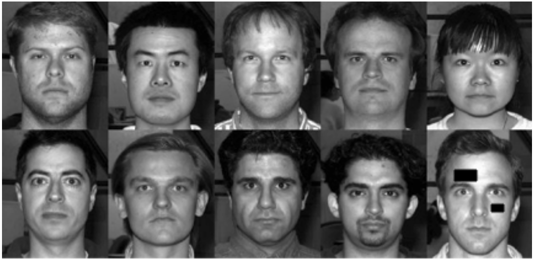
Local Computations with Probabilities on Graphical Structures

Lauritzen and Spiegelhalter, *JRSS-B*, 1988

October 10, 2014

Typical Problems: Pictures

Given some images of faces with missing part. Want to find most likely values of obscured pixels.



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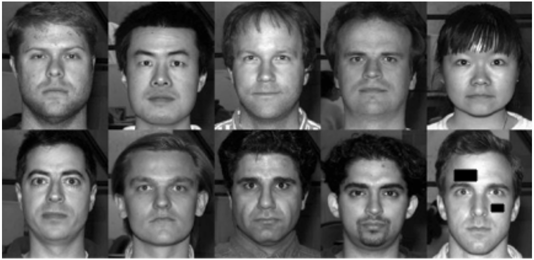

Typical small image has 100×100 pixels; modelling full joint distribution of 10,000 random variables is not realistic.

Typical Problems: Expert Systems

Consider a medical diagnosis tool, with list of symptoms, diseases, patient history:

- patient has a cough;
- patient is a non-smoker;
- chest x-ray has dark patches;
- $\cdots$

Given partial information, want to know probability patient has tuberculosis.

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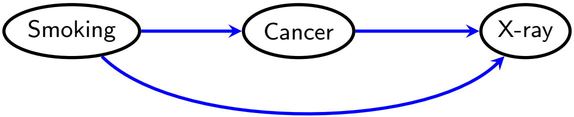
Idea is to assume each variable is dependent only upon some others, and perform computations locally.

Outline

1. Directed Acyclic Graphs
2. Undirected Graphical Models
3. Junction Tree Algorithms
4. Project

A Simple DAG



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In other words,

$$p(\text{X-ray} \mid \text{Smoking}, \text{Cancer}) = p(\text{X-ray} \mid \text{Cancer}).$$

Factorisation

We can always write a joint probability distribution in factors

$$p(x_1, \ldots, x_k) = p(x_1)\, p(x_2 \mid x_1)\, p(x_3 \mid x_1, x_2) \cdots p(x_k \mid x_1, \ldots, x_{k-1}).$$

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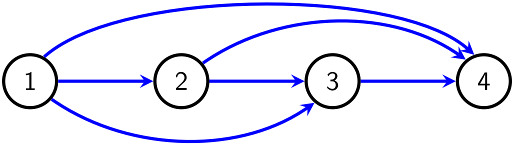
Draw a graph on $1, \ldots, k$ with edges $i \to j$ for each $i < j$.



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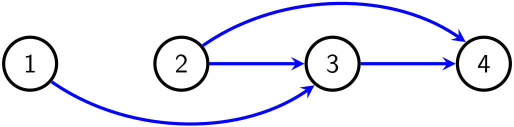

Remove $i \to j$ if x_i does not appear in conditioning set for x_j.

E.g.:

$$p(x_2 \mid x_1) = p(x_2)$$
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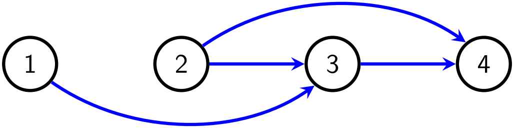

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This is equivalent to $X_2 \perp\!\!\!\perp X_1$ and $X_4 \perp\!\!\!\perp X_1 \mid X_2, X_3$.

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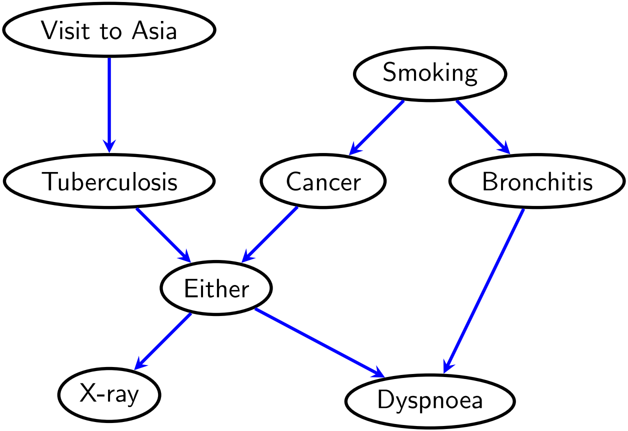
A joint probability density p **factorizes according to** $\mathcal{G}$ if

