Around the Brownian continuum random tree

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1. INTRODUCTION: BINARY TREES
Binary leaf-labelled trees

- Let $\mathcal{T}_n$ be the set of planted binary leaf-labelled trees with $n$ labelled leaves (note: we don’t distinguish a planar ordering around each vertex).
- The root, labelled 0 is, by convention, not a leaf.
- Note that every element of $\mathcal{T}_n$ has $n - 1$ internal vertices (which are not labelled) and $2n - 1$ edges.

![Diagram of a binary leaf-labelled tree]
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\[
|\mathcal{T}_n| = \frac{1}{n} \binom{2n - 2}{n - 1} \quad \text{(Catalan numbers)}
\]
Uniform binary leaf-labelled trees

\[ |T_n| = \frac{1}{n} \binom{2n-2}{n-1} \sim \frac{4^{n-1}}{n^{3/2} \sqrt{\pi}} \quad \text{as } n \to \infty. \]
Uniform binary leaf-labelled trees

$$|\mathbb{T}_n| = \frac{1}{n} \binom{2n-2}{n-1} \sim \frac{4^{n-1}}{n^{3/2} \sqrt{\pi}}$$ as $n \to \infty$.

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Our first object of interest is a uniform random element of \( \mathbb{T}_n \).

Rémy’s algorithm recursively constructs a sequence \((T_n)_{n \geq 1}\) of trees such that \( T_n \) is uniform on \( \mathbb{T}_n \) for each \( n \).
Rémy’s algorithm

- Start from a single edge with endpoints labelled 0 and 1.
- At step $n \geq 2$, pick an edge uniformly at random, divide it into two edges, insert a new vertex in the middle and attach to that vertex a new edge with a leaf labelled $n$ at its other end.
Rémy’s algorithm
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Claim: for each \( n \), \( T_n \) is a uniform element of \( \mathbb{T}_n \).

Vague question: what can we say about $T_n$ as $n \to \infty$?
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Concrete first question: as $n \to \infty$, how does the distance between 0 and 1 behave?
An urn in Rémy’s algorithm

The total number of edges present at step $n$ is equal to $2n - 1$.

Consider the number of edges in the path between 0 and 1:

- If we add our new leaf somewhere along that path, it gets longer by 1.
- If we add our new leaf anywhere else, the length of the path remains the same.
An urn in Rémy’s algorithm

We have an urn process with two colours, say black and white, where each black ball represents an edge in the path between 0 and 1, and each white ball represents an edge elsewhere.

When we pick a black ball, we replace it in the urn together with one black and one white ball.

When we pick a white ball, we replace it in the urn together with two new white balls.

We start with a single black ball. At step $n$, we always have $2n - 1$ balls present.
An urn in Rémy’s algorithm

Let $B_n$ be the number of black balls at step $n$. 
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We have $B_1 = 1$.

For $n \geq 1$,

$$
\mathbb{E} [B_{n+1} | \mathcal{F}_n] = \frac{B_n}{2n-1} (B_n + 1) + \frac{2n - 1 - B_n}{2n - 1} B_n
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Define a sequence by $b_1 = 1$ and $b_{n+1} = \frac{2^{2n}(n!)^2}{(2n)!}$ for $n \geq 1$. Then

$$b_{n+1} = \frac{2n}{2n-1}b_n.$$
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$$b_{n+1} = \frac{2n}{2n - 1} b_n.$$

Then we have that

$$\left( \frac{B_n}{b_n} \right)_{n \geq 1}$$

is a non-negative martingale.
Martingale limit

\((B_n/b_n)_{n \geq 1}\) is also bounded in \(L^2\), so it has an almost sure limit.

Since

\[ b_{n+1} = \frac{2^{2n}(n!)^2}{(2n)!} \sim \sqrt{\pi n}, \]

we get that

\[ \frac{B_n}{\sqrt{2n}} \to L \text{ a.s. as } n \to \infty. \]

Limiting distribution for the length

It also turns out that the law of $B_{n+1}$ is explicit:

\[ \mathbb{P}(B_{n+1} = k) = \frac{k-1}{n} 2^{k-1} \binom{2n-k}{n-1} \binom{2n}{n} \]

and so

\[ \mathbb{P} \left( B_{n+1} = \lfloor x \sqrt{2n} \rfloor \right) \sim \frac{x}{\sqrt{2n}} e^{-x^2/2}, \quad x > 0. \]
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$$\mathbb{P} \left( B_{n+1} = \lfloor x \sqrt{2n} \rfloor \right) \sim \frac{x}{\sqrt{2n}} e^{-x^2/2}, \quad x > 0.$$ 

In other words, we get

$$\frac{B_n}{\sqrt{2n}} \to L \text{ a.s. as } n \to \infty,$$

where the limit $L$ has the Rayleigh distribution, with density $xe^{-x^2/2}$ on $\mathbb{R}_+$.

Consequences

The distance between 0 and 1 varies as \( \sqrt{2n} \), with a nice almost sure limit. What can we say about the distances between the other leaves as \( n \to \infty \)?
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For example, let’s think about the distance from 2 to the path between 0 and 1, and the position along that path at which it branches off.
More urns: self-similarity

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Each of the three parts here behaves precisely as a little copy of Rémy’s algorithm, although the numbers of leaves we add to each copy are dependent. A useful consequence is that given the three sets of leaves, these three trees are themselves uniform binary leaf-labelled trees.
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How many leaves end up in each of the three copies?
Consider Pólya’s urn with three colours, red, green and blue. We start with one ball of each colour. We pick a ball at random and replace it in the urn with two more of the same colour. Let $R_n, G_n, B_n$ be the numbers of red, green and blue balls respectively at step $n$ (let us now re-number the steps from 0, so that $R_0 = G_0 = B_0 = 1$).
More urns: self-similarity

It is then standard that

\[ \frac{1}{2n + 3} (R_n, G_n, B_n) \to (\Delta_1, \Delta_2, \Delta_3) \quad \text{a.s. as } n \to \infty, \]

where \((\Delta_1, \Delta_2, \Delta_3) \sim \text{Dirichlet}(1/2, 1/2, 1/2)\).
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The Dirichlet distribution with parameters \(\alpha_1, \alpha_2, \ldots, \alpha_k > 0\) has density

\[
\frac{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} x_1^{\alpha_1-1} \ldots x_k^{\alpha_k-1}
\]

with respect to Lebesgue measure on

\[
\left\{ x = (x_1, \ldots, x_k) \in \mathbb{R}_+^k : \sum_{i=1}^{k} x_i = 1 \right\}.
\]
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where \((\Delta_1, \Delta_2, \Delta_3) \sim \text{Dirichlet}(1/2, 1/2, 1/2)\).

Let \(\gamma_i \sim \text{Gamma}(\alpha_i, 1)\) for \(1 \leq i \leq k\) independently. Then

\[
\frac{1}{\sum_{i=1}^{k} \gamma_i} (\gamma_1, \gamma_2, \ldots, \gamma_k) \sim \text{Dir}(\alpha_1, \ldots, \alpha_k),
\]

(and is independent of \(\sum_{i=1}^{k} \gamma_i\)).
More urns: self-similarity

The numbers of leaves in each of the three subtrees are given by

\[ N^R_n = \frac{(R_n + 1)}{2}, \quad N^G_n = \frac{(G_n + 1)}{2}, \quad N^B_n = \frac{(B_n + 1)}{2}. \]

So we have

\[ \frac{1}{n}(N^R_n, N^G_n, N^B_n) \rightarrow (\Delta_1, \Delta_2, \Delta_3) \quad \text{a.s.} \]
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Writing \( L^R_n, L^G_n, L^B_n \) for the lengths of the three paths at step \( n \), we see that they look like small copies of the first urn model run for numbers of steps which are approximately \( n\Delta_1, n\Delta_2 \) and \( n\Delta_3 \).
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\[ \frac{1}{\sqrt{2n}}(L_R^n, L_G^n, L_B^n) \rightarrow (\sqrt{\Delta_1}L_1, \sqrt{\Delta_2}L_2, \sqrt{\Delta_3}L_3) \text{ a.s.} \]

where \( L_1, L_2, L_3 \) are i.i.d. Rayleigh random variables, independent of \( (\Delta_1, \Delta_2, \Delta_3) \).
Limiting subtree lengths

An elementary calculation yields that

\[(\sqrt{\Delta_1 L_1}, \sqrt{\Delta_2 L_2}, \sqrt{\Delta_3 L_3}) \overset{d}{=} \sqrt{\Gamma_2} \times \text{Dir}(1, 1, 1),\]

where \(\Gamma_2 \sim \text{Gamma}(2, 1/2)\) and the two factors are independent.
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More generally, if we consider the subtree spanned by 0 and the leaves labelled 1, 2, \ldots, \(k\), we get \(2k - 1\) edges whose lengths are distributed as

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Note that the \( k = 1 \) case fits into this pattern, since Rayleigh \( \overset{d}{=} \sqrt{\Gamma_1} \).
A limiting version of Rémy’s algorithm: Aldous’ line-breaking construction of the Brownian CRT

Take an inhomogeneous Poisson process on $\mathbb{R}_+$ of intensity $t$ at $t$. 

Consider the line-segments $[0, C_1), [C_1, C_2), [C_2, C_3), [C_3, C_4), [C_4, C_5), [C_5, C_6), ...$.

Start from $[0, C_1)$ and proceed inductively.

For $i \geq 2$, attach $[C_{i-1}, C_i)$ at a random point chosen uniformly over the existing tree.
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A useful way of constructing this is to let $E_1, E_2, \ldots$ be i.i.d. $\text{Exp}(1/2)$ and set $C_i = \sqrt{\sum_{j=1}^{i} E_j}$. 

![Diagram of line segments]
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![Line diagram with points C1, C2, C3, C4, C5, C6]

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Line-breaking construction
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Why is this the right limit?

**Claim:** this gives the almost sure limit of the subtree spanned by 0 and the leaves 1, 2, \ldots, \( k \) in the rescaled version of Rémy’s algorithm.

- The tree at step \( k \geq 1 \) has total length

  \[
  C_k = \sqrt{\sum_{i=1}^{k} E_i} = \sqrt{\text{Gamma}(k, 1/2)}.
  \]

- The combinatorics of the attachment mechanism are exactly the same as in Rémy’s algorithm – so the underlying binary leaf-labelled tree has the right distribution.

- A calculation shows that the cut-points and attachment points split up the interval \([0, C_k]\) uniformly.
The line-breaking definition of the Brownian CRT

- Start from \([0, C_1)\) and proceed inductively.
- For \(i \geq 1\), sample \(B_i\) uniformly from \([0, C_i)\) and attach \([C_i, C_{i+1})\) at the corresponding point of the tree constructed so far (this is a point chosen uniformly at random over the existing tree).
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Now take the union of all the branches, thought of as a path metric space, and then take its completion.

This procedure gives (slightly informally expressed) definition of the Brownian continuum random tree (CRT) which is the key object in this minicourse.
The line-breaking definition of the Brownian CRT

[Picture by Igor Kortchemski]
The scaling limit of the uniform binary leaf-labelled tree

In the next section, we will make sense of the following statement.

**Theorem.** (Marchal (2003), Curien and Haas (2013))

As $n \to \infty$, 

$$ \frac{1}{\sqrt{2n}} T_n \to \mathcal{T} \quad \text{a.s.} $$

where $\mathcal{T}$ is the Brownian CRT.
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We need to know what sort of objects we’re really dealing with, and what is the topology in which the convergence occurs.
2. \textit{R-TREES AND CONVERGENCE}

Key reference:

Continuous trees

We want a continuous notion of a tree. We don’t really care about vertices: the important aspects are the shape of the tree and the distances. So it makes sense to think in terms of metric spaces.
R-trees

**Definition.** A compact metric space \((\mathcal{T}, d)\) is an \(\mathbb{R}\)-tree if for all \(x, y \in \mathcal{T}\),

- There exists a unique shortest path \([x, y]\) from \(x\) to \(y\) (of length \(d(x, y)\)).

- The only non-self-intersecting path from \(x\) to \(y\) is \([x, y]\).
Definition. A compact metric space \((T, d)\) is an \(\mathbb{R}\)-tree if for all \(x, y \in T\),

- There exists a unique shortest path \([[x, y]]\) from \(x\) to \(y\) (of length \(d(x, y)\)). (There is a unique isometric map \(f_{x,y}\) from \([0, d(x, y)]\) into \(T\) such that \(f(0) = x\) and \(f(d(x, y)) = y\). We write \(f_{x,y}([0, d(x, y)]) = [[x, y]]\).)

- The only non-self-intersecting path from \(x\) to \(y\) is \([[x, y]]\). (If \(g\) is a continuous injective map from \([0, 1]\) into \(T\), such that \(g(0) = x\) and \(g(1) = y\), then \(g([0, 1]) = [[x, y]]\).)
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An element \(v \in \mathcal{T}\) is called a vertex. A rooted \(\mathbb{R}\)-tree has a distinguished vertex \(\rho\) called the root. The height of a vertex \(v\) is its distance \(d(\rho, v)\) from the root. A leaf is a vertex \(v\) such that \(v \notin [[\rho, w]]\) for any \(w \neq v\).
Coding $\mathbb{R}$-trees

Let $h : [0, 1] \rightarrow \mathbb{R}^+$ be an excursion, that is a continuous function such that $h(0) = h(1) = 0$ and $h(x) > 0$ for $x \in (0, 1)$. 
Coding $\mathbb{R}$-trees

Now put glue on the underside of the excursion and push the two sides together...
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Coding R-trees

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Coding $\mathbb{R}$-trees

Now put glue on the underside of the excursion and push the two sides together to get a tree.
Formally, use $h$ to define a distance:

$$d_h(x, y) = h(x) + h(y) - 2 \inf_{x \land y \leq z \leq x \lor y} h(z).$$
Let $y \sim y'$ if $d_h(y, y') = 0$ and take the quotient $\mathcal{T}_h = [0, 1]/\sim$. 
Theorem. For any excursion $h$, $(\mathcal{T}_h, d_h)$ is an $\mathbb{R}$-tree.

