Theorem 1. *Let*

$$\mu(k) = \frac{1}{1+f(k)} \prod_{l=0}^{k-1} \frac{f(l)}{1+f(l)} \qquad \text{for } k \in \mathbb{N} \cup \{0\},$$

which is a sequence of probability weights. Then, almost surely,

$$\lim_{N\to\infty} X_N^{\mathrm{in}} = \mu$$

in total variation norm.

We start by showing that μ is a probability distribution with $\mu Q = 0$, where

$$Q = \begin{pmatrix} -f(0) & f(0) & & & \\ 1 & -(f(1)+1) & f(1) & & \\ 1 & & -(f(2)+1) & f(2) & \\ \vdots & & & \ddots & \ddots \end{pmatrix}$$

Indeed, by induction, we get that

$$1 - \sum_{l=0}^{k} \mu(l) = \prod_{l=0}^{k} \frac{f(l)}{1+f(l)}$$

for any $k \in \mathbb{N} \cup \{0\}$. Since $\sum_{l=0}^{\infty} 1/f(l) \geqslant \sum_{l=0}^{\infty} 1/(l+1) = \infty$ it follows that μ is a probability measure on the set $\mathbb{N} \cup \{0\}$. Moreover, it is straightforward to verify that

$$f(0)\mu(0) = 1 - \mu(0) = \sum_{l=1}^{\infty} \mu(l)$$
$$f(k-1)\,\mu(k-1) = (1+f(k))\,\mu(k),$$

and hence $\mu Q = 0$.

We define an inhomogeneous Markov process such that at every time N the state is the indegree of a uniformly chosen vertex from $\mathcal{G}_N$. In each time step, starting with state k we move to the newly added vertex with probability $1/(N+1)$, hence adapting state 0. Otherwise the indegree is increased by one with unconditional probability $f(k)/(N+1)$, or stays the same. Note that the transition matrix of this Markov chain at the time step $N \mapsto N+1$ is given by

$$P^{(N)} := I + \frac{1}{N+1}\, Q,$$

and that

$$\mu_N(k) := \mathbb{E}[X_N^{\mathrm{in}}(k)] = \mathbb{P}(Y_N^{0,1} = k),$$

where $(Y_N^{l,m})_{N \geqslant m}$ is the chain started at time $m \in \mathbb{N}$ in state $l \leqslant m-1$.

Next, fix $k \in \mathbb{N} \cup \{0\}$, let $m > k$ arbitrary, and denote by ν the restriction of μ to the set $\{m, m+1, \dots\}$. Since μ is invariant under each $P^{(N)}$ we get

$$\mu(k) = \mu P^{(m)} \cdots P^{(N)}(k) = \sum_{l=0}^{m-1} \mu(l)\, \mathbb{P}(Y_N^{l,m} = k) + \nu P^{(m)} \cdots P^{(N)}(k).$$

Note that in the Nth step of the Markov chain, the probability to jump to state zero is $1/(N+1)$ for all states in $\{1,\dots,N-1\}$ and bigger than $1/(N+1)$ for the state 0. Thus one can couple the Markov chains $(Y_N^{l,m})$ and $(Y_N^{0,1})$ in such a way that

$$\mathbb{P}(Y_{N+1}^{l,m} = Y_{N+1}^{0,1} = 0 \,|\, Y_N^{l,m} \neq Y_N^{0,1}) = \tfrac{1}{N+1},$$

and that once the processes meet at one site they stay together. Then

$$\mathbb{P}(Y_N^{l,m} = Y_N^{0,1}) \geqslant 1 - \prod_{i=m}^{N-1} \frac{i}{i+1} \longrightarrow 1.$$

Since, looking at the matrix products, we see $0 \leqslant \nu P^{(m)} \cdots P^{(N)}(k) \leqslant \mu([m,\infty))$, we get

$$\limsup_{N\to\infty} \Big| \mu(k) - \mathbb{P}(Y_N^{0,1} = k) \sum_{l=0}^{m-1} \mu(l) \Big| \leqslant \mu([m,\infty)).$$

As $m \to \infty$ we thus get that

$$\lim_{N\to\infty} \mu_N(k) = \mu(k).$$

In the next step we show that the sequence of the empirical indegree distributions $(X_N^{\mathrm{in}})_{N\in\mathbb{N}}$ converges almost surely to μ. Note that $NX_N^{\mathrm{in}}(k)$ is a sum of n independent Bernoulli random variables. Thus Chernoff's inequality implies that for any $t > 0$

$$\mathbb{P}\big(X_N^{\mathrm{in}}(k) \leqslant \mathbb{E}[X_N^{\mathrm{in}}(k)] - t\big) \leqslant e^{-Nt^2/(2\mathbb{E}[X_N^{\mathrm{in}}(k)])} = e^{-Nt^2/(2\mu_N(k))}.$$

Since

$$\sum_{N=1}^{\infty} e^{-Nt^2/(2\mu_N(k))} < \infty,$$

the Borel-Cantelli lemma implies that almost surely $\liminf_{N\to\infty} X_N^{\mathrm{in}}(k) \geqslant \mu(k)$ for all $k \in \mathbb{N}\cup\{0\}$. If $A \subset \mathbb{N}\cup\{0\}$ we thus have by Fatou's lemma

$$\liminf_{N\to\infty} \sum_{k\in A} X_N^{\mathrm{in}}(k) \geqslant \sum_{k\in A} \liminf_{N\to\infty} X_N^{\mathrm{in}}(k) = \mu(A).$$

Noting that μ is a probability measure and passing to the complementary events, we also get

$$\limsup_{N\to\infty} \sum_{k\in A} X_N^{\mathrm{in}}(k) \leqslant \mu(A).$$

Hence, given $\epsilon > 0$, we can pick $M \in \mathbb{N}$ so large that $\mu((M,\infty)) < \epsilon$, and obtain for the total variation norm

$$\begin{aligned} &\limsup_{N\uparrow\infty} \tfrac{1}{2} \sum_{k=0}^{\infty} \big| X_N^{\mathrm{in}}(k) - \mu(k) \big| \\ &\qquad \leqslant \limsup_{N\uparrow\infty} \tfrac{1}{2} \sum_{k=0}^{M} \big| X_N^{\mathrm{in}}(k) - \mu(k) \big| + \tfrac{1}{2} \lim_{N\uparrow\infty} \sum_{k=M+1}^{\infty} X_N^{\mathrm{in}}(k) + \tfrac{1}{2}\mu((M,\infty)) \leqslant \epsilon. \end{aligned}$$

This establishes almost sure convergence of (X_N^{in}) to μ in the total variation norm.