$$p(x_1, \dots, x_k) = \prod_{i=1}^{k} p(x_i \mid x_{\mathrm{pa}(i)}).$$

Example


Visit to Asia
Smoking
Tuberculosis
Cancer
Bronchitis
Either
X-ray
Dyspnoea

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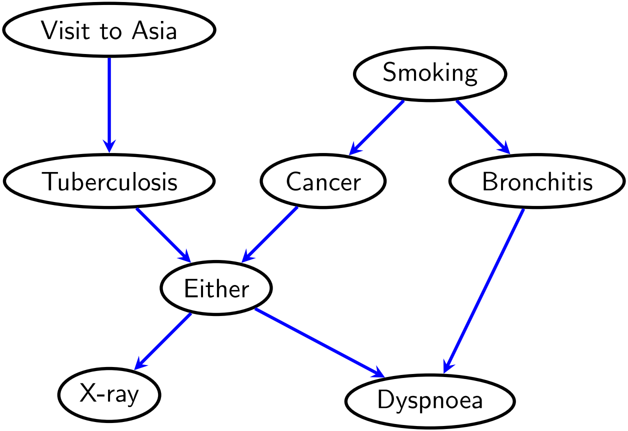

Only need to record conditional probability tables for each node.

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So in general, how do we know which sums to 'push' into the middle?

Outline

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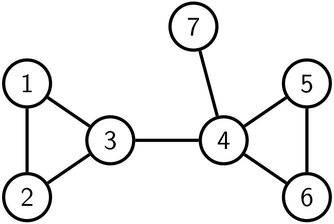

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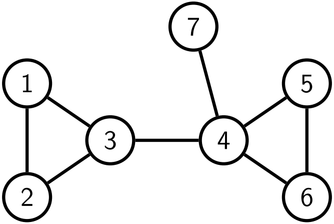

Many ways in which a distribution can 'obey' an undirected graph. Let $\mathcal{C}(\mathcal{G})$ be the **cliques** (maximal fully connected subgraphs).

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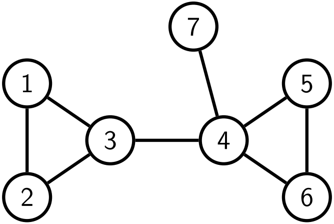

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$P \in \mathcal{P}_f(\mathcal{G})$ **factorizes** according to $\mathcal{G}$ if

$$p(x_1, \ldots, x_k) = \prod_{C \in \mathcal{C}(\mathcal{G})} \psi_C(x_C)$$

for some potential functions $\psi_C \geq 0$.

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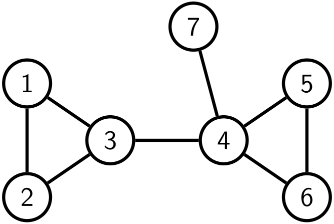

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for some potential functions $\psi_C \geq 0$.

So $p = \psi_{123} \cdot \psi_{34} \cdot \psi_{456} \cdot \psi_{47}$ above.

Decomposable Graphs

Say a graph is **decomposable** if either it is complete, or there exist three disjoint non-empty subsets $A \cup B \cup S = V$, where:

(i) $\mathcal{G}_S$ is complete;

(ii) S separates A from B in $\mathcal{G}$;

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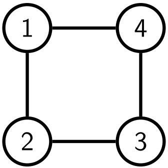

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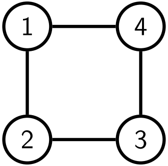

Theorem

An undirected graph $\mathcal{G}$ is decomposable iff it contains no chordless cycles of length ≥ 4.

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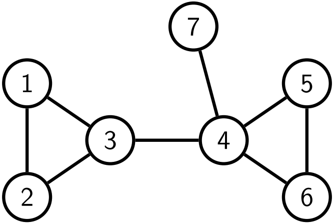
So by induction

$$= \frac{\prod_{C \in \mathcal{C}} p_C(x_C)}{\prod_{S \in \mathcal{S}} p_S(x_S)}$$

where $\mathcal{S}$ is a collection of separators (not necessarily disjoint).