Write $\pi_h : [0, 1] \to \mathcal{T}_h$ for the projection map.

We will often root $\mathcal{T}_h$ at $\rho = \pi_h(0) = \pi_h(1)$. 

Coding $\mathbb{R}$-trees
A natural measure

We will want to be able to sample random points in our trees. There is a natural “uniform” measure $\mu_h$ which is the push-forward of the Lebesgue measure on $[0, 1]$ onto $\mathcal{T}_h$. 
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To pick a point of $\mathcal{T}_h$ according to $\mu_h$, we simply sample $U \sim U[0, 1]$ and then take our point to be $\pi_h(U)$. 
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We will typically think of our continuous trees as triples $(T_h, d_h, \mu_h)$. 

A natural measure
Let $\mathcal{M}$ be the space of compact metric spaces endowed with a Borel probability measure, up to measure-preserving isometry.

We will define a metric $d_{\text{GHP}}$, the Gromov-Hausdorff-Prokhorov distance on $\mathcal{M}$. 
Topological considerations

Suppose that \((X, d)\) and \((X', d')\) are compact metric spaces.
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A correspondence \(R\) is a subset of \(X \times X'\) such that for every \(x \in X\), there exists \(x' \in X'\) with \((x, x') \in R\) and vice versa.
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Topological considerations

The distortion of $R$ is

$$\text{dis}(R) = \sup\{|d(x, y) - d'(x', y')| : (x, x'), (y, y') \in R\}.$$
Suppose that $\mu$ is a Borel probability measure on $(X, d)$ and that $\mu'$ is a Borel probability measure on $(X', d')$.

A measure $\nu$ on $X \times X'$ is a **coupling** of $\mu$ and $\mu'$ if $\nu(\cdot, X') = \mu(\cdot)$ and $\nu(X, \cdot) = \mu'(\cdot)$.
Suppose that $\mu$ is a Borel probability measure on $(X, d)$ and that $\mu'$ is a Borel probability measure on $(X', d')$.

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**Idea:** find a correspondence and a coupling such that the correspondence has small distortion and the coupling “lines up” well with the correspondence i.e. if $(V, V') \sim \nu$ then $\mathbb{P}((V, V') \in R) = \nu(R)$ is close to 1.
The Gromov-Hausdorff-Prokhorov distance between \((X, d, \mu)\) and \((X', d', \mu')\) is defined to be

\[
d_{\text{GHP}}((X, d, \mu), (X', d', \mu')) = \frac{1}{2} \inf_{R, \nu} \max\{\text{dis}(R), \nu(R^c)\}.
\]
Topological considerations

The Gromov-Hausdorff-Prokhorov distance between \((X, d, \mu)\) and \((X', d', \mu')\) is defined to be

\[
d_{\text{GHP}}((X, d, \mu), (X', d', \mu')) = \frac{1}{2} \inf_{R, \nu} \max \{ \text{dis}(R), \nu(R^c) \}.
\]

**Theorem.** \((\mathbb{M}, d_{\text{GHP}})\) is a complete separable metric space.


The Brownian CRT

**Definition.** The Brownian continuum random tree is \((T_{2e}, d_{2e}, \mu_{2e})\), where \(e\) is a standard Brownian excursion.

[Pictures by Igor Kortchemski]
A planar ordering

Observe that the excursion comes with slightly more information than the tree: if $s < t$ and $\pi_{2e}(s)$ and $\pi_{2e}(t)$ are leaves, it is natural to think of $\pi_{2e}(s)$ appearing to the left of $\pi_{2e}(t)$. 
Discrete trees as metric spaces

We want to think of \((T_n, n \geq 1)\) as metric spaces.

The vertices of \(T_n\) (labelled and unlabelled) come equipped with a natural metric: the graph distance \(d_n\).

We sometimes write \(aT_n\) for the metric space \((T_n, ad_n)\) given by the vertices of \(T_n\) with the graph distance scaled by \(a\).
We will endow $T_n$ with $\mu_n$, the measure which puts mass $1/(2n)$ on each of the $2n$ vertices.
Convergence

**Theorem.** As $n \to \infty$, 

$$\left( T_n, \frac{d_n}{\sqrt{2n}}, \mu_n \right) \to (T_{2e}, d_{2e}, \mu_{2e}) \text{ a.s.}$$

with respect to the Gromov-Hausdorff-Prokhorov topology.


A plane version of our binary trees

In order to see where the Brownian excursion comes from, it will be helpful for us to now give our binary trees a planar ordering. We achieve this in Rémy’s algorithm by simply gluing each new branch on the left or right with equal probability.
Binary trees and lattice excursions

There is a well-known bijection between planted binary plane trees with \( n \) leaves and lattice excursions with \( 2n \) steps.

Start every excursion with a +1 step. Now travel round the tree from left to right, recording a step whenever you see a vertex for the first time. The step is +1 if the vertex is a branch-point and −1 if the vertex is a leaf.
Binary trees and lattice excursions

To go back the other way, it’s easy to recover the tree:
Binary trees and lattice excursions

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Binary trees and lattice excursions

Since our trees are uniform, so are the lattice excursions. In other words, they are excursions of simple random walk away from 0.
Since our trees are uniform, so are the lattice excursions. In other words, they are excursions of **simple random walk** away from 0. So (at least in distribution), it’s clear that, suitably rescaled, they should converge to a Brownian excursion.
Rémy’s algorithm then corresponds to a sequence of simple operations on such lattice excursions.
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Binary trees and lattice excursions

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Let $(E_n)_{n \geq 1}$ be the sequence of lattice excursions.

**Theorem.** (Marchal (2003))
As $n \to \infty$, we have

$$\frac{1}{\sqrt{2n}} (E_n(\lfloor 2nt \rfloor), 0 \leq t \leq 1) \to (e(t), 0 \leq t \leq 1)$$

uniformly on $[0, 1]$, almost surely.
Convergence of the trees

This is not quite enough to conclude that the trees converge in the GHP sense. The embedding of the tree in the excursion distorts distances.
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\[
H_n(k) \quad \text{for the distance from the root to the vertex visited at time } k. \quad \text{Then } H_n(k) = \left| \left\{ 0 \leq i \leq k - 1 : E_n(i) = \min_{i \leq j \leq k} E_n(k) \right\} \right|.
\]

It turns out that \( H_n(k) \approx 2 E_n(k) \).
Convergence of the trees

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Write $H_n(k)$ for the distance from the root to the vertex visited at time $k$. Then

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Convergence of the trees

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It turns out that $H_n(k) \approx 2E_n(k)$. 
Convergence of the trees

**Theorem.** As $n \to \infty$, 

\[
\frac{1}{\sqrt{2n}}(H_n(\lfloor 2nt \rfloor), 0 \leq t \leq 1) \to (2e(t), 0 \leq t \leq 1)
\]

uniformly on $[0, 1]$, almost surely.

Convergence of the trees

Let's call the vertices be \( v_0, v_1, \ldots, v_{2n-1} \) in the order we visit them, where \( v_0 \) is the root.
Convergence of the trees

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By definition,

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d_n(v_0, v_k) = H_n(k).
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Let's call the vertices be $v_0, v_1, \ldots, v_{2n-1}$ in the order we visit them, where $v_0$ is the root.

By definition,

$$d_n(v_0, v_k) = H_n(k).$$

More generally, for $0 \leq i < j \leq 2n - 1$, write $v_i \wedge v_j$ for the most recent common ancestor of $v_i$ and $v_j$ in the tree. Then

$$d_n(v_i, v_j) = d_n(v_0, v_i) + d_n(v_0, v_j) - 2d_n(v_0, v_i \wedge v_j).$$
Convergence of the trees

\[ d_n(v_0, v_i \land v_j) = \begin{cases} 
\min_{i \leq k \leq j} H_n(k) - 1 & \text{if } v_i \text{ not an ancestor of } v_j \\
\min_{i \leq k \leq j} H_n(k) = H_n(i) & \text{if } v_i \text{ an ancestor of } v_j.
\end{cases} \]

So

\[ \left| d_n(v_0, v_i \land v_j) - \min_{i \leq k \leq j} H_n(k) \right| \leq 1. \]
A correspondence

Define a correspondence $R_n$ between \{${v_0, v_1, \ldots, v_{2n-1}}$\} and $[0, 1]$ by declaring $(v_i, s) \in R_n$ if $i = \lfloor 2ns \rfloor$.

Endow $[0, 1]$ with the pseudo-metric $d_{2e}$. We will bound $\text{dis}(R_n)$. 
A correspondence

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Endow $[0, 1]$ with the pseudo-metric $d_{2e}$. We will bound $\text{dis}(R_n)$.

Suppose that $(v_i, s), (v_j, t) \in R_n$ with $s \leq t$. Then

$$|d_n(v_i, v_j) - d_{2e}(s, t)| \leq \left| \frac{1}{\sqrt{2n}} \left( H_n(\lfloor 2ns \rfloor) + H_n(\lfloor 2nt \rfloor) - 2 \min_{s \leq u \leq t} H_n(\lfloor 2nu \rfloor) \right) - \left( 2e(s) + 2e(t) - 4 \min_{s \leq u \leq t} e(u) \right) \right| + \frac{2}{\sqrt{2n}}.$$
A correspondence

Define a correspondence $R_n$ between $\{v_0, v_1, \ldots, v_{2n-1}\}$ and $[0, 1]$ by declaring $(v_i, s) \in R_n$ if $i = [2ns]$.

Endow $[0, 1]$ with the pseudo-metric $d_{2e}$. We will bound $\text{dis}(R_n)$.

Suppose that $(v_i, s), (v_j, t) \in R_n$ with $s \leq t$. Then

$$|d_n(v_i, v_j) - d_{2e}(s, t)| \leq \frac{1}{\sqrt{2n}} \left( H_n([2ns]) + H_n([2nt]) - 2 \min_{s \leq u \leq t} H_n([2nu]) \right)$$

$$-\left( 2e(s) + 2e(t) - 4 \min_{s \leq u \leq t} e(u) \right) \right| + \frac{2}{\sqrt{2n}}.$$

The right-hand side converges to 0 uniformly in $s, t \in [0, 1]$. So

$$\text{dis}(R_n) \to 0 \quad \text{a.s.}$$
A coupling

Recall that $\mu_n$ is the measure which puts mass $1/(2n)$ on each of the vertices $v_0, v_1, \ldots, v_{2n-1}$. Then we may couple $\mu_n$ and $\mu_{2e}$ by taking $U \sim U[0,1]$ and taking $\nu$ to be the law of the pair

$$(\nu_{[2nU]}, \pi_{2e}(U)).$$

This is precisely the natural coupling $\nu_n$ induced by the correspondence $R_n$, and so $\nu_n(R_n^c) = 0$. 
GHP convergence

But then

\[ d_{\text{GHP}} \left( \left( T_n, \frac{d_n}{\sqrt{2n}}, \mu_n \right), (T_{2e}, d_{2e}, \mu_{2e}) \right) \leq \frac{1}{2} \max \{ \text{dis}(R_n), \nu_n(R_n^c) \} \to 0, \]

almost surely as \( n \to \infty. \)
3. PROPERTIES OF THE BROWNIAN CRT

Key references:


Recap: Rémy’s algorithm
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Recap: Rémy’s algorithm
Recap: line-breaking construction

Take an inhomogeneous Poisson process on $\mathbb{R}_+$ of intensity $t$ at $t$.

Consider the line-segments $[0, C_1)$, $[C_1, C_2)$, \ldots.

Start from $[0, C_1)$ and proceed inductively.

For $i \geq 2$, attach $[C_{i-1}, C_i)$ at a random point chosen uniformly over the existing tree.
Recap: convergence theorem

Recall that $T_n$ is a uniform binary leaf-labelled tree and that $T_{2e}$ is the Brownian CRT.

**Theorem.** As $n \to \infty$,\[ \left( T_n, \frac{d_n}{\sqrt{2n}}, \mu_n \right) \to \left( T_{2e}, d_{2e}, \mu_{2e} \right) \quad \text{a.s.} \]

with respect to the Gromov-Hausdorff-Prokhorov topology.
Uniform measure on the leaves

The same is true if we take the measure to be the uniform measure just on the leaves, $\tilde{\mu}_n$.

**Theorem.** (Curien & Haas (2013))

As $n \to \infty$,

$$\left(T_n, \frac{d_n}{\sqrt{2n}}, \tilde{\mu}_n\right) \to (\mathcal{T}_{2e}, d_{2e}, \mu_{2e}) \text{ a.s.}$$

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As $n \to \infty$,

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with respect to the Gromov-Hausdorff-Prokhorov topology.

Let $\mathcal{L}(T_{2e})$ be the set of leaves of $T_{2e}$. Then $\mu_{2e}(\mathcal{L}(T_{2e})) = 1$. 
Imagine permuting the labels 0, 1, . . . , n in the binary tree $T_n$. It's straightforward to see that this does not change its law. In particular, the root 0 acts just like a uniformly chosen leaf. So the same must also be true for $T_{2e}$. 
What is a continuum random tree?

