write $m(\varepsilon) = [(\log_k \varepsilon)/\alpha]$ so that $m(\varepsilon) + x(\varepsilon) = (\log_k \varepsilon)/\alpha$. Then

$$\begin{aligned} N_\varepsilon - m(\varepsilon) &= \sup\big\{n \ge -m(\varepsilon) : Z_{m(\varepsilon)+n} \ge k^{-(m(\varepsilon)+n)\alpha}\varepsilon\big\} \\ &= \sup\big\{n \ge -m(\varepsilon) : Z_{m(\varepsilon)+n} \ge k^{-n\alpha} \cdot k^{\alpha x(\varepsilon)}\big\}. \end{aligned}$$

Now take $\varepsilon = \varepsilon_n$ so that $\varepsilon_n \to 0$ and $x(\varepsilon_n) \to x$ as $n \to \infty$. Then for all $p \in \mathbb{Z}$ and all n such that $p > -m(\varepsilon_n)$,

$$\mathbb{P}\big(N_{\varepsilon_n} - m(\varepsilon_n) \geq p\big) = \mathbb{P}\big(Z_{m(\varepsilon_n)+p} \geq k^{-p\alpha} \cdot k^{\alpha x(\varepsilon_n)}\big)$$

since the sequence $(Z_n k^{n\alpha})$ is nonincreasing in n a.s. (indeed, $Z_n k^{n\alpha} = \zeta - T_n$). Similarly,

$$\mathbb{P}\big(N(x) \geq p\big) = \mathbb{P}\big(Z_p^{\mathrm{stat}} \geq k^{-p\alpha} k^{\alpha x}\big) = \pi_{\mathrm{stat}}\big(\big[k^{-p\alpha} k^{\alpha x}, \infty)\big).$$

Then, since $x(\varepsilon_n) \to x$, $Z_{m(\varepsilon_n)+p}$ converges in law to π_{stat} as $n \to \infty$ and as π_{stat} is nonatomic, we get that

$$\mathbb{P}\big(N_{\varepsilon_n} - m(\varepsilon_n) \geq p\big) \to \mathbb{P}\big(N(x) \geq p\big),$$

for all $p \in \mathbb{Z}$. In other words, $N_{\varepsilon_n} - m(\varepsilon_n)$ converges in law to $N(x)$ as $n \to \infty$. So $N_{\varepsilon_n} - (\log_k \varepsilon_n)/\alpha$ converges in law to $N(x) - x$, which entails that

$$\varepsilon_n^{1/\alpha} k^{-N_{\varepsilon_n}} \overset{\mathrm{law}}{\to} k^{x - N(x)},$$

as $n \to \infty$, as required. $\square$

APPENDIX

A.1. Convergence criteria. In this section, we record various technical lemmas concerning criteria for convergence in $(\mathcal{S}, d)$ and in the Skorohod topology on càdlàg processes taking values in $(\mathcal{S}, d)$. The proofs of the first two lemmas are straightforward, and so we omit them.

LEMMA A.1. *Let $(\mathbf{s}^{(n)}, n \geq 1)$ be a sequence of nonnegative elements of ℓ_1 converging to $\mathbf{s}^{(\infty)} \in \ell_1$ for the ℓ_1-topology. For every integer $n \in \mathbb{N} \cup \{\infty\}$, let $\mathbf{s}^{(n),\downarrow}$ denote the decreasing rearrangement of the terms of $\mathbf{s}^{(n)}$. Then $\mathbf{s}^{(n),\downarrow} \to \mathbf{s}^{(\infty),\downarrow}$ in $(\mathcal{S}, d)$.*

LEMMA A.2. *Let $n \in \mathbb{N}$. The two following functions are continuous*:

(i) $(\mathbf{s}^{(1)}, \dots, \mathbf{s}^{(n)}) \in \mathcal{S}^n \mapsto \{s_j^{(i)}, 1 \leq i \leq n, j \geq 1\}^{\downarrow} \in \mathcal{S}$, *where $\mathcal{S}^n$ is endowed with the product topology*;

(ii) $(x, \mathbf{s}) \in \mathbb{R}_+ \times \mathcal{S} \mapsto \{x s_j, j \geq 1\} \in \mathcal{S}$.

We next recall a classical result on Skorokhod convergence (see Proposition 3.6.5 of Ethier and Kurtz [16]) which we will use repeatedly.

PROPOSITION A.3. *Consider a metric space (E, d_E), and let f_n, f be càdlàg paths with values in E. Then $f_n \to f$ with respect to the Skorokhod topology if and only if the three following assertions are satisfied for all sequences $t_n \to t$, $t_n, t \geq 0$:*

(a) $\min(d_E(f_n(t_n), f(t)), d_E(f_n(t_n), f(t-))) \to 0$;

(b) $d_E(f_n(t_n), f(t)) \to 0 \Rightarrow d_E(f_n(s_n), f(t)) \to 0$ *for all sequences* $s_n \to t$, $s_n \geq t_n$;

(c) $d_E(f_n(t_n), f(t-)) \to 0 \Rightarrow d_E(f_n(s_n), f(t-)) \to 0$ *for all sequences* $s_n \to t$, $s_n \leq t_n$.

Of course, if t is not a jump time of f, then (a), (b), (c) are equivalent to $d_E(f_n(t_n), f(t)) \to 0$.

We now establish three lemmas on Skorokhod convergence, which are used in the main body of the paper.

LEMMA A.4. *Consider* $(c_n)_{n\in\mathbb{Z}_+\cup\{\infty\}}$, *a sequence of real-valued nondecreasing piecewise constant càdlàg functions defined on* $\mathbb{R}_+$ *by* $c_n(0) = 0$ *and, for* $t > 0$,

$$c_n(t) = b_n(k) \qquad \text{if } r_n(k) > t \geq r_n(k+1),$$

where $(r_n(k))_{k\in\mathbb{Z}}$ *is strictly decreasing in* k *and such that* $r_n(k) \to 0$ *as* $k \to \infty$ *and* $r_n(k) \to \infty$ *as* $k \to -\infty$. *Suppose that for all* $k \in \mathbb{Z}$, $r_n(k) \to r_\infty(k)$ *and* $b_n(k) \to b_\infty(k)$ *as* $n \to \infty$. *Then* $c_n \to c_\infty$ *for the Skorokhod topology on the set of real-valued càdlàg functions on* $\mathbb{R}_+$.