Forming a Junction Tree



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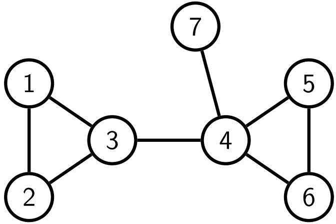

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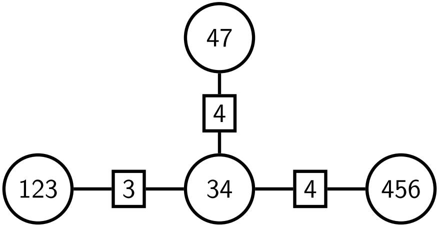

Idea

Let $\mathcal{T}$ be a junction tree. Initially have

$$p(x_V) = \frac{\prod_C \psi_C(x_C)}{\prod_S \psi_S(x_S)}$$

for cliques C and separators S (separators may not be unique).

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Lemma

If the ψ_C and ψ_S functions have consistent margins, then $\psi_C(x_C) = p_C(x_C)$.

If margins are not consistent, then we can make them so!

Message Passing

To 'pass a message' from clique C to clique D (with separator S), set:

- $\psi'_S(x_S) = \sum_{x_{C \setminus S}} \psi_C(x_C)$
- $\psi'_D(x_D) = \psi_D(x_D) \frac{\psi'_S(x_S)}{\psi_S(x_S)}$

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Lemma

After passing a message from C to D, the probability distribution remains unchanged, and

$$\sum_{x_{C \setminus S}} \psi_C(x_C) = \psi'_S(x_S).$$

If, in addition, $\sum_{x_{D \setminus S}} \psi_D(x_D) = \psi_S(x_S)$, then

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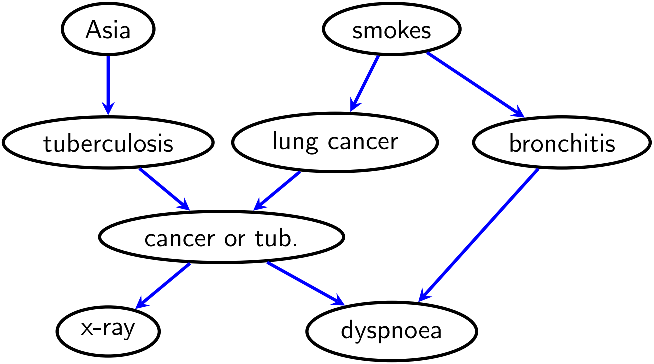
From DAGs to Decomposable Graphs



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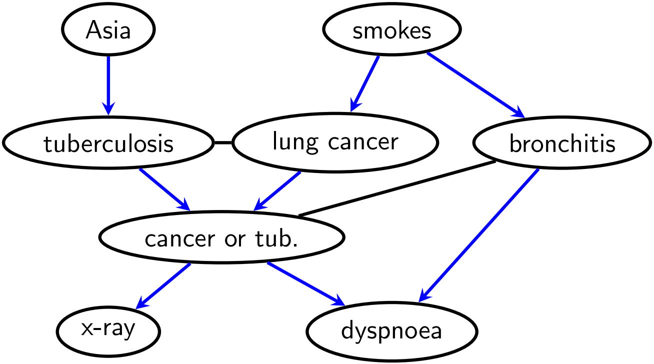

1. 'Marry' parents.

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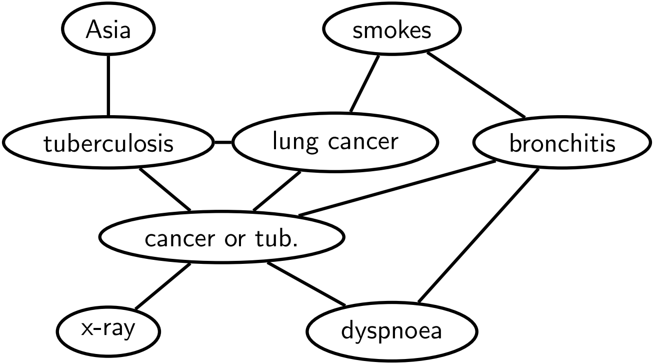

1. ‘Marry’ parents.
2. Drop arrows.

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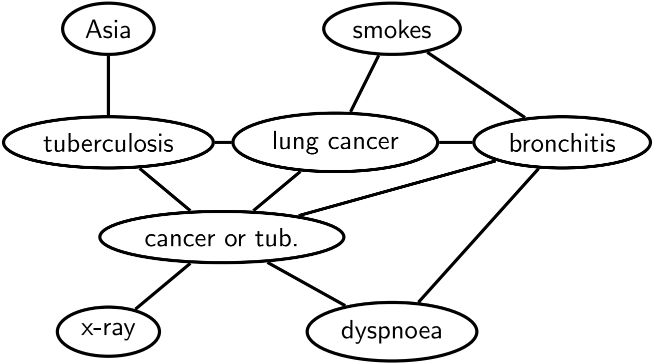

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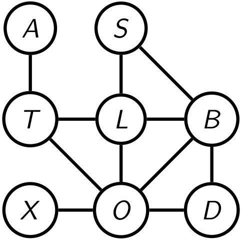
2. Drop arrows.
3. Triangulate (not unique or easy!)