A continuum tree is a triple \((\mathcal{T}, d, \mu)\) where \((\mathcal{T}, d)\) is an \(\mathbb{R}\)-tree with leaves \(\mathcal{L}(\mathcal{T})\) and \(\mu\) is a Borel probability measure on \(\mathcal{T}\) which is such that

- \(\mu\) is non-atomic
- \(\mu(\mathcal{L}(\mathcal{T})) = 1\)
- for every \(v \in \mathcal{T}\) of degree \(k \geq 2\), let \(\mathcal{T}_1, \ldots, \mathcal{T}_k\) be the connected components of \(\mathcal{T} \setminus \{v\}\). Then \(\mu(\mathcal{T}_i) > 0\) for all \(1 \leq i \leq k\).
What is a continuum random tree?

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A **continuum random tree** (CRT) is a random variable taking values in the set of (equivalence classes of) continuum trees.
Characterising a CRT via sampling

Take a CRT \((T, d, \mu)\) and suppose that \(V_0, V_1, \ldots\) are i.i.d. samples from the measure \(\mu\). (Note: these are a.s. leaves.) For \(k \geq 1\), let \(R_k\) be the subtree of \(T\) spanned by \(V_0, V_1, \ldots, V_k\).
Characterising a CRT via sampling

Take a CRT \((\mathcal{T}, d, \mu)\) and suppose that \(V_0, V_1, \ldots\) are i.i.d. samples from the measure \(\mu\). (Note: these are a.s. leaves.) For \(k \geq 1\), let \(\mathcal{R}_k\) be the subtree of \(\mathcal{T}\) spanned by \(V_0, V_1, \ldots, V_k\).
Characterising a CRT via sampling

For every \( k \geq 1 \), \( \mathcal{R}_k \) can be regarded as a discrete tree, rooted at \( V_0 \), with edge-lengths and labelled leaves, and so its distribution is specified by its tree-shape, a rooted unordered tree with \( k \) labelled leaves, and its edge-lengths. The reduced trees are clearly consistent, in that \( \mathcal{R}_k \) is a subtree of \( \mathcal{R}_{k+1} \).
For every $k \geq 1$, $R_k$ can be regarded as a discrete tree, rooted at $V_0$, with edge-lengths and labelled leaves, and so its distribution is specified by its tree-shape, a rooted unordered tree with $k$ labelled leaves, and its edge-lengths. The reduced trees are clearly consistent, in that $R_k$ is a subtree of $R_{k+1}$.

**Theorem.** (Aldous (1993))
The law of a continuum random tree $(\mathcal{T}, d, \mu)$ is specified by its random finite-dimensional distributions, that is the laws of $(R_k, k \geq 1)$.

Moreover, if we let

\[ \hat{\mu}_k = \frac{1}{k} \sum_{i=1}^{k} \delta_{V_i} \]

then

\[ (R_k, d|_{R_k}, \hat{\mu}_k) \rightarrow (T, d, \mu) \]

almost surely in \( d_{GHP} \), as \( k \rightarrow \infty \).
The random finite-dimensional distributions of the Brownian CRT

Our earlier urn results translate into facts about the Brownian CRT.
Our earlier urn results translate into facts about the Brownian CRT.

Recall our planted binary leaf-labelled tree $T_n$. As $n$ gets large, the leaves $1, 2, \ldots, k$ behave like i.i.d. samples from $\tilde{\mu}_n$. But our urn arguments gave us the limiting distribution of the rescaled tree spanned by 0 and 1, 2, \ldots, $k$. 
The random finite-dimensional distributions of the Brownian CRT

**Theorem.** (Aldous (1993))

- \( \mathcal{R}_k \) is a uniform random planted binary leaf-labelled tree, with edge-lengths distributed as

\[
\sqrt{\Gamma_k} \times \text{Dir}(1, 1, \ldots, 1)_{2k-1},
\]

where \( \Gamma_k \sim \text{Gamma}(k, 1/2) \), independent of the Dirichlet vector.

- \( (\mathcal{R}_k)_{k \geq 1} \) evolves according to the line-breaking construction.

[See Le Gall (2005) for a direct proof from the Brownian excursion.]
The random finite-dimensional distributions of the Brownian CRT

Note that since

$$(\mathcal{R}_k, d_{2e}|_{\mathcal{R}_k}, \hat{\mu}_k) \to (\mathcal{T}_{2e}, d_{2e}, \mu_{2e})$$

as $k \to \infty$, it follows that the Brownian CRT is binary.

Another way to see this is to observe that the local minima of a Brownian excursion are unique almost surely.
Consider picking three independent points $V_1, V_2, V_3$ from $\mathcal{T}_{2e}$ according to $\mu_{2e}$. There is a unique branch-point between these three points, and it splits the tree into three subtrees, $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$. 

Recursive self-similarity
Consider picking three independent points $V_1, V_2, V_3$ from $\mathcal{T}_{2e}$ according to $\mu_{2e}$. There is a unique branch-point between these three points, and it splits the tree into three subtrees, $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$.

Write $d_1, d_2, d_3$ and $\mu_1, \mu_2, \mu_3$ for the restrictions of $d$ and $\mu$ to each of these subtrees respectively. Let

$$\Delta_1 = \mu(\mathcal{T}_1), \Delta_2 = \mu(\mathcal{T}_2), \Delta_3 = \mu(\mathcal{T}_3).$$
Theorem. (Aldous (1997))

- We have \((\Delta_1, \Delta_2, \Delta_3) \sim \text{Dir}(1/2, 1/2, 1/2)\).
- The rescaled subtrees \((T_1, d_1/\sqrt{\Delta_1}, \mu_1/\Delta_1), (T_2, d_2/\sqrt{\Delta_2}, \mu_2/\Delta_2), (T_3, d_3/\sqrt{\Delta_3}, \mu_3/\Delta_3)\) are i.i.d. Brownian CRTs, independent of \((\Delta_1, \Delta_2, \Delta_3)\).
- \(V_1\) and the original branch-point are independent samples from \(\mu_i/\Delta_i\) in subtree \(i = 1, 2, 3\).

The spine decomposition

Now look more closely at the tree from the perspective of the path between the root and a single uniform point (the spine).

What are the masses hanging off the spine? Where are they located?
The spine decomposition

**Theorem.** (Haas, Pitman & Winkel (2009))
The spinal mass partition is distributed as Poisson-Dirichlet PD(1/2, 1/2) and the trees corresponding to the different blocks are attached at i.i.d. uniform points along the spine. These little subtrees are randomly rescaled i.i.d. Brownian CRT’s.

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Fix $0 \leq \alpha < 1$ and $\theta > -\alpha$. Let $\beta_1, \beta_2, \ldots$ be independent random variables such that $\beta_i \sim \text{Beta}(1 - \alpha, \theta + i\alpha)$. Let

$$\tilde{P}_i = \beta_i \prod_{j=1}^{i-1}(1 - \beta_j), \ i \geq 1,$$

and let $P_1 \geq P_2 \geq \ldots \geq 0$ be the $\tilde{P}_i$ in decreasing order. Then $(P_i)_{i \geq 1} \sim \text{PD}(\alpha, \theta)$.

The Chinese restaurant process

Generate an exchangeable random partition of $\mathbb{N}$ as follows.

- The first customer arrives and sits at a table.
- Suppose that after $n$ customers have arrived, there are $n_i$ of them sitting at table $i$, for $1 \leq i \leq k$.
- Customer $n + 1$ arrives and sits at table $i$ with probability $\frac{n_i - \alpha}{n + \theta}$, for $1 \leq i \leq k$, or starts a new table with probability $\frac{\theta + k\alpha}{n + \theta}$. 

Let $K_n$ be the number of tables occupied by the first $n$ customers, and $\Pi(n)_1, \ldots, \Pi(n)_{K_n}$ the sets of customers sitting at the different tables. Then as $n \to \infty$, 

$$\frac{|\Pi(n)_1|, \ldots, |\Pi(n)_{K_n}|}{n} \to \text{PD}(\alpha, \theta)$$

where $\Pi(n)_{i} \geq 1$ a.s.
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- Customer \( n + 1 \) arrives and sits at table \( i \) with probability
  \[ \frac{n_i - \alpha}{n + \theta}, \] for \( 1 \leq i \leq k \), or starts a new table with probability
  \[ \frac{\theta + k\alpha}{n + \theta}. \]

Let \( K_n \) be the number of tables occupied by the first \( n \) customers, and \( \Pi_1^{(n)}, \ldots, \Pi_{K_n}^{(n)} \) the sets of customers sitting at the different tables. Then as \( n \to \infty \),

\[
\frac{1}{n} \left( |\Pi_1^{(n)}|, \ldots, |\Pi_{K_n}^{(n)}| \right) \downarrow (P_i)_{i \geq 1} \quad \text{a.s.}
\]

where \( (P_i)_{i \geq 1} \sim \text{PD}(\alpha, \theta) \).
Claim: the spinal mass partition is distributed as $\text{PD}(1/2, 1/2)$ and the trees corresponding to the different blocks are attached at i.i.d. uniform points along the spine.

Every time we add a new vertex in Rémy’s algorithm, we either add it to a subtree which is already hanging from the path between 0 and 1, or we create a new such subtree.
The spine decomposition

Suppose we are at step $n$ and that the current length of the path from 0 to 1 is $k + 1$, so that there are $k$ subtrees hanging off. Suppose these subtrees contain $n_1, \ldots, n_k$ leaves, listing from top to bottom.

- We add our new vertex to the $i$th existing subtree (containing $n_i$ vertices) with probability $\frac{2n_i - 1}{2n - 1}$.
- We add a new subtree with probability proportional to $\frac{k+1}{2n-1}$.
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These are precisely the probabilities in the Chinese restaurant process with parameters $\alpha = 1/2, \theta = 1/2$. 
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- We add a new subtree with probability proportional to $\frac{k+1}{2n-1}$.

These are precisely the probabilities in the Chinese restaurant process with parameters $\alpha = 1/2, \theta = 1/2$. So the labels in the subtrees behave exactly as tables in a $(1/2, 1/2)$-Chinese restaurant process (on $\{2, 3, 4, \ldots\}$) and it follows that their ordered sizes, divided by $n$, converge almost surely to a PD$(1/2, 1/2)$ vector. Since each new subtree gets attached at a uniform position along the path, the same is also true in the limit.
An alternative viewpoint: via a path transformation

**Theorem.** (Bertoin & Pitman (1994))

Let $B^{ex}$ be a standard Brownian excursion and let $U \sim U[0, 1]$. Let

$$K_t = \begin{cases} \min_{t \leq s \leq U} B^e_x & \text{for } 0 \leq t \leq U \\ \min_{U \leq s \leq t} B^e_x & \text{for } U \leq t \leq 1. \end{cases}$$

Then $B^{br} := B^{ex} - K$ is the modulus of a standard Brownian bridge.

An alternative viewpoint: via a path transformation

The excursions of the bridge encode the little subtrees hanging off the spine. The ordered lengths of these excursions are Poisson-Dirichlet($1/2, 1/2$) distributed. The local time at 0 of the Brownian bridge is Rayleigh distributed, which represents the length of the path between the root and uniform leaf. $U$ sits exactly halfway through the local time.
A random fractal

The self-similarity of the Brownian CRT tells us, in particular, that it is a random fractal. What is its dimension?
The self-similarity of the Brownian CRT tells us, in particular, that it is a random fractal. What is its dimension?

The Minkowski (or box-counting) dimension is defined to be
\[ \lim_{\epsilon \downarrow 0} \frac{\log N(T_{2e}, \epsilon)}{\log(1/\epsilon)} \] (if the limit exists) where \( N(T_{2e}, \epsilon) \) is the number of balls of radius \( \epsilon \) needed to cover \( T_{2e} \).

**Theorem.** (Duquesne & Le Gall (2005)) The Brownian CRT has Minkowski dimension 2, almost surely.
A random fractal

The self-similarity of the Brownian CRT tells us, in particular, that it is a random fractal. What is its dimension?

The Minkowski (or box-counting) dimension is defined to be
$$\lim_{\epsilon \downarrow 0} \frac{\log N(T_{2e}, \epsilon)}{\log(1/\epsilon)}$$
(if the limit exists) where $N(T_{2e}, \epsilon)$ is the number of balls of radius $\epsilon$ needed to cover $T_{2e}$.

**Theorem.** (Duquesne & Le Gall (2005))
The Brownian CRT has Minkowski dimension 2, almost surely.

Minkowski dimension: heuristic for lower bound

Consider $\mathcal{R}_k$, the tree subtree spanned by the root and $k$ uniform leaves.

For a lower bound, we cover parts of this subtree. Recall that the lengths are

$$\sqrt{\Gamma_k} \times \text{Dir}(1, 1, \ldots, 1),$$

where $\Gamma_k \sim \text{Gamma}(k, 1/2)$. 
Minkowski dimension: heuristic for lower bound

We have $\mathbb{E} \left[ \sqrt{\Gamma_k} \right] = \frac{\sqrt{2} \Gamma(k+1/2)}{\Gamma(k)} \sim \sqrt{2k/e}$. 
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For a well-chosen \( \delta > 0 \), there is exponentially small probability of not having at least \( \delta k \) of the elements of the \( \text{Dir}(1, 1, \ldots, 1) \) vector larger than \( 1/(4k) \).
Minkowski dimension: heuristic for lower bound

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For a well-chosen \( \delta > 0 \), there is exponentially small probability of not having at least \( \delta k \) of the elements of the \( \text{Dir}(1, 1, \ldots, 1) \) vector larger than \( 1/(4k) \).

So we can find \( \Omega(k) \) disjoint balls of radius \( \Theta(1/\sqrt{k}) \) which cover a strict subset of \( \mathcal{R}_k \).
Minkowski dimension: heuristic for upper bound

Consider again picking $k$ uniform leaves.