PROOF. This is nearly obvious from the definition of the Skorokhod topology. To prove it carefully, we use Proposition A.3. It is easy to see that for a fixed $t > 0$ and all sequences $t_n \to t$, conditions (a), (b) and (c) of this proposition are satisfied for the sequence $(c_n)_{n\in\mathbb{Z}_+}$, with c_∞ at the limit. It remains to check them for $t = 0$, which consists then in checking that $c_n(t_n) \to c_\infty(0) = 0$. This is immediate, using monotonicity. Indeed, let $\varepsilon > 0$; for large n, $t_n \leq \varepsilon$, and so $c_n(t_n) \leq c_n(\varepsilon)$. The sequence $(c_n(\varepsilon))$ might not converge, but clearly $\limsup_n c_n(\varepsilon) \leq c_\infty(\varepsilon)$. Since c_∞ is right-continuous, we get, letting $\varepsilon \to 0$, that $\limsup_n c_n(\varepsilon) = 0$. $\square$

The next lemma concerns the time-reversed conditioned fragmentation process $\bar{F}^{(x)}$ introduced in Section 5.

LEMMA A.5. *Let* $(a_n), (b_n), (c_n), a_\infty, b_\infty, c_\infty$ *be nonnegative numbers such that* $a_n \to a_\infty$, $b_n \to b_\infty$ *and* $c_n \to c_\infty$. *Then*

$$\left(c_n \bar{F}^{(a_n)}(b_n t+), t \geq 0\right) \overset{law}{\to} \left(c_\infty \bar{F}^{(a_\infty)}(b_\infty t+), t \geq 0\right)$$

in sense of the Skorokhod topology on càdlàg processes taking values in $(\mathcal{S}, d)$.

PROOF. Let F be a fragmentation process and, for all $n \in \mathbb{N} \cup \{\infty\}$, let $G^{(n)}$ be defined by

$$G^{(n)}(t) = \begin{cases} c_n F(a_n - b_n t), & \text{if } 0 \leq b_n t \leq a_n, \\ \mathbf{0}, & \text{if } b_n t > a_n. \end{cases}$$

Then observe that for all $u \geq 0$:

- if (u_n) is a sequence converging to u, with $u_n > u$ for all n, then $F(u_n-) \to F(u)$;
- if (u_n) is a sequence converging to u, with $u_n \leq u$ for all n, then $F(u_n-) \to F(u-)$.

We can deduce from this [together with Lemma A.2(ii)] that for all $t \geq 0$:

- $G^{(n)}(t_n+) \to G^{(\infty)}(t)$ when $t_n \to t$ and $a_n - b_n t_n > a_\infty - b_\infty t$ for all n large enough;
- $G^{(n)}(t_n+) \to G^{(\infty)}(t+)$ when $t_n \to t$ and $a_n - b_n t_n \leq a_\infty - b_\infty t$ for all n large enough.

From these observations and Proposition A.3, we get that $(G^{(n)}(t+), t \geq 0)$ converges to $(G^{(\infty)}(t+), t \geq 0)$ as $n \to \infty$ for the Skorokhod topology on $\mathcal{S}$, almost surely. Since the extinction time ζ of F has a continuous cumulative distribution function, we also have $\mathbb{1}_{\{\zeta < a_n\}} \to \mathbb{1}_{\{\zeta < a_\infty\}}$, almost surely. Hence, for all bounded continuous test functions $f : \mathcal{S} \to \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}\big[f\big((c_n \bar{F}^{(a_n)}(b_n t+), t \geq 0)\big)\big] &= \frac{\mathbb{E}[f((G^{(n)}(t+), t \geq 0))\mathbb{1}_{\{\zeta < a_n\}}]}{\mathbb{P}(\zeta < a_n)} \\
&\underset{n\to\infty}{\to} \frac{\mathbb{E}[f((G^{(\infty)}(t+), t \geq 0))\mathbb{1}_{\{\zeta < a_\infty\}}]}{\mathbb{P}(\zeta < a_\infty)} \\
&= \mathbb{E}\big[f\big((c_\infty \bar{F}^{(a_\infty)}(b_\infty t+), t \geq 0)\big)\big]. \qquad \square
\end{aligned}$$

Finally, the following lemma is an easy consequence of Proposition A.3 and the continuity property for the decreasing rearrangement of a finite number of elements of $\mathcal{S}$ [Lemma A.2(i)]. Its proof is omitted.

LEMMA A.6. (i) *Consider càdlàg functions $u^{(n)}, u : [0, \infty) \to \mathcal{S}$ such that $u^{(n)} \to u$ as $n \to \infty$ with respect to the Skorokhod topology. Let (t_n) be a sequence of nonnegative numbers converging to $t \geq 0$, and consider another family of càdlàg functions $v^{(n)}, v : [0, \infty) \to \mathcal{S}$ such that $v^{(n)}(t_n) \to v(t)$. For $s \geq 0$, set $f_n(s) = \{u_j^{(n)}(s), v_k^{(n)}(s), j \geq 1, k \geq 1\}^{\downarrow}$ and similarly $f(s) = \{u_j(s), v_k(s), j \geq 1, k \geq 1\}^{\downarrow}$. Then the functions f_n, f are càdlàg and satisfy assertions* (a), (b) *and* (c) *of Proposition* A.3 *for the sequence (t_n).*

(ii) *Let $u^{(n,i)}, u^{(i)}, n \in \mathbb{N}, i \in I$ be càdlàg functions from $[0, \infty)$ to $\mathcal{S}$, with I a finite set. For $t \geq 0$, set $g_n(t) = \{u_j^{(n,i)}(t), j \geq 1, i \in I\}^{\downarrow}$ and $g(t) = \{u_j^{(n,i)}(t), j \geq 1, i \in I\}^{\downarrow}$. These functions are càdlàg. Moreover, if $u^{(n,i)} \to u^{(i)}$ as $n \to \infty$ in the Skorokhod sense for all $i \in I$ and if the functions $u^{(i)}, i \in I$ do not jump simultaneously on $[0, \infty)$, then g_n converges in the Skorokhod sense to g as $n \to \infty$.*

A.2. Properties of the stationary and biased Markov chains. We collect here various technical results about the stationary and biased Markov chains (introduced in Section 3.3) which are used in the body of the paper.