Forming a Junction Tree



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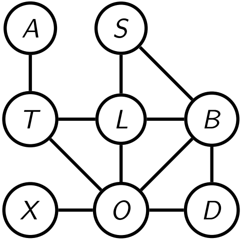

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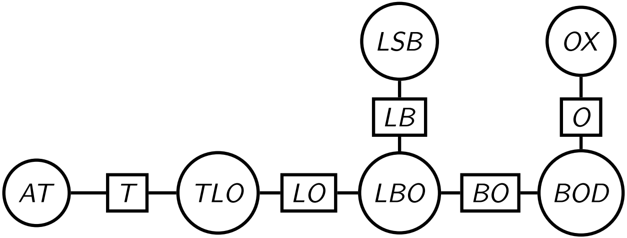

Initialization

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$$p(x_V) = \prod_v p(x_v \mid x_{\mathrm{pa}(v)})$$

and each set $\{v\} \cup \mathrm{pa}(v)$ is contained in a clique.

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So can set $\prod_C \psi_C(x_C) = \prod_v p(x_v \mid x_{\mathrm{pa}(v)})$ and $\psi_S(x_S) = 1$ to get valid ψ representation.

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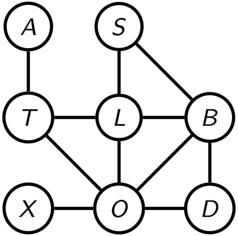
Not locally consistent, but:

Theorem

After collecting and distributing messages in a junction tree, the potentials are locally consistent.

Outline

Conditioning



$$p(a,t,s,l,b,o,d,x) = \psi(a,t) \cdot \psi(t,l,o) \cdot \psi(s,l,b) \cdot \psi(b,o,d) \cdot \psi(x,o)$$

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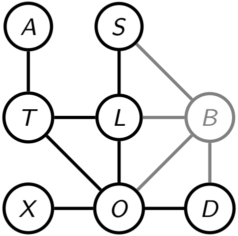

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Graph structure is preserved under conditioning:

$$p(a,t,s,l,o,d,x \mid b) \propto \psi(a,t)\cdot\psi(t,l,o)\cdot\psi^*(s,l)\cdot\psi^*(o,d)\cdot\psi(x,o).$$

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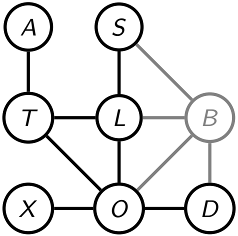

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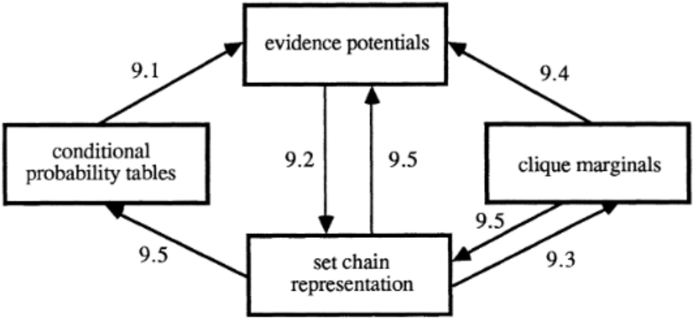
Gives something proportional to conditional distribution: use structure to calculate normalizing constant.

Outline

Project Idea

The idea would be to produce a collection of functions for switching between different representations of a DAG model/Bayesian Network.



Project Idea

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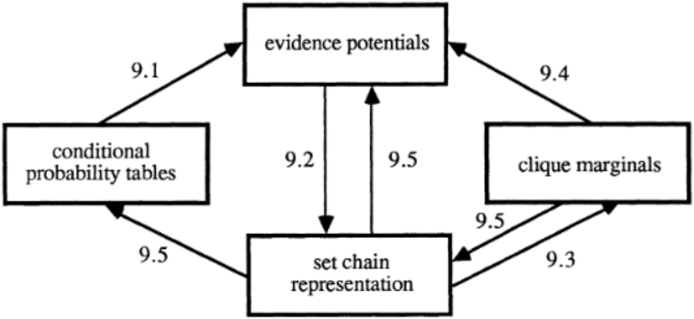

Can also write functions for the introduction of evidence, and efficient calculation of arbitrary conditional and marginal distributions.

[Won't attempt to find good decomposable representations.]