We know that the $2k - 1$ subtrees have masses $(\Delta_1, \ldots, \Delta_{2k-1}) \sim \text{Dir}(1/2, 1/2, \ldots, 1/2)$. Inside each blob is a rescaled independent Brownian CRT. Let $R_1, \ldots, R_{2k-1}$ be i.i.d. copies of the maximum distance from the root in a Brownian CRT.
Minkowski dimension: heuristic for upper bound

\( R_1 \) has the distribution of the maximum of a standard Brownian excursion, which is such that

\[
P(R_1 \geq x) = \sum_{k \geq 1} (-1)^{k+1} e^{-k^2 x^2} \leq e^{-x^2}.
\]

So we have covered \( \mathcal{T}_{2e} \) with balls of random radius at most

\[
\max_{1 \leq i \leq 2k-1} \sqrt{\Delta_i R_i}.
\]

We may realise the Dirichlet vector as

\[
(\Delta_1, \ldots, \Delta_{2k-1}) = \frac{1}{\sum_{i=1}^{2k-1} \gamma_i} (\gamma_1, \ldots, \gamma_{2k-1}),
\]

where \( \gamma_1, \ldots, \gamma_{2k-1} \) are i.i.d. Gamma(1/2, 1).
Minkowski dimension: heuristic for upper bound

So

\[
\max_{1 \leq i \leq 2k-1} \sqrt{\Delta_i R_i} = \sqrt{\frac{\max_{1 \leq i \leq 2k-1} \gamma_i R_i^2}{\sum_{i=1}^{2k-1} \gamma_i}}.
\]
Minkowski dimension: heuristic for upper bound

So

$$\max_{1 \leq i \leq 2k-1} \sqrt{\Delta_i} R_i = \sqrt{\max_{1 \leq i \leq 2k-1} \frac{\gamma_i R_i^2}{\sum_{i=1}^{2k-1} \gamma_i}}.$$ 

Now

$$\mathbb{P} \left( \gamma_1 R_1^2 > x \right) \leq \mathbb{E} \left[ \exp \left( -x / \gamma_1 \right) \right] = \exp(-2\sqrt{x}).$$
Minkowski dimension: heuristic for upper bound

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\]

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P(\gamma_1 R_1^2 > x) \leq \mathbb{E} \left[ \exp(-x/\gamma_1) \right] = \exp(-2\sqrt{x}).
\]

So \(\max_{1 \leq i \leq 2k-1} \gamma_i R_i^2 \sim (\log k)^2\). Since \(\sum_{i=1}^{2k-1} \gamma_i \sim k\), we get

\[
\max_{1 \leq i \leq 2k-1} \sqrt{\Delta_i R_i} \sim \frac{\log k}{\sqrt{k}}.
\]
Minkowski dimension: heuristic for upper bound

So
\[
\max_{1 \leq i \leq 2k-1} \sqrt{\Delta_i} R_i = \sqrt{\frac{\max_{1 \leq i \leq 2k-1} \gamma_i R_i^2}{\sum_{i=1}^{2k-1} \gamma_i}}.
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Now
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P(\gamma_1 R_1^2 > x) \leq \mathbb{E} \left[ \exp\left( -x/\gamma_1 \right) \right] = \exp(-2\sqrt{x}).
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So \( \max_{1 \leq i \leq 2k-1} \gamma_i R_i^2 \sim (\log k)^2 \). Since \( \sum_{i=1}^{2k-1} \gamma_i \sim k \), we get
\[
\max_{1 \leq i \leq 2k-1} \sqrt{\Delta_i} R_i \sim \frac{\log k}{\sqrt{k}}.
\]

So we need \( O(k) \) balls of radius approximately \( k^{-1/2} \log k \) to cover \( \mathcal{T}_{2e} \).

A different perspective

Croydon & Hambly (2008) showed that we can also view \((T_{2e}, d_{2e})\) as this familiar deterministic fractal endowed with a random metric.

4. VORONOI CELLS IN THE BROWNIAN CRT

Joint work with Louigi Addario-Berry (McGill), Omer Angel (UBC), Guillaume Chapuy (Paris 7) and Éric Fusy (École polytechnique)

Voronoi cells in a metric space

Let \((M, d)\) be a metric space.

Fix \(k \geq 1\) and let \(S = \{x_i : 1 \leq i \leq k\}\) be a collection of points in \(M\), the centres.

For \(1 \leq i \leq k\), the Voronoi cells are

\[
V_i = \{y \in M : d(y, S) = d(y, x_i)\}.
\]

(Note that the Voronoi cells are not necessarily disjoint.)
Standard example: Voronoi cells in $\mathbb{R}^2$

Euclidean distance

Picture by Balu Ertl (CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=38534275)
Standard example: Voronoi cells in $\mathbb{R}^2$

Manhattan distance
Voronoi supermarkets

See https://chriszetter.com/voronoi-map/examples/uk-supermarkets/
Voronoi supermarkets

See https://chriszetter.com/voronoimap/examples/uk-supermarkets/
General set-up: Voronoi cells in a metric space

Let $(M, d)$ be a metric space endowed with a Borel probability measure $\mu$.

Fix $k \geq 1$ and let $S = \{x_i : 1 \leq i \leq k\}$ be a collection of points in $M$, the centres. Typically these will be random and i.i.d. samples from $\mu$.

For $1 \leq i \leq k$, the Voronoi cells are

$$V_i = \{y \in M : d(y, S) = d(y, x_i)\}.$$  

(Note that the Voronoi cells are not necessarily disjoint.)

We will be interested in the “masses” of these cells, as measured by $\mu$, i.e.

$$(\mu(V_1), \mu(V_2), \ldots, \mu(V_k))$$.
Warm-up: circle

Circle of circumference 1, Euclidean distance, Lebesgue measure. Any two points.
Warm-up: circle

Circle of circumference 1, Euclidean distance, Lebesgue measure. Any two points.
Warm-up: circle

Circle of circumference 1, Euclidean distance, Lebesgue measure. Any two points.

$$(\mu(V_1), \mu(V_2)) = (1/2, 1/2).$$
Warm-up: circle

Circle of circumference 1, Euclidean distance, Lebesgue measure.
Three uniform points.
Warm-up: circle

Circle of circumference 1, Euclidean distance, Lebesgue measure. Three uniform points.
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Circle of circumference 1, Euclidean distance, Lebesgue measure. Three uniform points.

The lengths of these intervals are uniform on the 2-dimensional simplex i.e. have \( \text{Dir}(1, 1, 1) \) distribution.
Warm-up: circle

Circle of circumference 1, Euclidean distance, Lebesgue measure. Three uniform points.

The lengths of these intervals are uniform on the 2-dimensional simplex i.e. have Dir(1, 1, 1) distribution.

We get that the Lebesgue measures of the Voronoi cells are

\[(\mu(V_1), \mu(V_2), \mu(V_3)) = (\frac{1}{2} U(2), \frac{1}{2} (1 - U(1)), \frac{1}{2} (1 - U(1) - U(2)))\]

(exchangeable with marginals distributed as \(\frac{1}{2} \text{Beta}(2, 1)\)).
Question: what if we take the metric space to be the Brownian CRT?
Theorem. (Addario-Berry, Angel, Chapuy, Fusy & G. (2018))
Let \((\mathcal{T}, d, \mu)\) be the Brownian CRT. Fix \(k \geq 2\) and let \(X_1, X_2, \ldots, X_k\) be i.i.d. samples from \(\mu\). Let \(V_1, V_2, \ldots, V_k\) be the corresponding Voronoi cells. Then

\[
(\mu(V_1), \mu(V_2), \ldots, \mu(V_k)) \sim \text{Dir}(1, 1, \ldots, 1).
\]
**Theorem.** (Addario-Berry, Angel, Chapuy, Fusy & G. (2018))

Let \((T, d, \mu)\) be the Brownian CRT. Fix \(k \geq 2\) and let \(X_1, X_2, \ldots, X_k\) be i.i.d. samples from \(\mu\). Let \(V_1, V_2, \ldots, V_k\) be the corresponding Voronoi cells. Then

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If you want to chop up the Brownian CRT in a uniform manner, pick uniform points and find their Voronoi cells!
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\]

If you want to chop up the Brownian CRT in a uniform manner, pick uniform points and find their Voronoi cells!

(Compare to the \(\text{Dir}(1/2, 1/2, 1/2)\) mass-split we get by cutting at the branch-point between three uniform points.)
Our original motivation

**Conjecture.** (Chapuy (2016))
Let $(B, d, \mu)$ be the Brownian map (or Brownian surface of genus $g \geq 0$). Let $X_1, X_2, \ldots, X_k$ be i.i.d. points sampled from $\mu$ and $V_1, V_2, \ldots, V_k$ be the corresponding Voronoi cells. Then

$$(\mu(V_1), \mu(V_2), \ldots, \mu(V_k)) \sim \text{Dir}(1, 1, \ldots).$$

The Brownian map (sphere)

[Picture by Jérémie Bettinelli]
The Brownian double torus

[Picture by Jérémie Bettinelli]
**Conjecture.** (Chapuy (2016))

Let $(\mathcal{B}, d, \mu)$ be the Brownian map (or Brownian surface of genus $g \geq 0$). Let $X_1, X_2, \ldots, X_k$ be i.i.d. points sampled from $\mu$ and $V_1, V_2, \ldots, V_k$ be the corresponding Voronoi cells. Then

$$(\mu(V_1), \mu(V_2), \ldots, \mu(V_k)) \sim \text{Dir}(1, 1, \ldots).$$

Proved for $g = 0, k = 2$ by Emmanuel Guitter (but proof does not generalise).

**Open problem.** Which properties of a random metric space give rise to uniform Voronoi mass-partitions?
Suppose we start from the subtree spanned by $X_1, \ldots, X_k$. 
Suppose we start from the subtree spanned by $X_1, \ldots, X_k$. In order to get back to the whole tree, we need to take i.i.d. copies of the Brownian CRT, randomly rescaled by an exchangeable vector with sum 1, and glued onto the subtree at i.i.d. uniform positions.
Base case: $k = 2$

The proof goes via induction, with the base case being $k = 2$.

We wish to find the masses of the blue and red parts.
$k = 2$: an observation

Call the masses above and below the backbone the contour cells.
Call the masses above and below the backbone the contour cells. These are equal to \( U_1 \) and \( 1 - U_1 \), with \( U_1 \sim U[0, 1] \). The little trees attached to the backbone have exchangeable masses.
$k = 2$: a bijection

We may convert the Voronoi cells into the contour cells of a different tree:

Since the subtree masses are exchangeable, the new tree is again a Brownian CRT. But the contour cells in a Brownian CRT have $(U, 1 - U)$ mass split, so the same must be true for the Voronoi cells. (This may be read off from results of Lévy (1939) or Bertoin and Pitman (1994).)
$k = 2$: a bijection

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Reductions for $k \geq 3$: contour cells

Consider the subtree spanned by our uniform points.
Reductions for $k \geq 3$: contour cells

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We will show that the lengths of the coloured intervals (the contour intervals) have the same joint law as the lengths of the Voronoi cells in the subtree.
Reductions for $k \geq 3$: contour cells

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Reductions for \( k \geq 3 \): contour cells

Consider the subtree spanned by our uniform points.

We will show that the lengths of the coloured intervals (the contour intervals) have the same joint law as the lengths of the Voronoi cells in the subtree. Since the mass attached to the contour intervals yields a uniform split of unity, the same must then be true for the Voronoi cells.
Reductions for $k \geq 3$: scaling

Since we’re now only interested in showing that two vectors of lengths have the same distribution, it makes no difference if we rescale the whole tree.

So by the properties of the Brownian CRT, we may take the edge-lengths in the subtree spanned by our uniform points to be i.i.d. $\text{Exp}(1)$.
$k = 3$: contour lengths $\leftrightarrow$ Voronoi lengths
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$k = 3$: contour lengths ↔ Voronoi lengths

So again we have a bijection between the contour lengths and the Voronoi lengths.
General $k \geq 3$: by induction

Suppose the result is true for all smaller $k$. We start with a uniform binary plane leaf-labelled tree with i.i.d. Exp(1) edge-lengths.

Start from the shortest branch incident to a leaf. This branch is uniform among all those incident to leaves. Call its leaf $i$ and its length $\ell$. Call the “opposite leaf” $j$. 
General $k \geq 3$: by induction

Voronoi lengths: $(L_0, L_1, \ldots, L_{k-1})$
Contour lengths: $(C_0, C_1, \ldots, C_{k-1})$. 
General $k \geq 3$: by induction

Now burn in from every leaf to remove length $\ell$:
General $k \geq 3$: by induction

Now burn in from every leaf to remove length $\ell$:

By the memoryless property of the exponential, and the uniformity of the shortest leaf, we split into two uniform binary leaf-labelled trees with i.i.d. exponential edge-lengths, each with $< k$ leaves.

So, by the induction hypothesis, the Voronoi and contour lengths have the same laws in each of the subtrees.
General $k \geq 3$: by induction

- For each leaf other than $j$, we can get back the original contour length $C_r$ from $r$ to $r + 1$ by simply adding $2\ell$ to the contours in the smaller problems.
- For the contour from $j$ to $j + 1$, we must add two contours together and add $2\ell$. 
General $k \geq 3$: by induction

For the Voronoi cells, add $2\ell$ to the new lengths of the cells to get $L_r$, $r \neq i$.

For the cell of $i$, add two Voronoi cells from the smaller trees, plus $2\ell$.