LEMMA A.7. *If* $\int_{\mathcal{S}_1} s_1^{-1}\nu(\mathrm{d}\mathbf{s}) < \infty$, *then for all* $c > 0$,

$$\int_0^\infty \frac{\exp(-cx)}{f_\zeta(x)}\pi_{\mathrm{stat}}(\mathrm{d}x) < \infty.$$

PROOF. It suffices to prove the result for small values of $c > 0$. As in the proof of Lemma 3.8, let $V(x) = \exp(-cx)/f_\zeta(x)$, $x > 0$, with $c \in (0, 1/2)$ small enough so that $\exp(cx) f_\zeta(x) \to 0$ as $x \to \infty$. Then, as a direct consequence of (3.10) and Theorem 14.0.1 of [23], we have that $\int_0^\infty V(x)\pi_{\mathrm{stat}}(\mathrm{d}x) < \infty$. The result follows. □

LEMMA A.8. *If* $\int_{\mathcal{S}_1} s_1^{-1}\nu(\mathrm{d}\mathbf{s}) < \infty$, *then for* $a > 0$ *sufficiently small and all* $b < 1 + 1/|\alpha|$,

$$\int_1^\infty \exp(ax)\pi_{\mathrm{stat}}(\mathrm{d}x) < \infty \quad \textit{and} \quad \int_0^1 x^{-b}\pi_{\mathrm{stat}}(\mathrm{d}x) < \infty.$$

In particular, for all $p > 0$,

$$\mathbb{E}\big[\big|\log\big(Z_0^{\mathrm{stat}}\big)\big|^p\big] < \infty.$$

PROOF. To see the first assertion, note that by Lemma 2.2, there exist constants $C_1 > 0$ and $c > 0$ such that

$$f_\zeta(x) \le C_1 \exp(-cx)$$

for all $x > 0$. Hence, for all $a < c$, by Lemma A.7,

$$\int_0^\infty \exp(ax)\pi_{\mathrm{stat}}(\mathrm{d}x) \le C_1 \int_0^\infty \frac{\exp(-(c-a)x)}{f_\zeta(x)}\pi_{\mathrm{stat}}(\mathrm{d}x) < \infty.$$

Next, from (3.6), we have that

$$\frac{\pi_{\mathrm{stat}}(x)}{f_\zeta(x)} = \int_{\mathcal{S}_1}\left(\sum_{i=1}^\infty e^{s_i^{-\alpha}x}\prod_{j\neq i}\mathbb{F}_\zeta\big(s_j^\alpha s_i^{-\alpha}x\big)\left(\int_{s_i^{-\alpha}x}^\infty \frac{e^{-y}\pi_{\mathrm{stat}}(y)}{f_\zeta(y)}\,\mathrm{d}y\right)\right)\nu(\mathrm{d}\mathbf{s}).$$

Recall the definition of $\zeta^{(I)}$ from just below equation (3.3). Since $\int_0^\infty \frac{\exp(-x)}{f_\zeta(x)} \times \pi_{\mathrm{stat}}(\mathrm{d}x) < \infty$ and $e^{s_i^{-\alpha}x} \le e^x$, there exists a constant C such that

$$\begin{aligned}\frac{\pi_{\mathrm{stat}}(x)}{f_\zeta(x)} &\le \frac{Ce^x}{1-\mathbb{F}_\zeta(x)}\int_{\mathcal{S}_1}\left(\sum_{i=1}^\infty\big(1-\mathbb{F}_\zeta(x)\big)\prod_{j\neq i}\mathbb{F}_\zeta\big(s_j^\alpha s_i^{-\alpha}x\big)\right)\nu(\mathrm{d}\mathbf{s})\\ &\le \frac{Ce^x}{1-\mathbb{F}_\zeta(x)}\mathbb{P}\big(\zeta^{(I)} > x\big) \le \frac{Ce^x}{1-\mathbb{F}_\zeta(x)}.\end{aligned}$$

Then, for $x \in (0, 1]$, $\pi_{\mathrm{stat}}(x)/f_\zeta(x)$ is bounded by some constant C_2. It follows that

$$\int_0^1 x^{-b}\pi_{\mathrm{stat}}(\mathrm{d}x) \le C_2 \int_0^1 x^{-b} f_\zeta(x)\,\mathrm{d}x,$$

and this upper bound is finite by Lemma 2.2(iii) when $b < 1 + 1/|\alpha|$. $\square$

LEMMA A.9. *If* $\int_{\mathcal{S}_1} s_1^{-1}\nu(\mathrm{d}\mathbf{s}) < \infty$, *then* $\mathbb{E}[(\log(Y_1^{\mathrm{stat}}))^p] < \infty$ *for all* $p > 0$.

PROOF. Fix $p > 0$. By definition,

$$\begin{aligned}
&\mathbb{E}\big[(\log(Y_1^{\mathrm{stat}}))^p\big]\\
&\quad = \frac{1}{|\alpha|^p}\int_0^\infty \mathbb{E}\biggl[\biggl(\log\biggl(\frac{\zeta}{\zeta - T_1}\biggr)\biggr)^p \Big| \zeta = x\biggr]\pi_{\mathrm{stat}}(\mathrm{d}x)\\
&\quad \le \frac{C_p}{|\alpha|^p}\int_0^\infty \bigl(|\log(x)|^p + \mathbb{E}\bigl[|\log(\zeta - T_1)|^p \mathbb{1}_{\{\zeta - T_1 \le 1\}} | \zeta = x\bigr]\bigr)\pi_{\mathrm{stat}}(\mathrm{d}x),
\end{aligned}$$

for some constant C_p. By Lemma A.8, $\int_0^\infty |\log(x)|^p\pi_{\mathrm{stat}}(\mathrm{d}x) < \infty$. Next, using the notation introduced in Lemma 2.2, we write $\zeta = T_1 + \xi$ where $\xi = \max_{i\ge1}\{F_i^{-\alpha}(T_1)\zeta^{(i)}\}$. Since T_1 is independent of ξ and is exponentially distributed with mean 1, the joint distribution of (ξ, ζ) is $\exp(-x + y) \times \mathbb{1}_{0\le y\le x} f_\xi(y)\,\mathrm{d}y\,\mathrm{d}x$, where we recall that f_ξ denotes the density of ξ. Hence

$$\begin{aligned}
&\int_0^\infty \mathbb{E}\bigl[|\log(\zeta - T_1)|^p \mathbb{1}_{\{\zeta - T_1 \le 1\}} | \zeta = x\bigr]\pi_{\mathrm{stat}}(\mathrm{d}x)\\
&\qquad = \int_0^\infty \frac{\exp(-x)}{f_\zeta(x)}\biggl(\int_0^{\min(x,1)} \exp(y)|\log y|^p f_\xi(y)\,\mathrm{d}y\biggr)\pi_{\mathrm{stat}}(\mathrm{d}x)\\
&\qquad \le e\int_0^1 |\log y|^p f_\xi(y)\,\mathrm{d}y \int_0^\infty \frac{\exp(-x)}{f_\zeta(x)}\pi_{\mathrm{stat}}(\mathrm{d}x).
\end{aligned}$$