By induction, the vectors of lengths therefore have the same law.
5. THE BROWNIAN CRT AS A UNIQUE FIXED POINT

Joint work with Marie Albenque (École polytechnique)

[The Brownian continuum random tree as the unique solution to a fixed point equation, Electronic Communications in Probability 20, 2015, paper no. 61, pp.1-14.]
By a **recursive distributional equation** (RDE) for a random variable \( X \) taking values in some Polish space \( S \), we mean an equation of the form

\[
X \overset{d}{=} f((\xi_i, X_i)_{i \geq 1})
\]

where \( X_1, X_2, \ldots \) are i.i.d. copies of \( X \), independent of the family of random variables \( (\xi_i)_{i \geq 1} \) and \( f \) is a suitable \( S \)-valued function.
Recursive distributional equations

**Example.** Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. real-valued r.v.'s with finite variance such that

$$X_1 \overset{d}{=} \frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n}}.$$ 

Then $X_1 \sim N(0, \sigma^2)$ for some $\sigma^2 > 0$. 
Recursive distributional equations

**Example.** Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. real-valued r.v.’s with finite variance such that

$$X_1 \overset{d}{=} \frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n}}.$$ 

Then $X_1 \sim N(0, \sigma^2)$ for some $\sigma^2 > 0$.

The centred normal distributions are the fixed points of this RDE.
Recap: recursive self-similarity

Consider picking three independent points $U_1, U_2, U_3$ from the Brownian CRT $\mathcal{T}$ according to $\mu$. There is a unique branch-point between these three points, and it splits the tree into three subtrees, $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$. 

Theorem. (Aldous (1993))

$$(\Delta_1, \Delta_2, \Delta_3) \sim \text{Dir}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}).$$

The rescaled subtrees $(\mathcal{T}_1, d_1/\sqrt{\Delta_1}, \mu_1/\Delta_1)$, $(\mathcal{T}_2, d_2/\sqrt{\Delta_2}, \mu_2/\Delta_2)$, $(\mathcal{T}_3, d_3/\sqrt{\Delta_3}, \mu_3/\Delta_3)$ are i.i.d. Brownian CRTs, independent of $(\Delta_1, \Delta_2, \Delta_3)$.

$U_i$ and the original branch-point are independent samples from $\mu_i/\Delta_i$ in subtree $i = 1, 2, 3$. 
Recap: recursive self-similarity

Consider picking three independent points $U_1, U_2, U_3$ from the Brownian CRT $\mathcal{T}$ according to $\mu$. There is a unique branch-point between these three points, and it splits the tree into three subtrees, $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$.

Write $d_1, d_2, d_3$ and $\mu_1, \mu_2, \mu_3$ for the restrictions of $d$ and $\mu$ to each of these subtrees respectively. Let

$$\Delta_1 = \mu(\mathcal{T}_1), \Delta_2 = \mu(\mathcal{T}_2), \Delta_3 = \mu(\mathcal{T}_3).$$
Recap: recursive self-similarity

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**Theorem.** (Aldous (1993))

- We have $(\Delta_1, \Delta_2, \Delta_3) \sim \text{Dir}(1/2, 1/2, 1/2)$.
- The rescaled subtrees $(\mathcal{T}_1, d_1/\sqrt{\Delta_1}, \mu_1/\Delta_1)$, $(\mathcal{T}_2, d_2/\sqrt{\Delta_2}, \mu_2/\Delta_2)$, $(\mathcal{T}_3, d_3/\sqrt{\Delta_3}, \mu_3/\Delta_3)$ are i.i.d. Brownian CRTs, independent of $(\Delta_1, \Delta_2, \Delta_3)$.
- $U_i$ and the original branch-point are independent samples from $\mu_i/\Delta_i$ in subtree $i = 1, 2, 3$. 
An operator on \((\text{laws of}) \text{ CRTs}\)

Let \(\mathcal{M}\) be the set of probability measures on continuum trees. Define an operator \(\mathcal{F} : \mathcal{M} \to \mathcal{M}\) as follows: for \(M \in \mathcal{M}\),

1. Sample independent trees \((T_1, d_1, \mu_1), (T_2, d_2, \mu_3), (T_3, d_3, \mu_3)\) having distribution \(M\);
2. For \(1 \leq i \leq 3\), sample \(U_i\) according to \(\mu_i\);
3. Independently sample \((\Delta_1, \Delta_2, \Delta_3) \sim \text{Dir}(1/2, 1/2, 1/2)\);
4. Rescale to obtain \((T_1, \Delta_1^{1/2}d_1, \Delta_1\mu_1), (T_2, \Delta_2^{1/2}d_2, \Delta_2\mu_2), (T_3, \Delta_3^{1/2}d_3, \Delta_3\mu_3)\).
5. Identify the vertices \(U_1, U_2, U_3\) in order to obtain a single larger tree \(T\) with a marked branch-point \(B\); the metrics and measures naturally induce a metric \(d\) and a measure \(\mu\) on \(T\).
6. Forget the branch-point in order to obtain \((T, d, \mu)\).

\(\mathcal{F}(M)\) is the distribution of \((T, d, \mu)\).
The previous theorem told us that the law of the Brownian CRT is a fixed point of $\mathcal{F}$. 

Theorem. (Albenque & G. (2015))

- [Unique fixed point] Suppose that $M$ is a law on continuum trees which is a fixed point of $\mathcal{F}$. Then there exists $\alpha > 0$ such that if $(T, d, \mu) \sim M$ then $(T, \alpha d, \mu)$ is a Brownian CRT.

- [Attractive] Suppose that $M$ is a law on continuum trees such that if $(T, d, \mu) \sim M$, given $(T, d, \mu)$, $V_1, V_2$ are sampled independently from $\mu$, then $E[d(V_1, V_2)]$ exists and is equal to $\pi/2$. Let $M_n = \mathcal{F}_n M$. Then $M_n$ converges weakly to the law of the Brownian CRT in the sense of the Gromov-Prokhorov topology.
The Brownian CRT as a unique fixed point

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- **[Attractive]** Suppose that $M$ is a law on continuum trees such that if $(T, d, \mu) \sim M$, given $(T, d, \mu)$, $V_1, V_2$ are sampled independently from $\mu$, then $\mathbb{E}[d(V_1, V_2)]$ exists and is equal to $\pi/2$. Let $M_n = \mathcal{F}^n M$. Then $M_n$ converges weakly to the law of the Brownian CRT in the sense of the Gromov-Prokhorov topology.
Lemma. Suppose that $M \in \mathcal{M}$ is a fixed point of $\mathfrak{F}$. Let $(T, d, \mu) \sim M$ and, conditionally on $(T, d, \mu)$, let $V_1, V_2 \overset{i.i.d.}{\sim} \mu$. Then there exists a constant $\alpha > 0$ such that $\alpha d(V_1, V_2) \sim \text{Rayleigh}$. 
Two-point distances

\((T, d, \mu) \sim M\) and \(V_1, V_2 \overset{\text{i.i.d.}}{\sim} \mu\). Let \(L = d(V_1, V_2)\).

Since \(M\) is a fixed point of \(\mathfrak{F}\), we can think of \((T, d, \mu)\) as having been built out of three scaled independent copies, \((T_1, d_1, \mu_1)\), \((T_2, d_2, \mu_2)\) and \((T_3, d_3, \mu_3)\):
Two-point distances

There are two possibilities (plus symmetries) for what happens to the two points $V_1$ and $V_2$: 

Diagram: Two separate diagrams showing two possible scenarios for the two points $V_1$ and $V_2$. One scenario places $V_1$ and $V_2$ closer together, while the other places them farther apart. Both diagrams include labels for $T_1$, $T_2$, and $T_3$.
Two-point distances

Let $P_1, P_2, P_3$ denote the numbers of points falling in each of $T_1, T_2, T_3$ respectively.

Conditionally on $(\Delta_1, \Delta_2, \Delta_3)$, we have

$$(P_1, P_2, P_3) \sim \text{Multinomial}(2; \Delta_1, \Delta_2, \Delta_3).$$
Two-point distances

Let $P_1$, $P_2$, $P_3$ denote the numbers of points falling in each of $T_1$, $T_2$, $T_3$ respectively.

Conditionally on $(\Delta_1, \Delta_2, \Delta_3)$, we have

$$(P_1, P_2, P_3) \sim \text{Multinomial}(2; \Delta_1, \Delta_2, \Delta_3).$$

Then

$$L \overset{d}{=} \sqrt{\Delta_1} L_1 1_{\{P_1 > 0\}} + \sqrt{\Delta_2} L_2 1_{\{P_2 > 0\}} + \sqrt{\Delta_3} L_3 1_{\{P_3 > 0\}},$$

where $L_1, L_2, L_3$ are i.i.d. copies of $L$, independent of $(\Delta_1, \Delta_2, \Delta_3)$ and $(P_1, P_2, P_3)$. 
Two-point distances

Let $P_1, P_2, P_3$ denote the numbers of points falling in each of $T_1, T_2, T_3$ respectively.

Conditionally on $(\Delta_1, \Delta_2, \Delta_3)$, we have

$$(P_1, P_2, P_3) \sim \text{Multinomial}(2; \Delta_1, \Delta_2, \Delta_3).$$

Then

$$L \overset{d}{=} \sqrt{\Delta_1} L_1 \mathbb{1}_{\{P_1 > 0\}} + \sqrt{\Delta_2} L_2 \mathbb{1}_{\{P_2 > 0\}} + \sqrt{\Delta_3} L_3 \mathbb{1}_{\{P_3 > 0\}},$$

where $L_1, L_2, L_3$ are i.i.d. copies of $L$, independent of $(\Delta_1, \Delta_2, \Delta_3)$ and $(P_1, P_2, P_3)$.

This recursive distributional equation is an instance of the so-called smoothing transform.
The smoothing transform

Suppose that

\[ X_1 \overset{d}{=} \sum_{i=1}^{n} W_i X_i, \]

where \( X_1, X_2, \ldots, X_n \) are i.i.d. non-negative r.v.'s, independent of the non-negative r.v.'s \( W_1, W_2, \ldots, W_n \) which are such that

\[ \mathbb{E} \left[ W_i^\gamma \right] < \infty \]

for all \( 1 \leq i \leq n \) and some \( \gamma > 1 \).
The smoothing transform, \( X_1 \overset{d}{=} \sum_{i=1}^{n} W_i X_i \)

Let \( g(s) = \log (\sum_{i=1}^{n} \mathbb{E} [W_i^s \mathbb{1}_{\{W_i > 0\}}]) \), \( s \geq 0 \). Write \( \mathcal{F}(\mu) \) for the distribution of \( \sum_{i=1}^{n} W_i X_i \) when \( X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} \mu \).

**Theorem.** (Durrett & Liggett (1983))

1. Suppose that \( g \) has a unique zero \( \alpha \in (0, 1] \). If \( \alpha = 1 \) and \( g'(1) < 0 \) then the RDE has a unique fixed point \( \mu \), up to a deterministic scaling factor. Write \( \mu_m \) for the fixed point with mean \( m \).

2. Suppose \( \nu \) is any law on \( \mathbb{R}_+ \) such that \( \int_{0}^{\infty} x d\nu(x) = m \). Then

\[
\mathcal{F}^k(\nu) \to \mu_m
\]

as \( k \to \infty \).

Two-point distances

For any fixed point of $\mathcal{F}$, the two-point distances satisfy

$$L \overset{d}{=} \sqrt{\Delta} L \mathbb{1}_{\{P_1>0\}} + \sqrt{\Delta} L \mathbb{1}_{\{P_2>0\}} + \sqrt{\Delta} L \mathbb{1}_{\{P_3>0\}}.$$
Two-point distances

For any fixed point of $\mathcal{F}$, the two-point distances satisfy

$$L \overset{d}{=} \sqrt{\Delta_1} L_1 \mathbb{1}_{\{P_1>0\}} + \sqrt{\Delta_2} L_2 \mathbb{1}_{\{P_2>0\}} + \sqrt{\Delta_3} L_3 \mathbb{1}_{\{P_3>0\}}.$$

We get $g(s) = \log \left( \frac{3(s+7)}{(s+3)(s+5)} \right)$, which has its unique zero in $s \geq 0$ at $s = 1$, with $g'(1) = -7/24 < 0$. So the theorem of Durrett and Liggett applies to give a unique distributional solution, up to a deterministic constant.
Two-point distances

For any fixed point of $\mathcal{F}$, the two-point distances satisfy

$$L \overset{d}{=} \sqrt{\Delta_1} L_1 \mathbb{I}_{\{P_1 > 0\}} + \sqrt{\Delta_2} L_2 \mathbb{I}_{\{P_2 > 0\}} + \sqrt{\Delta_3} L_3 \mathbb{I}_{\{P_3 > 0\}}.$$  

We get $g(s) = \log \left( \frac{3(s+7)}{(s+3)(s+5)} \right)$, which has its unique zero in $s \geq 0$ at $s = 1$, with $g'(1) = -7/24 < 0$. So the theorem of Durrett and Liggett applies to give a unique distributional solution, up to a deterministic constant.

Moreover, since the Rayleigh distribution must be a solution, and $\mathbb{E} \left[ \text{Rayleigh} \right] = \sqrt{\frac{\pi}{2}}$, the value $\alpha$ in the lemma is $\sqrt{\pi/2}/\mathbb{E} \left[ L \right]$. 

Theorem. Suppose that $M \in \mathcal{M}$ is a fixed point of $\mathcal{F}$. Then the random finite-dimensional distributions of $M$ are the same as those of the Brownian CRT, up to a strictly positive scaling factor $\alpha$. 
Random finite-dimensional distributions

**Theorem.** Suppose that $M \in \mathcal{M}$ is a fixed point of $\mathcal{F}$. Then the random finite-dimensional distributions of $M$ are the same as those of the Brownian CRT, up to a strictly positive scaling factor $\alpha$.

Since the law of a continuum random tree is uniquely determined by its random finite-dimensional distributions, this will be enough to give the first part of our fixed point theorem.
Three-point distances

There are three cases:
Three-point distances

Case 1:

Accounting for symmetries, this event occurs with probability $6\mathbb{E} [\Delta_1 \Delta_2 \Delta_3] = 2/35$. 
Three-point distances

Case 1:

Accounting for symmetries, this event occurs with probability $\mathbb{P}[\Delta_1 \Delta_2 \Delta_3] = 2/35$.