The integral $\int_0^\infty \exp(-x)/f_\zeta(x)\pi_{\mathrm{stat}}(\mathrm{d}x)$ is finite, by Lemma A.7. Finally, note that $\xi \ge F_1(T_1)^{-\alpha}\zeta^{(1)}$ and so

$$\mathbb{E}\bigl[|\log(\xi)|^p \mathbb{1}_{\{\xi\le1\}}\bigr] \le C_p\bigl(|\alpha|^p\mathbb{E}\bigl[|\log(F_1(T_1))|^p\bigr] + \mathbb{E}\bigl[|\log(\zeta^{(1)})|^p\bigr]\bigr).$$

The first expectation on the right-hand side is equal to $\int_{\mathcal{S}_1} |\log(s_1)|^p\nu(\mathrm{d}\mathbf{s})$ and is finite since $\int_{\mathcal{S}_1} s_1^{-1}\nu(\mathrm{d}\mathbf{s}) < \infty$. The second expectation is also finite, by assertions (i) and (ii) of Lemma 2.2. $\square$

The following result is the only place that we need the extra condition $\int_{\mathcal{S}_1} s_1^{-1-\rho}\nu(\mathrm{d}\mathbf{s}) < \infty$ for some $\rho > 0$.

LEMMA A.10. *Assume that* $\int_{\mathcal{S}_1} s_1^{-1-\rho}\nu(d\mathbf{s}) < \infty$ *for some* $\rho > 0$. *Then there exits* $\delta_\rho > 0$ *such that for all* $\delta \in [0, \delta_\rho)$,

$$\mathbb{E}\big[\big|\log\big(\mathbb{F}_\zeta\big(Z_0^{\mathrm{stat}}\big(Y_1^{\mathrm{stat}}\big)^\alpha\big)\big)\big|^{1+\delta}\big] < \infty.$$

PROOF. Again we let $\xi = \sup_{i\ge 1}\{F_i(T_1)^{-\alpha}\zeta^{(i)}\}$. The first step of our proof is to show that, for $0 < x \le 1$,

$$\big|\log\big(\mathbb{F}_\zeta(x)\big)\big| \le C\big(x^{1/\alpha}|\log x| + x^{1/\alpha}\big|\log\big(\mathbb{F}_\xi(x)\big)\big|\big). \tag{A.1}$$

For $x > 0$, let $K(x) = \sup\{k \ge 1 : F_k(T_1) > x^{-1/\alpha}\}$, and let $C_1 > 1$ be such that $1 - t \ge \exp(-C_1 t)$ for all $t \in [0, \mathbb{P}(\zeta > 1))$. Then

$$\begin{aligned}\prod_{i\ge K(x)+1}\mathbb{F}_\zeta\big(xF_i(T_1)^\alpha\big) &\ge \exp\Big(-C_1\sum_{i\ge K(x)+1}\mathbb{P}\big(\zeta > xF_i(T_1)^\alpha\big)\Big)\\ &\ge \exp\Big(-C_1\mathbb{E}\big[\zeta^{-1/\alpha}\big]x^{1/\alpha}\sum_{i\ge K(x)+1}F_i(T_1)\Big)\\ &\ge \exp\big(-C_2x^{1/\alpha}\big),\end{aligned}$$

where we have used Markov's inequality to get the second inequality and the fact that $\sum_{i\ge K(x)+1}F_i(T_1) \le 1$ to get the third. Now note that $K(x) \le x^{1/\alpha}$ since $F_k(T_1) \le 1/k$ for all $k \ge 1$. So, for $c \in (0, 1)$ such that $\nu(s_1 \le c) > 0$,

$$\begin{aligned}\mathbb{F}_\xi(x) &= \mathbb{E}\Big[\prod_{i\ge 1}\mathbb{F}_\zeta\big(xF_i(T_1)^\alpha\big)\Big]\\ &\ge \mathbb{E}\Bigg[\prod_{i=1}^{K(x)}\mathbb{F}_\zeta\big(xF_1(T_i)^\alpha\big)\Bigg]\exp\big(-C_2x^{1/\alpha}\big)\\ &\ge \mathbb{E}\big[\mathbb{F}_\zeta\big(xc^\alpha\big)^{K(x)}\mathbb{1}_{\{F_1(T_1)\le c\}}\big]\exp\big(-C_2x^{1/\alpha}\big)\\ &\ge \nu(s_1 \le c)\mathbb{F}_\zeta\big(xc^\alpha\big)^{x^{1/\alpha}}\exp\big(-C_2x^{1/\alpha}\big).\end{aligned}\tag{A.2}$$

Next, since $\mathbb{F}_\zeta(x) = \exp(-x)\int_0^x \exp(y)\mathbb{F}_\xi(y)\,dy$, we have that for all $0 < x \le 1$,

$$\mathbb{F}_\zeta(x) \ge \exp(-1)\big(1 - c^{-\alpha/2}\big)x\mathbb{F}_\xi\big(c^{-\alpha/2}x\big). \tag{A.3}$$

Using (A.2), we get

$$\mathbb{F}_\zeta(x) \ge C_3x\mathbb{F}_\zeta\big(c^{-\alpha/2}xc^\alpha\big)^{c^{-1/2}x^{1/\alpha}}\exp\big(-C_2c^{-1/2}x^{1/\alpha}\big),$$

and another application of (A.3) yields

$$\mathbb{F}_\zeta(x) \ge C_3x\big(C_4x\mathbb{F}_\xi\big(c^{-\alpha}xc^\alpha\big)\big)^{c^{-1/2}x^{1/\alpha}}\exp\big(-C_2c^{-1/2}x^{1/\alpha}\big).$$

All of the constants here are strictly positive, and so (A.1) follows.

From (A.1) and Hölder's inequality, we deduce that for all $\delta > 0$,

$$\begin{aligned}&\mathbb{E}\big[\big|\log\big(\mathbb{F}_\zeta(\xi)\big)\mathbb{1}_{\{\xi\le 1\}}\big|^{1+\delta}\big]\\&\quad\le C'\big(\mathbb{E}\big[\xi^{(1+\delta)^2/\alpha}\big]+\mathbb{E}\big[\xi^{(1+\delta)^2/\alpha}\big]^{1/(1+\delta)}\mathbb{E}\big[\big|\log\big(\mathbb{F}_\xi(\xi)\big)\big|^{(1+\delta)^2/\delta}\big]^{\delta/(1+\delta)}\big).\end{aligned}$$

Since $\mathbb{F}_\xi(\xi)$ has a uniform distribution, $|\log(\mathbb{F}_\xi(\xi))| \sim \mathrm{Exp}(1)$ and so has finite positive moments of all orders. Moreover, since $\xi \ge F_1(T_1)^{-\alpha}\zeta^{(1)}$, we have

$$\mathbb{E}\big[\xi^{(1+\delta)^2/\alpha}\big]\le\mathbb{E}\big[\zeta^{(1+\delta)^2/\alpha}\big]\int_{\mathcal{S}_1}s_1^{-(1+\delta)^2}\nu(\mathrm{d}\mathbf{s}).$$

Let $\rho > 0$ be such that $\int_{\mathcal{S}_1} s_1^{-1-\rho}\nu(\mathrm{d}\mathbf{s}) < \infty$. By Lemma 2.2(iii), $\mathbb{E}[\zeta^{-a}] < \infty$ for all $a < 1 + (1+\rho)/|\alpha|$. So for all $\delta \ge 0$ such that $(1+\delta)^2 \le 1+\rho$, the expectation $\mathbb{E}[\xi^{(1+\delta)^2/\alpha}]$ is finite and thus

$$\mathbb{E}\big[\big|\log\big(\mathbb{F}_\zeta(\xi)\big)\big|^{1+\delta}\big]<\infty$$

[since $|\log(\mathbb{F}_\zeta(\xi))| \le |\log(\mathbb{F}_\zeta(1))|$ when $\xi \ge 1$].

In particular, we can deduce that $\mathbb{E}[|\log(\mathbb{F}_\zeta(\xi))|^{1+\delta}|\zeta = x_0] < \infty$ for some $x_0 > 0$. Our goal now is to check that

$$\mathbb{E}\big[\big|\log\big(\mathbb{F}_\zeta\big(Z_0^{\mathrm{stat}}\big(Y_1^{\mathrm{stat}}\big)^\alpha\big)\big)\big|^{1+\delta}\big]<\infty.$$

Recall that $\xi = \zeta - T_1$ and so $Z_0Y_1^\alpha = Z_1\Theta_1^{-\alpha} = \xi$. Hence,

$$\mathbb{E}\big[\big|\log\big(\mathbb{F}_\zeta\big(Z_0^{\mathrm{stat}}\big(Y_1^{\mathrm{stat}}\big)^\alpha\big)\big)\big|^{1+\delta}\big]=\int_0^\infty\mathbb{E}\big[\big|\log\big(\mathbb{F}_\zeta(\xi)\big)\big|^{1+\delta}|\zeta=x\big]\pi_{\mathrm{stat}}(\mathrm{d}x).$$

Write $\int_0^\infty = \int_0^{x_0} + \int_{x_0}^\infty$, where x_0 is chosen so that $\mathbb{E}[|\log(\mathbb{F}_\zeta(\xi))|^{1+\delta}|\zeta = x_0] < \infty$. As seen in the proof of Lemma A.8, $\pi_{\mathrm{stat}}(x) \le C_{x_0} f_\zeta(x)$ on $(0, x_0)$. Hence,

$$\int_0^{x_0}\mathbb{E}\big[\big|\log\big(\mathbb{F}_\zeta(\xi)\big)\big|^{1+\delta}|\zeta=x\big]\pi_{\mathrm{stat}}(\mathrm{d}x)\le C_{x_0}\mathbb{E}\big[\big|\log\big(\mathbb{F}_\zeta(\xi)\big)\big|^{1+\delta}\big]<\infty.$$

Next, for $x > x_0$, we use the fact that the joint distribution of (ξ, ζ) is $\exp(-z + y)\mathbb{1}_{0\le y\le z} f_\xi(y)\,\mathrm{d}y\,\mathrm{d}z$, to obtain that

$$\begin{aligned}&\mathbb{E}\big[\big|\log\big(\mathbb{F}_\zeta(\xi)\big)\big|^{1+\delta}|\zeta=x\big]\\&\quad=\frac{e^{-x}}{f_\zeta(x)}\int_0^x e^y\big|\log\big(\mathbb{F}_\zeta(y)\big)\big|^{1+\delta}f_\xi(y)\,\mathrm{d}y\\&\quad\le\frac{e^{-x}}{f_\zeta(x)}\int_0^{x_0} e^y\big|\log\big(\mathbb{F}_\zeta(y)\big)\big|^{1+\delta}f_\xi(y)\,\mathrm{d}y\\&\qquad+\big|\log\big(\mathbb{F}_\zeta(x_0)\big)\big|^{1+\delta}\frac{e^{-x}}{f_\zeta(x)}\int_{x_0}^x e^y f_\xi(y)\,\mathrm{d}y\\&\quad\le\frac{e^{-x}f_\zeta(x_0)}{f_\zeta(x)e^{-x_0}}\mathbb{E}\big[\big|\log\big(\mathbb{F}_\zeta(\xi)\big)\big|^{1+\delta}|\zeta=x_0\big]+\big|\log\big(\mathbb{F}_\zeta(x_0)\big)\big|^{1+\delta}.\end{aligned}$$

The integral of this upper bound with respect to $\pi_{\text{stat}}(\mathrm{d}x)$ on (x_0, ∞) is finite, by Lemma A.7. $\square$

We now prove some almost sure limits for the biased chain.