Conditioning on its occurrence yields biased sizes $(\Delta_1^*, \Delta_2^*, \Delta_3^*)$ and we get $(\sqrt{\Delta_1^*} L_1, \sqrt{\Delta_2^*} L_2, \sqrt{\Delta_3^*} L_3)$ for the three distances.
Three-point distances

Cases 2 and 3:

Here, in order to understand the distances, we need to figure out what’s happening inside one of the level-1 subtrees. Note that, in the two cases, the problem is really the same: we have three uniform points within one of the subtrees, and want to find the distances between them – it’s just that in case 2, one of the three points is the branchpoint at the centre.
Three-point distances

Cases 2 and 3:

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Three-point distances: an example
Three-point distances

It is always possible to split the paths between 3 points up into sums of randomly scaled copies of the path between 2 uniform points.
Three-point distances

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How deep do we need to go in order to find the decomposition? At each level, we either get that the three points are in different subtrees (which occurs with probability $\frac{2}{35}$) or they are not and we need to go one level deeper.
Three-point distances

It is always possible to split the paths between 3 points up into sums of randomly scaled copies of the path between 2 uniform points.

How deep do we need to go in order to find the decomposition? At each level, we either get that the three points are in different subtrees (which occurs with probability $2/35$) or they are not and we need to go one level deeper.

So the depth $N$ to which we need to go satisfies $N \sim \text{Geometric}(2/35)$ which is, in particular, almost surely finite.
Three-point distances

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So the depth $N$ to which we need to go satisfies $N \sim \text{Geometric}(2/35)$ which is, in particular, almost surely finite.

But since we know the two-point distribution is Rayleigh, the three-point distances must also be uniquely determined (and equal to their distribution for the Brownian CRT).
A similar inductive argument shows that the $k$-point distances may always be expressed in terms of sums of randomly scaled two-point distances, and so again must be uniquely fixed.

The attractiveness of the fixed point again makes use of Durrett and Liggett’s theorem for the two-point distances, as well as a slightly complicated coupling.
6. UNIVERSALITY

Key reference:

A universal scaling limit

Let $T_n$ be the family tree of a Galton-Watson process with critical offspring distribution of variance $\sigma^2 \in (0, \infty)$, conditioned to have total progeny $n$. Let $d_n$ be the graph distance on $T_n$ and let $\mu_n$ be the uniform measure on the vertices.

**Theorem.** (Aldous (1993), Le Gall (2005))

As $n \to \infty$,

$$\left( T_n, \frac{\sigma}{\sqrt{n}} d_n, \mu_n \right) \xrightarrow{d} \left( T_{2e}, d_{2e}, \mu_{2e} \right),$$

where convergence is in the Gromov-Hausdorff-Prokhorov sense.
A Galton-Watson branching process \((Z_n)_{n \geq 0}\) describes the size of a population which evolves as follows:

- Start with a single individual.
- This individual has a number of children distributed according to the offspring distribution \(p\), where \(p(k)\) gives the probability of \(k\) children, \(k \geq 0\).
- Each child reproduces as an independent copy of the original individual.

\(Z_n\) gives the number of individuals in generation \(n\) (in particular, \(Z_0 = 1\)).
Galton-Watson trees

A Galton-Watson tree is the family tree arising from a Galton-Watson branching process. We will think of this as a rooted ordered tree.
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Consider the case where the offspring distribution $p$ is critical i.e.

$$\sum_{k=1}^{\infty} kp(k) = 1.$$ 

This ensures, in particular, that the resulting tree, $T$, is finite.
Uniform random trees

**Proposition.** Let $T$ be a (rooted, ordered) Galton-Watson tree, with Poisson(1) offspring distribution and total progeny $N$. Assign the vertices labels uniformly at random from $\{1, 2, \ldots, N\}$ and then forget the ordering and the root. Let $\tilde{T}$ be the labelled tree obtained. Then, conditional on $N = n$, $\tilde{T}$ has the same distribution as $T_n$, a uniform random tree on $n$ labelled vertices.
Other combinatorial trees in disguise

Let $T$ be a Galton-Watson tree with offspring distribution $p$ and total progeny $N$.

- If $p(k) = 2^{-k-1}$, $k \geq 0$ (i.e. Geometric(1/2) offspring distribution) then conditional on $N = n$, the tree is uniform on the set of plane trees with $n$ vertices.
- If $p(0) = 1/2$ and $p(2) = 1/2$ then, conditional on $N = 2n$, the tree is uniform on the set of planted plane binary trees with $n$ leaves.
Two ways of encoding a tree

As we have seen, it is convenient to encode our trees in terms of discrete functions which are easier to manipulate.

We will do this in two different ways:

- the height function
- the depth-first walk.
Suppose that our tree has $n$ vertices. Let them be $v_0, v_1, \ldots, v_{n-1}$, listed in depth-first order.
Height function

Suppose that our tree has \( n \) vertices. Let them be \( v_0, v_1, \ldots, v_{n-1} \), listed in depth-first order.

Then the height function is defined by

\[
H(k) = d_{gr}(v_0, v_k), \quad 0 \leq k \leq n - 1.
\]
Height function

\[ H(k) \]

\[ H(k) = \begin{cases} 3 & \text{if } k = 1 \\ 2 & \text{if } k = 2 \\ 1 & \text{if } k = 3 \\ 0 & \text{otherwise} \end{cases} \]
Height function
Height function

\[ H(k) \]

Diagram showing a tree with labels and a scatter plot with points at coordinates (1, 1) and (2, 2).
Height function

\[ H(k) \]

![Graph of Height function]

\[ k \]

Graph showing the height function with points at \((1, 1), (2, 2), (3, 3), (4, 0), (5, -1), (6, 2)\).
Height function
Height function

\[ H(k) \]

\begin{itemize}
  \item \( H(0) = -1 \)
  \item \( H(1) = -1 \)
  \item \( H(2) = 1 \)
  \item \( H(3) = 3 \)
  \item \( H(4) = 2 \)
  \item \( H(5) = 3 \)
  \item \( H(6) = 3 \)
\end{itemize}
Height function

\[ H(k) \]
We can easily recover the tree from its height function.
Depth-first walk

Let $c(v)$ be the number of children of $v$, and that $v_0, v_1, \ldots, v_{n-1}$ is a list of the vertices in depth-first order.

Depth-first walk

Let $c(v)$ be the number of children of $v$, and that $v_0, v_1, \ldots, v_{n-1}$ is a list of the vertices in depth-first order.

Define

$$X(0) = 0,$$

$$X(i) = \sum_{j=0}^{i-1} (c(v_j) - 1), \text{ for } 1 \leq i \leq n.$$
Depth-first walk

Let \( c(v) \) be the number of children of \( v \), and that \( v_0, v_1, \ldots, v_{n-1} \) is a list of the vertices in depth-first order.

Define

\[
X(0) = 0, \\
X(i) = \sum_{j=0}^{i-1} (c(v_j) - 1), \text{ for } 1 \leq i \leq n.
\]

In other words,

\[
X(i + 1) = X(i) + c(v_i) - 1, \quad 0 \leq i \leq n - 1.
\]
Depth-first walk

\[ X(k) \]

\[ 42, 6, 53, 1-1, 0, 1, 2 \]
Depth-first walk

\[ X(k) \]

\[ k \]

\[ \theta \]
Depth-first walk
Depth-first walk

\[ X(k) = k \]

- Graph showing the depth-first walk with nodes labeled and edges connecting them.
- A plot illustrating \( X(k) \) against \( k \) with data points indicating the values of the function at different values of \( k \).
Depth-first walk

\[ X(k) \]

\[ X(1)\]  \[ X(2)\]  \[ X(3)\]  \[ X(4)\]  \[ X(5)\]  \[ X(6)\]  \[ X(7)\]  

\[ k \]  \[ 1 \]  \[ 2 \]  \[ 3 \]  \[ 4 \]  \[ 5 \]  \[ 6 \]  \[ 7 \]
Depth-first walk

\[ X(k) \]

\[ k \]

Graph and line plot showing a depth-first walk.
Depth-first walk

\begin{align*}
X(k) & = k - 1 \\
& \text{for } k = 1, 2, 3, 4, 5, 6, 7
\end{align*}
Depth-first walk

\[ X(k) \]
Proposition. For $0 \leq i \leq n - 1$, 

$$H(i) = \# \left\{ 0 \leq j \leq i - 1 : X(j) = \min_{j \leq k \leq i} X(k) \right\}.$$
The depth-first walk of a Galton-Watson process is a stopped random walk

Recall that $p$ is a distribution on $\mathbb{Z}_+$ such that $\sum_{k=1}^{\infty} kp(k) = 1$. 

Proposition. Let $(R(k), k \geq 0)$ be a random walk with initial value 0 and step distribution $\nu(k) = p(k+1)$, $k \geq -1$. Set $M = \inf \{k \geq 0 : R(k) = -1\}$.

Now suppose that $T$ is a Galton-Watson tree with offspring distribution $p$ and total progeny $N$. Then $(X(k), 0 \leq k \leq N)$ $d =$ $(R(k), 0 \leq k \leq M)$.

[Careful proof: see Le Gall (2005).]
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Now suppose that $T$ is a Galton-Watson tree with offspring distribution $p$ and total progeny $N$. Then

$$(X(k), 0 \leq k \leq N) \overset{d}{=} (R(k), 0 \leq k \leq M).$$
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$$(X(k), 0 \leq k \leq N) \overset{d}{=} (R(k), 0 \leq k \leq M).$$

[Careful proof: see Le Gall (2005).]
Suppose now that we have offspring variance
$$\sigma^2 := \sum_{k=1}^{\infty} (k - 1)^2 p(k) \in (0, \infty).$$

The depth-first walk $X$ is a random walk with step mean 0 and variance $\sigma^2$, stopped at the first time it hits $-1$. The underlying random walk has a Brownian motion as its scaling limit, by Donsker’s theorem.

The total progeny $N$ is equal to $\inf\{k \geq 0 : X(k) = -1\}$. We want to condition on the event $\{N = n\}$. 

Standing assumption: $P(N = n) > 0$ for all $n$ sufficiently large.
Suppose now that we have offspring variance 
\[ \sigma^2 := \sum_{k=1}^{\infty} (k - 1)^2 p(k) \in (0, \infty). \]

The depth-first walk \( X \) is a random walk with step mean 0 and variance \( \sigma^2 \), stopped at the first time it hits \(-1\). The underlying random walk has a Brownian motion as its scaling limit, by Donsker’s theorem.

The total progeny \( N \) is equal to \( \inf\{k \geq 0 : X(k) = -1\} \). We want to condition on the event \( \{N = n\} \).

**Standing assumption:** \( \mathbb{P}(N = n) > 0 \) for all \( n \) sufficiently large.
Write \((X_n(k), 0 \leq k \leq n)\) for the depth-first walk conditioned on \(\{N = n\}\). Then there is a conditional version of Donsker’s theorem.

**Theorem.** As \(n \to \infty\),

\[
\frac{1}{\sigma \sqrt{n}}(X_n(\lfloor nt \rfloor), 0 \leq t \leq 1) \xrightarrow{d} (e(t), 0 \leq t \leq 1),
\]

where \((e(t), 0 \leq t \leq 1)\) is a standard Brownian excursion.

Height process

Let \((H_n(i), 0 \leq i \leq n)\) be the height process of a critical Galton-Watson tree with offspring variance \(\sigma^2 \in (0, \infty)\), conditioned to have total progeny \(n\), so that

\[
H_n(i) = \# \left\{ 0 \leq j \leq i - 1 : X_n(j) = \min_{j \leq k \leq i} X_n(k) \right\}.
\]

**Theorem.** As \(n \to \infty\),

\[
\frac{\sigma}{\sqrt{n}} (H_n(\lfloor nt \rfloor), 0 \leq t \leq 1) \xrightarrow{d} 2 (e(t), 0 \leq t \leq 1),
\]

where \((e(t), 0 \leq t \leq 1)\) is a standard Brownian excursion.
The convergence

\[ \left( T_n, \frac{\sigma}{\sqrt{n}} d_n, \mu_n \right) \xrightarrow{d} (T_{2e}, d_{2e}, \mu_{2e}), \]

now follows by the same proof that we used in the case of binary trees.
Universality

The universality class of the Brownian CRT is, in fact, even larger. Some other examples of trees (and graphs!) with the Brownian CRT as their scaling limit are:

- uniform unordered unlabelled rooted trees
- uniform unordered unlabelled unrooted trees
- critical multi-type Galton-Watson trees
- random trees with a prescribed degree sequence satisfying certain conditions
- random dissections
- random graphs from subcritical classes.

A particularly useful tool: Markov branching trees.

Universal scaling limits often show up in other places, and the Brownian CRT is no exception. It appears, for example, as a building block in the scaling limit of random planar maps: the Brownian map is constructed as a (complicated) quotient of the Brownian CRT.
What if the offspring variance isn’t finite?

Suppose instead that the offspring distribution is critical but in the domain of attraction of an $\alpha$-stable law, for $\alpha \in (1, 2)$. For example,

$$p(k) \sim Ck^{-1-\alpha} \quad \text{as } k \to \infty$$

for $C > 0$. 
The limiting depth-first walk

We now get

\[ \frac{1}{n^{1/\alpha}}(X_n([nt]), 0 \leq t \leq 1) \overset{d}{\rightarrow} (e^{(\alpha)}(t), 0 \leq t \leq 1), \]

where \( e^{(\alpha)} \) is an excursion of a spectrally positive \( \alpha \)-stable Lévy process.