LEMMA A.11. *As $n \to \infty$, the following limits hold almost surely*:

$$\frac{1}{n}\sum_{j=1}^{n} \log(Y_j^{\text{bias}}) \to \mu, \qquad \frac{1}{n}\sum_{j=-n+1}^{0} \log(Y_j^{\text{bias}}) \to \mu,$$

$$\frac{1}{n}\log(Y_{-n}^{\text{bias}}) \to 0, \qquad \frac{1}{n}\log(Z_{-n}^{\text{bias}}) \to 0,$$

$$\frac{1}{n}\log(\mathbb{F}_{\zeta}(Z_{-n-1}^{\text{bias}}(Y_{-n}^{\text{bias}})^{\alpha})) \to 0.$$

PROOF. Suppose that $(X_k)_{k\geq 0}$ is any positive Harris chain possessing an invariant distribution. Then Theorem 17.0.1 of Meyn and Tweedie [23] gives the following law of large numbers: for any function g such that $\mathbb{E}[|g(X_0^{\text{stat}})|] < \infty$,

$$\frac{1}{n}\sum_{j=1}^{n} g(X_j) \to \mathbb{E}[g(X_0^{\text{stat}})]$$

almost surely, as $n \to \infty$, irrespective of the distribution of X_0. Moreover, it follows straightforwardly from this that $n^{-1}g(X_n) \to 0$ almost surely, as $n \to \infty$.

Now note that $(Z_k^{\text{bias}}, Y_k^{\text{bias}})_{k\geq 1}$ is a realization of the Markov chain $(Z_k, Y_k)_{k\geq 1}$ with initial state (Z_1, Y_1) having the distribution specified (for suitable test functions ϕ) by

$$\mathbb{E}[\phi(Z_1, Y_1)] = \frac{1}{\mu}\mathbb{E}[\log(Y_1^{\text{stat}})\phi(Z_1^{\text{stat}}, Y_1^{\text{stat}})].$$

Since $\mathbb{E}[\log(Y_1^{\text{stat}})] = \mu < \infty$, we get that a.s.

$$\frac{1}{n}\sum_{j=1}^{n} \log(Y_j^{\text{bias}}) \to \mu.$$

Observe next that $(Z_{-k}^{\text{bias}}, Y_{-k}^{\text{bias}})_{k\geq 0}$ is a realization of the (backward) Markov chain $(Z_{-k}, Y_{-k})_{k\geq 0}$ with initial distribution for (Z_0, Y_0) specified (for suitable test functions ϕ) by

$$\mathbb{E}[\phi(Z_0, Y_0)] = \frac{1}{\mu}\mathbb{E}[\log(Y_1^{\text{stat}})\phi(Z_0^{\text{stat}}, Y_0^{\text{stat}})].$$

The chain $(Z_{-k}, Y_{-k})_{k\geq 0}$ is also a positive Harris chain possessing the same invariant distribution as $(Z_k, Y_k)_{k\geq 1}$. Hence,

$$\frac{1}{n}\sum_{j=-n+1}^{0} \log(Y_j^{\text{bias}}) \to \mu \quad \text{and} \quad \frac{1}{n}\log(Y_{-n}^{\text{bias}}) \to 0$$

almost surely, as before. By Lemma A.8, $\mathbb{E}[|\log(Z_1^{\mathrm{stat}})|] < \infty$ and, by the $\delta = 0$ case of Lemma A.10, $\mathbb{E}[|\log(\mathbb{F}_\zeta(Z_0^{\mathrm{stat}}(Y_1^{\mathrm{stat}})^\alpha))|] < \infty$, and so we also have the almost sure convergences

$$\frac{1}{n}\log(Z_{-n}^{\mathrm{bias}}) \to 0 \quad \text{and} \quad \frac{1}{n}\big|\log(\mathbb{F}_\zeta(Z_{-n-1}^{\mathrm{bias}}(Y_{-n}^{\mathrm{stat}})^\alpha))\big| \to 0. \qquad \square$$

Finally, we show that $\mathbb{E}[\prod_{i=1}^n (Y_i^{\mathrm{stat}})^\alpha]$ decays exponentially in n.

LEMMA A.12. *For any $x > 0$, we have*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left[\prod_{i=1}^n (Y_i^{\mathrm{stat}})^\alpha\right] < 0.$$

In order to prove Lemma A.12, we use a renewal process derived from the biased Markov chain $(Z_n^{\mathrm{bias}})_{n\in\mathbb{Z}}$. We therefore begin with a result about general renewal processes.

Suppose that $(N(n))_{n\geq 0}$ is a delayed renewal process. Write τ_0 for the delay and $\tau_1, \tau_2, \ldots$ for the subsequent arrival times, so that $\tau_{k+1} - \tau_k$ are i.i.d. random variables for $k \geq 0$, independent of τ_0, and $N(n) = \#\{k \geq 1 : \tau_k \leq n\}$. We will say that a random variable X has *exponential tails* if there exists $r > 1$ such that $\mathbb{E}[r^X] < \infty$.

LEMMA A.13. *Suppose that τ_0 and $\tau_1 - \tau_0$ both have exponential tails. Then for any $s \in (0, 1)$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}[s^{N(n)}] < 0.$$

PROOF. The proof is elementary, and so we sketch it. Let $\chi = \mathbb{E}[\tau_1 - \tau_0]$ be the mean of the standard inter-arrival distribution and take $\varepsilon > 0$. Then

$$\begin{aligned}\mathbb{E}[s^{N(n)}] &\leq \mathbb{P}\big(N(n) < (\chi^{-1} - \varepsilon)n\big) + s^{(\chi^{-1}-\varepsilon)n} \\ &\leq \mathbb{P}(\tau_{k_n} \geq n) + s^{(\chi^{-1}-\varepsilon)n},\end{aligned}$$

where $k_n = \lfloor (\chi^{-1} - \varepsilon)n \rfloor$. But a simple application of the Gärtner–Ellis theorem then implies that

$$\mathbb{P}(\tau_{k_n} \geq n) \leq \mathbb{P}\big(\tau_{k_n} \geq k_n \chi/(1 - \chi\varepsilon)\big)$$

is exponentially small in n. The result follows. $\square$

Suppose now that we mark the kth inter-arrival interval with some probability which depends, in general, on its length $\tau_k - \tau_{k-1}$, but independently for different inter-arrival intervals. Let I_k be the indicator that the kth inter-arrival interval is marked, so that $I_1, I_2, \dots$ are independent Bernoulli random variables such that I_k depends on $\tau_i, i \geq 0$ only through $\tau_k - \tau_{k-1}$. Let

$$M(n) = \#\{k \geq 1 : \tau_k \leq n, I_k = 1\}. \tag{A.4}$$

$(M(n))_{n \geq 0}$ is again a delayed renewal process.