[Picture by Igor Korchemski]
The limiting height process

It’s no longer the case that $H_n$ has the same limit. We get

$$\frac{1}{n^{1-1/\alpha}}(H_n(\lfloor nt \rfloor), 0 \leq t \leq 1) \overset{d}{\to} (h^{(\alpha)}(t), 0 \leq t \leq 1)$$

for some much more complicated continuous excursion $h^{(\alpha)}$. [Pictures by Igor Kortchemski]
The stable trees

**Theorem.** (Duquesne (2003)) Suppose that $p(k) \sim ck^{-1-\alpha}$ as $k \to \infty$ for $\alpha \in (1, 2)$. Then as $n \to \infty$,

$$\frac{1}{n^{1-1/\alpha}} T_n \overset{d}{\to} c_\alpha T_\alpha,$$

where $T_\alpha$ is the stable tree of parameter $\alpha$ and $c_\alpha$ is a strictly positive constant.


The stable trees

An important difference between the stable trees for $\alpha \in (1, 2)$ and the Brownian CRT is that the Brownian CRT is binary. The stable trees, on the other hand, have only branch-points of infinite degree.

[Pictures by Igor Kortchemski]
7. CONNECTED GRAPHS

Joint work with Louigi Addario-Berry (McGill) and Nicolas Broutin (Sorbonne Université Paris).


Uniform connected graph with fixed surplus

Fix $k \geq 0$ and let $G_n^k$ be a uniform connected graph with vertices labelled by $1, 2, \ldots, n$ and $n + k - 1$ edges.
Uniform connected graph with fixed surplus

Fix $k \geq 0$ and let $G_n^k$ be a uniform connected graph with vertices labelled by $1, 2, \ldots, n$ and $n + k - 1$ edges. We say $G_n^k$ has surplus $k$. 
Uniform connected graph with fixed surplus

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(For $k = 0$, this is just a uniform random tree on $n$ vertices.)
Uniform connected graph with fixed surplus

Fix $k \geq 0$ and let $G^k_n$ be a uniform connected graph with vertices labelled by $1, 2, \ldots, n$ and $n + k - 1$ edges. We say $G^k_n$ has surplus $k$.

(For $k = 0$, this is just a uniform random tree on $n$ vertices.)

Write $d^k_n$ for the graph distance and $\mu^k_n$ for the uniform measure on the vertices.
Uniform connected graph with fixed surplus

Fix $k \geq 0$ and let $G_n^k$ be a uniform connected graph with vertices labelled by $1, 2, \ldots, n$ and $n + k - 1$ edges. We say $G_n^k$ has surplus $k$.

(For $k = 0$, this is just a uniform random tree on $n$ vertices.)

Write $d_n^k$ for the graph distance and $\mu_n^k$ for the uniform measure on the vertices.

**Theorem.** (Addario-Berry, Broutin & G. (2012))
There exists a random compact metric measure space $(\mathcal{G}^k, d^k, \mu^k)$ such that

$$\frac{1}{\sqrt{n}}(G_n^k, d_n^k, \mu_n^k) \xrightarrow{d} (\mathcal{G}^k, d^k, \mu^k)$$

as $n \to \infty$. 
Uniform connected graph with fixed surplus

Fix $k \geq 0$ and let $G_n^k$ be a uniform connected graph with vertices labelled by $1, 2, \ldots, n$ and $n + k - 1$ edges. We say $G_n^k$ has surplus $k$.

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\[
\frac{1}{\sqrt{n}}(G_n^k, d_n^k, \mu_n^k) \xrightarrow{d} (\mathcal{G}^k, d^k, \mu^k)
\]
as $n \to \infty$.

We can give an explicit description for the scaling limit.
Let $e$ be a standard Brownian excursion. Define a random excursion $\tilde{e}^k : [0, 1] \to \mathbb{R}_+$ via a change of measure as follows. For any suitable test-function $f : C([0, 1], \mathbb{R}_+) \to \mathbb{R}$,

$$
\mathbb{E} \left[ f(\tilde{e}^k(t), 0 \leq t \leq 1) \right] = \frac{\mathbb{E} \left[ f(e(t), 0 \leq t \leq 1) \left( \int_0^1 e(u) du \right)^k \right]}{\mathbb{E} \left[ \left( \int_0^1 e(u) du \right)^k \right]}$
$$
Scaling limit

Use $2\tilde{e}^k$ to encode a continuum random tree $(\tilde{T}^k, \tilde{d}^k, \tilde{\mu}^k)$. 

Pick $k$ independent uniform marks in the area under the curve. Each mark picks out two points of the tree.
Pick $k$ independent uniform marks in the area under the curve. Each mark picks out two points of the tree. Identify them.
Vertex identifications

Write $\pi^k$ for the usual projection $[0, 1] \to \tilde{T}^k$.

We have marks $(x_1, y_1), \ldots, (x_k, y_k)$ which are uniform in the area under the excursion. For $1 \leq i \leq k$, let

$$t_i = \inf\{t \geq x_i : 2\tilde{e}^k(t) = y_i\}.$$

Define another equivalence relation $\sim$ on $\tilde{T}^k$ by declaring $\pi^k(x_i) \sim \pi^k(t_i)$ and now let $G^k = \tilde{T}^k / \sim$.

We have $G^k = \tilde{T}^k / \sim$. Let $d^k$ be the metric and $\mu^k$ the measure induced from $\tilde{d}^k$ and $\tilde{\mu}^k$ respectively.
Scaling limit \((\mathcal{G}^k, d^k, \mu^k)\) for \(k = 4\)
For a tree of size $n$, we defined the depth-first walk by $X(0) = 0$ and, for $1 \leq k \leq n$,

$$X(k) = \sum_{i=0}^{k-1} (c(v_i) - 1),$$

where $c(v)$ is the number of children of vertex $v$ and $v_0, v_1, \ldots, v_{n-1}$ are the vertices in contour order.
Proof technique: depth-first exploration

For a tree of size $n$, we defined the depth-first walk by $X(0) = 0$ and, for $1 \leq k \leq n$,

$$X(k) = \sum_{i=0}^{k-1} (c(v_i) - 1),$$

where $c(v)$ is the number of children of vertex $v$ and $v_0, v_1, \ldots, v_{n-1}$ are the vertices in contour order.

$X$ tracks the number of vertices we have seen, but whose children we have not yet explored.
Depth-first exploration: an example

Step 0: $X(0) = 0$
Depth-first exploration: an example

Step 1: $X(1) = 2$
Depth-first exploration: an example

Step 2: $X(2) = 3$
Depth-first exploration: an example

Step 3: $X(3) = 3$
Depth-first exploration: an example

Step 4: $X(4) = 2$
Depth-first exploration: an example

Step 5: \( X(5) = 1 \)
Depth-first exploration: an example

Step 6: $X(6) = 0$
Depth-first exploration: an example

Step 7: $X(7) = 0$
Depth-first exploration: an example

Step 9: $X(8) = 1$
Depth-first exploration: an example

Step 10: $X(9) = 0$
Depth-first walk
Depth-first tree

In the depth-first exploration, we effectively explored this spanning tree; the dashed surplus edges made no difference.

Call the spanning tree the depth-first tree associated with the graph $G$, and write $T(G)$. $X$ is also the depth-first walk of $T$. 
Permitted edges

Look at things the other way round: for a given tree $T$, which connected graphs $G$ have depth-first tree $T(G) = T$?

In other words, where can we put surplus edges so that they don’t change $T$?

Call such edges permitted.
Depth-first walk and permitted edges

Step 0: $X(0) = 0$. 
Depth-first walk and permitted edges

Step 1: $X(1) = 2$. 
Depth-first walk and permitted edges

Step 2: $X(2) = 3$. 
Step 3: $X(3) = 3$. 
Depth-first walk and permitted edges

Step 4: $X(4) = 2$. 
Depth-first walk and permitted edges

Step 5: $X(5) = 1$. 

Depth-first walk and permitted edges

Step 6: \( X(6) = 0. \)
Step 7: $X(7) = 0$. 
Step 8: \( X(8) = 1 \).
Step 10: $X(9) = 0$. 
Area

At step $k \geq 0$, there are $X(k)$ permitted edges. So the total number is

$$a(T) = \sum_{k=0}^{n-1} X(k).$$

We call this the area of $T$. 
Let $\mathcal{G}_T$ be the set of graphs $G$ such that $T(G) = T$. It follows that $|\mathcal{G}_T| = 2^{a(T)}$, since each permitted edge may either be included or not.
Classifying graphs by depth-first tree

Let $G_T$ be the set of graphs $G$ such that $T(G) = T$. It follows that $|G_T| = 2^a(T)$, since each permitted edge may either be included or not.

Let $T[n]$ be the set of trees with label-set $[n] = \{1, 2, \ldots, n\}$. Then

$$\{G_T : T \in T[n]\}$$

is a partition of the set of connected graphs on $[n]$. 
Recipe for creating a uniform connected graph

Create a uniform connected graph $G_n^k$ as follows.
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Create a uniform connected graph $G_n^k$ as follows.

- Pick a random labelled tree $\tilde{T}_n^k$ such that
  \begin{equation}
  \mathbb{P}\left(\tilde{T}_m^k = T\right) \propto \binom{a(T)}{k}, \quad T \in \mathbb{T}[n].
  \end{equation}
Recipe for creating a uniform connected graph

Create a uniform connected graph $G_n^k$ as follows.

- Pick a random labelled tree $\tilde{T}_n^k$ such that

$$\mathbb{P}\left(\tilde{T}_m^k = T\right) \propto \binom{a(T)}{k}, \ T \in \mathbb{T}[n].$$

- Choose a uniform $k$-set from among the $a(\tilde{T}_n^k)$ permitted edges and add them to the tree.
We essentially need to show

- the tree $\tilde{T}_n^k$ converges to a CRT coded by the excursion $\tilde{e}^k$;
- the locations of the surplus edges converge to the locations in the limiting picture.
Taking limits for the tree

Write $\tilde{X}_n^k$ for the depth-first walk associated with $\tilde{T}_n^k$. Then

$$a\left(\tilde{T}_n^k\right) = \sum_{i=0}^{n-1} \tilde{X}_n^k(i) = \int_0^n \tilde{X}_n^k(\lfloor s \rfloor) ds = n^{3/2} \int_0^1 n^{-1/2} \tilde{X}_n^k(\lfloor nu \rfloor) du,$$

by changing variable in the integral.
Taking limits for the tree

Write \( \tilde{X}^k_n \) for the depth-first walk associated with \( \tilde{T}^k_n \). Then

\[
a\left( \tilde{T}^k_n \right) = \sum_{i=0}^{n-1} \tilde{X}^k_n(i) = \int_0^n \tilde{X}^k_n([s])ds = n^{3/2} \int_0^1 n^{-1/2} \tilde{X}^k_n([nu])du,
\]

by changing variable in the integral.

If \( T_n \) is a uniform random tree on \([n]\) and \( X_n \) is its depth-first walk, then

\[
(n^{-1/2}X_n([nt]), 0 \leq t \leq 1) \xrightarrow{d} (e(t), 0 \leq t \leq 1).
\]
Taking limits for the tree

Write $\tilde{X}_n^k$ for the depth-first walk associated with $\tilde{T}_n^k$. Then

$$a(\tilde{T}_n^k) = \sum_{i=0}^{n-1} \tilde{X}_n^k(i) = \int_0^n \tilde{X}_n^k([s])ds = n^{3/2} \int_0^1 n^{-1/2} \tilde{X}_n^k([nu])du,$$

by changing variable in the integral.

If $T_n$ is a uniform random tree on $[n]$ and $X_n$ is its depth-first walk, then

$$(n^{-1/2}X_n([nt]), 0 \leq t \leq 1) \overset{d}{\rightarrow} (e(t), 0 \leq t \leq 1).$$

So by the continuous mapping theorem,

$$\int_0^1 n^{-1/2}X_n([nu])du \overset{d}{\rightarrow} \int_0^1 e(u)du.$$
Taking limits for the tree

Use the change of measure to get from $\tilde{X}_n^k$ to $X_n$: for any bounded continuous function $f$,

$$
\mathbb{E} \left[ f \left( n^{-1/2} \tilde{X}_n^k(\lfloor nt \rfloor), 0 \leq t \leq 1 \right) \right]
$$

$$
= \frac{\mathbb{E} \left[ f \left( n^{-1/2} X_n(\lfloor nt \rfloor), 0 \leq t \leq 1 \right) \left( n^{3/2} \int_0^1 n^{-1/2} X_n(\lfloor nu \rfloor) du \right) \right]}{\mathbb{E} \left[ \left( n^{3/2} \int_0^1 n^{-1/2} X_n(\lfloor nu \rfloor) du \right) \right]}
$$
Taking limits for the tree

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$$
\mathbb{E} \left[ f \left( n^{-1/2} \tilde{X}_n^k(\lfloor nt \rfloor), 0 \leq t \leq 1 \right) \right]
= \mathbb{E} \left[ f \left( n^{-1/2} X_n(\lfloor nt \rfloor), 0 \leq t \leq 1 \right) \left( n^{3/2} \int_0^1 n^{-1/2} X_n(\lfloor nu \rfloor) du \right)^k \right]
\mathbb{E} \left[ \left( n^{3/2} \int_0^1 n^{-1/2} X_n(\lfloor nu \rfloor) du \right)^k \right]
$$

We have

$$
n^{-3k/2} \left( n^{3/2} \int_0^1 n^{-1/2} X_n(\lfloor nu \rfloor) du \right) \xrightarrow{d} \left( \int_0^1 e(s) ds \right)^k / k! \quad \text{as } n \to \infty.
$$
Taking limits for the tree

We also have uniform integrability, so we obtain

\[ \mathbb{E} \left[ f \left( n^{-1/2} \tilde{X}_n^k(nt), 0 \leq t \leq 1 \right) \right] \rightarrow \frac{\mathbb{E} \left[ f(e) \left( \int_0^1 e(u)du \right)^k \right]}{\mathbb{E} \left[ \left( \int_0^1 e(u)du \right)^k \right]} \]

\[ = \mathbb{E} \left[ f(\tilde{e}^k) \right]. \]
Taking limits for the tree

We also have uniform integrability, so we obtain

\[ \mathbb{E} \left[ f \left( n^{-1/2} \tilde{X}^k_n(nt), 0 \leq t \leq 1 \right) \right] \rightarrow \frac{\mathbb{E} \left[ f(e) \left( \int_0^1 e(u)du \right)^k \right]}{\mathbb{E} \left[ \left( \int_0^1 e(u)du \right)^k \right]} \]

\[ = \mathbb{E} \left[ f(\tilde{e}^k) \right]. \]

This (after converting to the height process) entails that

\[ \frac{1}{\sqrt{n}} \tilde{T}^k_n \xrightarrow{d} \tilde{T}^k. \]
Taking limits for the surplus edges

The permitted edges are in bijective correspondence with the integer points under the graph of the depth-first walk.