LEMMA A.14. *Suppose that τ_0 and $\tau_1 - \tau_0$ have exponential tails and that $q := \mathbb{P}(I_1 = 1) > 0$. Then the delay and inter-arrival distributions of $(M(n))_{n \geq 0}$ have exponential tails. Hence, for any $s \in (0, 1)$,*

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{E}[s^{M(n)}] < 0.$$

PROOF. The case $q = 1$ follows immediately from Lemma A.13, and so we henceforth assume that $q < 1$. Let $\sigma_1, \sigma_2, \dots, \tilde{\sigma}$ and G be mutually independent random variables, independent of τ_0. Let $\sigma_1, \sigma_2, \dots$ have common distribution given by $\mathbb{P}(\sigma_1 = i) = \mathbb{P}(\tau_1 - \tau_0 = i | I_1 = 0)$, $i \geq 1$. Let $\tilde{\sigma}$ have distribution $\mathbb{P}(\tilde{\sigma} = i) = \mathbb{P}(\tau_1 - \tau_0 = i | I_1 = 1)$, $i \geq 1$. Finally, let G be such that $\mathbb{P}(G = i) = q(1 - q)^i$ for $i \geq 0$. Then the delay has the same distribution as

$$\tau_0 + \sum_{i=1}^{G} \sigma_i + \tilde{\sigma}$$

and the inter-arrival intervals have the same distribution as

$$\sum_{i=1}^{G} \sigma_i + \tilde{\sigma}.$$

By Lemma A.13, it will be sufficient to prove that $\sum_{i=1}^{G} \sigma_i$ and $\tilde{\sigma}$ are random variables with exponential tails. For $r \geq 0$,

$$\mathbb{E}[r^{\sigma_1}] = \mathbb{E}[r^{\tau_1 - \tau_0} | I_1 = 0] \leq \frac{\mathbb{E}[r^{\tau_1 - \tau_0}]}{1 - q}$$

and, similarly,

$$\mathbb{E}[r^{\tilde{\sigma}}] = \mathbb{E}[r^{\tau_1 - \tau_0} | I_1 = 1] \leq \frac{\mathbb{E}[r^{\tau_1 - \tau_0}]}{q}.$$

By assumption, there exists $r > 1$ such that $\mathbb{E}[r^{\tau_1 - \tau_0}] < \infty$. Hence, there exists $r > 1$ such that $\mathbb{E}[r^{\sigma_1}] < \infty$ and $\mathbb{E}[r^{\tilde{\sigma}}] < \infty$. Moreover,

$$\mathbb{E}[r^{\sum_{i=1}^{G} \sigma_i}] = \frac{rq}{1 - (1 - q)\mathbb{E}[r^{\sigma_1}]}.$$

Now $\mathbb{E}[r^{\sigma_1}] \to 1$ as $r \downarrow 1$, and so we can find $r > 1$ sufficiently small that $\mathbb{E}[r^{\sigma_1}] < (1 - q)^{-1}$. Hence, for such a value of r,

$$\mathbb{E}[r^{\sum_{i=1}^{G} \sigma_i}] < \infty.$$

The result follows. $\square$

Recall from Lemma 3.8 the Foster–Lyapunov criterion for the Markov chain $(Z_k)_{k\geq 0}$: there exist a function $V:(0,\infty)\to[1,\infty)$, a small set C and constants $\beta\in(0,1)$ and $b>0$ such that

$$\mathbb{E}\big[V(Z_1)|Z_0=x\big]\leq(1-\beta)V(x)+b\mathbb{1}_{\{x\in C\}}.$$

Since C is small, there exist $p\in(0,1)$ and a probability measure $\tilde{\mu}_C$ [which is a version of the measure μ_C given explicitly at (3.9) normalized to have total mass 1] such that

$$P(x,B)=\mathbb{P}(Z_1\in B|Z_0\in x)\geq p\tilde{\mu}_C(B)$$

for all $x\in C$ and any B any Borel subset of $(0,\infty)$. Consider now constructing the process $(Z_k)_{k\geq 0}$ via the standard *split chain construction*: whenever $Z_k\in C$, we flip a coin with probability $p\in(0,1)$. If the coin comes up heads, we sample Z_{k+1} from the measure $\tilde{\mu}_C$. Otherwise, sample Z_{k+1} from the probability measure $(P(Z_k,\cdot)-p\tilde{\mu}_C(\cdot))/(1-p)$. If $Z_k\notin C$, we simply sample Z_{k+1} from $P(Z_k,\cdot)$. If $Z_k\in C$ and the coin comes up heads, we say that there is a *regeneration* at time k. (In particular, a regeneration can only occur at k if $Z_k\in C$.) Let

$$\tau_0=\inf\{i\geq 0:\text{there is a regeneration at } i\}$$

and for $k\geq 0$,

$$\tau_{k+1}=\inf\{i>\tau_k:\text{there is a regeneration at } i\}.$$

Then τ_0 and $\{\tau_{k+1}-\tau_k:k\geq 0\}$ are all independent, and $\{\tau_{k+1}-\tau_k:k\geq 0\}$ are identically distributed. Hence, $N(n):=\#\{k\geq 1:\tau_k\leq n\}$ is a delayed renewal process.

The following lemma is a standard consequence of geometric ergodicity; see, for example, equation (22) of Roberts and Rosenthal [27] for the precise formulation given here.

LEMMA A.15. *There exists $\theta>1$ such that*

$$\int_0^\infty\mathbb{E}\big[\theta^{\tau_0}|Z_0=x\big]\pi_{\text{stat}}(\mathrm{d}x)<\infty\quad\textit{and}\quad\mathbb{E}\big[\theta^{\tau_1-\tau_0}\big]<\infty.$$

Hence, if the chain is begun in stationarity, $(N(n))_{n\geq 0}$ is a delayed renewal process such that both delay and inter-arrival distributions have exponential tails.

PROOF OF LEMMA A.12. Let $f:(0,\infty)^2\to(0,1)$ be defined by

$$f(x,y)=\mathbb{E}\big[Y_1^\alpha|Z_0=x,Z_1=y\big].$$

Using the fact that $(Z_n)_{n\geq 0}$ acts a driving chain for $(Z_n,Y_n)_{n\geq 0}$, we have that $Y_1,Y_2,\dots,Y_n$ are conditionally independent given $Z_0,Z_1,\dots,Z_n$ and, for $1\leq i\leq n$, the distribution of Y_i depends only on the values of Z_{i-1} and Z_i. Hence, for all $x > 0$,

$$\mathbb{E}\left[\prod_{i=1}^{n} Y_i^{\alpha} \middle| Z_0 = x\right] = \mathbb{E}\left[\prod_{i=1}^{n} f(Z_{i-1}, Z_i) \middle| Z_0 = x\right]$$

and, therefore,

$$\mathbb{E}\left[\prod_{i=1}^{n} (Y_i^{\mathrm{stat}})^{\alpha}\right] = \mathbb{E}\left[\prod_{i=1}^{n} f(Z_{i-1}^{\mathrm{stat}}, Z_i^{\mathrm{stat}})\right].$$

The function f takes values in $(0, 1)$ and is continuous, so for any compact set $K \subseteq (0, \infty)^2$ we can find a constant $\gamma \in (0, 1)$ such that $f(x, y) \le \gamma$ on K. Take $K = K_1 \times K_2$, where $K_1, K_2 \subseteq (0, \infty)$ are compact and have strictly positive Lebesgue measure. Let $\tilde{N}(n) = \#\{1 \le i \le n : (Z_{i-1}^{\mathrm{stat}}, Z_i^{\mathrm{stat}}) \in K\}$. Then

$$\mathbb{E}\left[\prod_{i=1}^{n} (Y_i^{\mathrm{stat}})^{\alpha}\right] \le \mathbb{E}[\gamma^{\tilde{N}(n)}].$$

We will bound $\tilde{N}(n)$ below by the number of renewals between which there is a visit to K, that is,

$$M(n) = \#\{k \ge 1 : \tau_k \le n, (Z_{i-1}^{\mathrm{stat}}, Z_i^{\mathrm{stat}}) \in K \text{ for some } \tau_{k-1} + 1 < i \le \tau_k\}.$$

This clearly has the effect of independently marking the renewal intervals, as at (A.4). Note that since $P(x, B) > 0$ for any $x \in (0, \infty)$ and any Borel set $B \subseteq (0, \infty)$ of positive Lebesgue measure, there is positive probability of visiting K between any two renewals. The result then follows from Lemmas A.14 and A.15. $\square$

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REFERENCES

[1] ALDOUS, D. and SHIELDS, P. (1988). A diffusion limit for a class of randomly-growing binary trees. *Probab. Theory Related Fields* **79** 509–542. MR0966174

[2] ALDOUS, D. J. and BANDYOPADHYAY, A. (2005). A survey of max-type recursive distributional equations. *Ann. Appl. Probab.* **15** 1047–1110. MR2134098

[3] ALSMEYER, G. (1994). On the Markov renewal theorem. *Stochastic Process. Appl.* **50** 37–56. MR1262329

[4] ALSMEYER, G. (1997). The Markov renewal theorem and related results. *Markov Process. Related Fields* **3** 103–127. MR1446921

[5] ATHREYA, K. B. (1985). Discounted branching random walks. *Adv. in Appl. Probab.* **17** 53–66. MR0778593

[6] ATHREYA, K. B., MCDONALD, D. and NEY, P. (1978). Limit theorems for semi-Markov processes and renewal theory for Markov chains. *Ann. Probab.* **6** 788–797. MR0503952

[7] BARLOW, M. T., PEMANTLE, R. and PERKINS, E. A. (1997). Diffusion-limited aggregation on a tree. *Probab. Theory Related Fields* **107** 1–60. MR1427716

[8] BERTOIN, J. (2001). Homogeneous fragmentation processes. *Probab. Theory Related Fields* **121** 301–318. MR1867425

[9] BERTOIN, J. (2002). Self-similar fragmentations. *Ann. Inst. Henri Poincaré Probab. Stat.* **38** 319–340. MR1899456

[10] BERTOIN, J. (2003). The asymptotic behavior of fragmentation processes. *J. Eur. Math. Soc.* (*JEMS*) **5** 395–416. MR2017852

[11] BERTOIN, J. (2006). *Random Fragmentation and Coagulation Processes. Cambridge Studies in Advanced Mathematics* **102**. Cambridge Univ. Press, Cambridge. MR2253162

[12] DEAN, D. S. and MAJUMDAR, S. N. (2006). Phase transition in a generalized Eden growth model on a tree. *J. Stat. Phys.* **124** 1351–1376. MR2266447

[13] DEVROYE, L. (1986). A note on the height of binary search trees. *J. Assoc. Comput. Mach.* **33** 489–498. MR0849025

[14] DUQUESNE, T. and LE GALL, J.-F. (2002). Random trees, Lévy processes and spatial branching processes. *Astérisque* **281** vi+147. MR1954248

[15] DUQUESNE, T. and LE GALL, J.-F. (2005). Probabilistic and fractal aspects of Lévy trees. *Probab. Theory Related Fields* **131** 553–603. MR2147221

[16] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence*. Wiley, New York. MR0838085

[17] FILIPPOV, A. F. (1961). On the distribution of the sizes of particles which undergo splitting. *Theory Probab. Appl.* **6** 275–294.

[18] GOLDSCHMIDT, C. and HAAS, B. (2010). Behavior near the extinction time in self-similar fragmentations. I. The stable case. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** 338–368. MR2667702

[19] HAAS, B. (2003). Loss of mass in deterministic and random fragmentations. *Stochastic Process. Appl.* **106** 245–277. MR1989629

[20] HAAS, B. (2004). Regularity of formation of dust in self-similar fragmentations. *Ann. Inst. Henri Poincaré Probab. Stat.* **40** 411–438. MR2070333

[21] JACOD, J. (1971). Théorème de renouvellement et classification pour les chaînes semi-markoviennes. *Ann. Inst. H. Poincaré Sect. B* (*N.S.*) **7** 83–129. MR0305496

[22] KESTEN, H. (1974). Renewal theory for functionals of a Markov chain with general state space. *Ann. Probab.* **2** 355–386. MR0365740

[23] MEYN, S. P. and TWEEDIE, R. L. (1993). *Markov Chains and Stochastic Stability*. Springer London, London. MR1287609

[24] MIERMONT, G. (2003). Self-similar fragmentations derived from the stable tree. I. Splitting at heights. *Probab. Theory Related Fields* **127** 423–454. MR2018924

[25] OREY, S. (1961). Change of time scale for Markov processes. *Trans. Amer. Math. Soc.* **99** 384–397. MR0145586

[26] PITTEL, B. (1984). On growing random binary trees. *J. Math. Anal. Appl.* **103** 461–480. MR0762569

[27] ROBERTS, G. O. and ROSENTHAL, J. S. (2004). General state space Markov chains and MCMC algorithms. *Probab. Surv.* **1** 20–71. MR2095565

[28] SHURENKOV, V. (1985). On the theory of Markov renewal. *Theory Probab. Appl.* **29** 247–265.

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