![Graph with integer points and depth-first walk](image)
Taking limits for the surplus edges

The permitted edges are in bijective correspondence with the integer points under the graph of the depth-first walk. Since we pick a uniform $k$-set from among these points, in the limit what we see is just $k$ points picked independently and uniformly from the area under the limit curve.
Taking limits for the surplus edges

Surplus edges almost go to ancestors... In fact, they always go to younger children of ancestors of the current vertex.
Taking limits for the surplus edges

When we rescale, the distance between a vertex and one of its children vanishes and so, in the limit, surplus “edges” do go to ancestors of the current vertex (i.e. vertices on the path down to the root).
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The marks corresponding to the surplus edges, when rescaled, straightforwardly converge to the required independent uniform points.
Taking limits for the surplus edges

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The marks corresponding to the surplus edges, when rescaled, straightforwardly converge to the required independent uniform points.

Taking care over the details, this completes the proof.
A limiting component can have quite a complicated cycle structure:
A limiting component can have quite a complicated cycle structure:

What more can we say about it?
Cycle structure of a graph

Fix a connected graph $G$. The core $C(G)$ consists of the edges in cycles and those joining the cycles. If $G$ is a tree, $C(G)$ is empty.
Cycle structure: an example

Graph $G$
Cycle structure: an example

Core $C(G)$
The kernel $K(G)$ is the multigraph which gives the “shape of the core”:
Cycle structure of a graph

The kernel $K(G)$ is the multigraph which gives the “shape of the core”: take the vertices of the core of degree 3 or more; contract the paths between them to a single edge.
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By convention, the kernel of a tree or unicyclic component is empty.
Cycle structure: an example

Vertices of degree at least 3 in the core
Cycle structure: an example

Contract paths between them
Cycle structure: an example

Kernel $K(G)$
Cycle structure: an example

Kernel $K(G)$

Note that the kernel has the same surplus as the original graph.
Cycle structure of a real tree with vertex identifications

It still makes sense to talk about the degree of a point in a real tree with vertex identifications.
Cycle structure of a real tree with vertex identifications

It still makes sense to talk about the degree of a point in a real tree with vertex identifications.

It’s not hard to see that the core and kernel also make sense in the real tree context as a path metric space and a discrete multigraph respectively.
Law of the kernel

The kernel is distributed as a 3-regular multigraph sampled from the configuration model, conditioned to be connected.

In other words, take the vertices of the kernel, attach 3 half-edges to each, and take a uniformly random pairing of the half-edges to create full edges. Condition the resulting multigraph to be connected.

Sample a kernel according to the 3-regular configuration model, conditioned to be connected and to have surplus \( k \). Such a kernel always has \( 3k - 3 \) edges.
Alternative construction of a limit component

Sample independent rooted Brownian CRT’s $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_{3k-3}$. 
Alternative construction of a limit component

Sample a uniform point in each.
Alternative construction of a limit component

Randomly rescale so that the mass of $T_i$ becomes $X_i$, where $(X_1, X_2, \ldots, X_{3k-3}) \sim \text{Dir}(\frac{1}{2}, \ldots, \frac{1}{2})$. 
Alternative construction of a limit component

Glue the trees to the kernel.
Alternative construction of a limit component

Glue the trees to the kernel.

This has the same distribution as $(G^k, d^k, \mu^k)$. 
The core

We get core paths of lengths

$$\sqrt{\Gamma} \times \text{Dir}(1, 1, \ldots, 1),$$

where the two factors are independent and

$$\Gamma \sim \text{Gamma}((3k - 2)/2, 1/2).$$
We get core paths of lengths

\[ \sqrt{\Gamma} \times \text{Dir}(1, 1, \ldots, 1), \]

where the two factors are independent and

\[ \Gamma \sim \text{Gamma}((3k - 2)/2, 1/2). \]

(Compare to the random $k$-dimensional distributions in the Brownian CRT, where we had $\sqrt{\Gamma_k}$ with $\Gamma_k \sim \text{Gamma}(k, 1/2)$, $k \geq 1$.)
Line-breaking construction

Starting from the core, it turns out we can give a line-breaking construction for the rest of the limit component.
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Take an inhomogeneous Poisson process of rate $t$ at time $t$, conditioned to have its first point at $\sqrt{\Gamma}$. Write $C_0 = \sqrt{\Gamma}$; subsequent points occur at times $C_1, C_2, \ldots$. 
Starting from the core, it turns out we can give a line-breaking construction for the rest of the limit component.

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Consider the line-segments $[C_0, C_1), [C_1, C_2), \ldots$ and proceed inductively.
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Consider the line-segments $[C_0, C_1), [C_1, C_2), \ldots$ and proceed inductively.

For $i \geq 0$, attach $[C_i, C_{i+1})$ at a random point chosen uniformly over the existing structure.
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For $i \geq 0$, attach $[C_i, C_{i+1})$ at a random point chosen uniformly over the existing structure.

Take the completion of the metric space obtained.
Line-breaking construction
Line-breaking construction
Line-breaking construction
Line-breaking construction
Line-breaking construction
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Line-breaking construction
Line-breaking construction
Line-breaking construction
8. PERSPECTIVES

[Picture by Evilbish (https://commons.wikimedia.org/wiki/File:Mam_Tor.jpg) "Mam Tor", https://creativecommons.org/licenses/by/3.0/legalcode]
(a) Unicellular random maps

Let $S$ be an arbitrary compact surface without boundary. Let $M_n$ be a uniform random map drawn on $S$ with $n$ vertices and a single face. ($M_n$ is unicellular.)
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If $S$ is the sphere then $M_n$ is a uniform random plane tree, and has the Brownian CRT as its scaling limit.

**Note:** this is a very different object to the Brownian surfaces, which are the scaling limits of random maps with diverging numbers of faces!
Unicellular random maps

If $S$ is a torus, we generically get:

This is an embedded version of a graph conditioned to have "theta" kernel (which is one of the two possible kernels with 2 surplus edges). The scaling limit can then be constructed out of three independent randomly rescaled Brownian CRT’s.
**Unicellular random maps**

On any surface, we get as a scaling limit (a copy of $G^k$ for some $k$, conditioned to have one or a mixture of kernels).

$\left(M_n, \frac{1}{\sqrt{n}}d_n, \mu_n\right) \xrightarrow{d} (\mathcal{M}, d, \mu)$.

$\mathcal{M}$ can always be constructed out of randomly rescaled independent Brownian CRT’s.

Genus 2:

![Picture by Igor Kortchemski]
A generalisation of our Voronoi theorem

**Theorem.** (Addario-Berry, Angel, Chapuy, Fusy & G. (2018+))

For any compact surface $S$ without boundary, the continuum random unicellular map $(\mathcal{M}, d, \mu)$ has uniform Voronoi mass-partitions.

Genus 2, $k = 5$:
(b) The critical Erdős-Rényi random graph

Consider the Erdős-Rényi random graph $G(n, p)$. There is a phase transition for the emergence of a giant component at $p = 1/n$. Aldous (1997) gives a description of the component sizes and surpluses in the critical window $p = 1/n + \lambda n^{-4/3}$. Here, the component sizes are on the order of $n^{2/3}$ and the surpluses are finite random variables.

Using the fact that components of the Erdős-Rényi random graph are uniform on their vertex-sets with the number of edges determined by the size of the vertex-set and the surplus, we can obtain a metric-space scaling limit for the whole graph.


The critical Erdős-Rényi random graph

Let $p = 1/n + \lambda n^{-4/3}$ for fixed $\lambda \in \mathbb{R}$. Let $C_1^n, C_2^n, \ldots$ be the components of $G(n, p)$ listed in decreasing order of size, and let $d_1^n, d_2^n, \ldots$ be the graph distances and $\mu_1^n, \mu_2^n, \ldots$ be the counting measures on $C_1^n, C_2^n, \ldots$ respectively.

**Theorem.** (Addario-Berry, Broutin & G. (2012))

As $n \to \infty$,

$$
\left( \left( C_1^n, \frac{d_1^n}{n^{1/3}}, \frac{1}{n^{2/3}} \mu_1^n \right), \left( C_2^n, \frac{d_2^n}{n^{1/3}}, \frac{1}{n^{2/3}} \mu_2^n \right), \ldots \right) \xrightarrow{d} ((C_1, d_1, \mu_1), (C_2, d_2, \mu_2), \ldots)
$$

in an $\ell_4$ version of GHP.

Here, the limit spaces are randomly scaled i.i.d. copies of $(G^k, d^k, \mu^k)$ with a certain random surplus $k$. 
The critical Erdős-Rényi random graph

[Picture by Nicolas Broutin]


(c) Critical random transposition random walk

Generate a random permutation of \{1, 2, \ldots, n\} by composing i.i.d. uniform random transpositions. This gives a Markov chain on \( \mathfrak{S}_n \) called the random transposition random walk (RTRW).
(c) Critical random transposition random walk

Generate a random permutation of \( \{1, 2, \ldots, n\} \) by composing i.i.d. uniform random transpositions. This gives a Markov chain on \( \mathcal{S}_n \) called the random transposition random walk (RTRW).

There is a natural coupling with the Erdős-Rényi random graph process, whereby we include an edge \( \{i, j\} \) in the graph iff we have multiplied by the transposition \((i, j)\). We can use our understanding of the graph process to deduce properties of the RTRW.


Critical random transposition random walk

Each component of the graph corresponds to at least one cycle of the permutation. So, for example, there cannot be giant cycles below the point of the phase transition.

**Question.** What are the lengths of the cycles of the RTRW walk in the critical window?
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**Question.** What are the lengths of the cycles of the RTRW walk in the critical window?

There are many tree components, each of which corresponds to a single cycle of the permutation. The other components all have finite surplus. Given a graph component, notice that all edge-arrival orders are equally likely.
Critical random transposition random walk

Each component of the graph corresponds to at least one cycle of the permutation. So, for example, there cannot be giant cycles below the point of the phase transition.

**Question.** What are the lengths of the cycles of the RTRW walk in the critical window?

There are many tree components, each of which corresponds to a single cycle of the permutation. The other components all have finite surplus. Given a graph component, notice that all edge-arrival orders are equally likely.

So it is sufficient to consider what happens for a uniform random connected graph on \( n \) vertices with surplus \( k \) and a uniform ordering of the edges. [→ Dominic’s talk]
Critical random transposition random walk

Example: \( n = 19 \), surplus 2, single permutation cycle.

Within a single such component, the limiting cycle-lengths can then be understood in terms of \((G^k, d^k, \mu^k)\).
(d) The scaling limit of the minimum spanning tree of the complete graph

Consider the complete graph on $n$ vertices with independent edge-weights which are uniformly distributed on $[0, 1]$. 
The scaling limit of the minimum spanning tree of the complete graph

Find the minimum spanning tree (MST).
The scaling limit of the minimum spanning tree of the complete graph

**Question.** Does the MST of the complete graph on $n$ vertices possess a **scaling limit**?
The scaling limit of the minimum spanning tree of the complete graph

Let $M_n$ be the MST of the complete graph on $n$ vertices, let $d_n$ be its graph distance, and $\mu_n$ its uniform measure.

**Theorem.** (Addario-Berry, Broutin, G. & Miermont (2017))
There exists a random compact measured real tree $(\mathcal{M}, d, \mu)$ such that

$$
\left( M_n, \frac{d_n}{n^{1/3}}, \mu_n \right) \xrightarrow{d} (\mathcal{M}, d, \mu)
$$

as $n \to \infty$, in GHP. $\mathcal{M}$ is binary and has Minkowski dimension 3 almost surely.

The key to understanding this result is a connection between the Erdős-Rényi random graph and Kruskal’s algorithm for constructing the MST.

(e) The stable trees

Recall that the $\alpha$-stable tree, for $\alpha \in (1, 2)$, is the scaling limit of a Galton-Watson tree with critical offspring distribution in the domain of attraction of an $\alpha$-stable law.

There is an analogue of Rémy’s algorithm due to Marchal (2008) and there is also a (more complicated) line-breaking construction.


The stable graphs

[Picture by Delphin Sénizergues]
The stable graphs

The natural graph model whose scaling limit involves the stable trees is the configuration model with i.i.d. power-law degrees. Work in progress...

[G. Conchon-Kerjan & C. Goldschmidt, The stable graph: the metric space scaling limit of a critical random graph with i.i.d. power-law degrees, in preparation]

[C. Goldschmidt, B. Haas and D. Sénizergues, Stable graphs: distributions and line-breaking construction, in preparation.]


Related work:


Thank you!