Theorem 1. Let
\[ \mu(k) = \frac{1}{1 + f(k)} \prod_{l=0}^{k-1} \frac{f(l)}{1 + f(l)} \quad \text{for } k \in \mathbb{N} \cup \{0\}, \]
which is a sequence of probability weights. Then, almost surely,
\[ \lim_{N \to \infty} X_N^{\text{in}} = \mu \]
in total variation norm.

We start by showing that \( \mu \) is a probability distribution with \( \mu Q = 0 \), where
\[ Q = \begin{pmatrix} -f(0) & f(0) & 1 \\ 1 & -(f(1) + 1) & f(1) \\ 1 & -(f(2) + 1) & f(2) \end{pmatrix} \]
Indeed, by induction, we get that
\[ 1 - \sum_{l=0}^{k} \mu(l) = \prod_{l=0}^{k} \frac{f(l)}{1 + f(l)} \]
for any \( k \in \mathbb{N} \cup \{0\} \). Since \( \sum_{l=0}^{\infty} 1/f(l) \geq \sum_{l=0}^{\infty} 1/(l+1) = \infty \) it follows that \( \mu \) is a probability measure on the set \( \mathbb{N} \cup \{0\} \). Moreover, it is straightforward to verify that
\[ f(0)\mu(0) = 1 - \mu(0) = \sum_{l=1}^{\infty} \mu(l) \]
\[ f(k-1)\mu(k-1) = (1 + f(k)) \mu(k), \]
and hence \( \mu Q = 0 \).

We define an inhomogeneous Markov process such that at every time \( N \) the state is the indegree of a uniformly chosen vertex from \( G_N \). In each time step, starting with state \( k \) we move to the newly added vertex with probability \( 1/(N+1) \), hence adapting state 0. Otherwise the indegree is increased by one with unconditional probability \( f(k)/(N+1) \), or stays the same. Note that the transition matrix of this Markov chain at the time step \( N \mapsto N+1 \) is given by
\[ P^{(N)} := I + \frac{1}{N+1} Q, \]
and that
\[ \mu_N(k) := \mathbb{E}[X_N^{\text{in}}(k)] = \mathbb{P}(Y_N^{0,1} = k), \]
where \( (Y_N^{l,m})_{N \geq m} \) is the chain started at time \( m \in \mathbb{N} \) in state \( l \leq m - 1 \).

Next, fix \( k \in \mathbb{N} \cup \{0\} \), let \( m > k \) arbitrary, and denote by \( \nu \) the restriction of \( \mu \) to the set \( \{m, m+1, \ldots\} \). Since \( \mu \) is invariant under each \( P^{(N)} \) we get
\[ \mu(k) = \mu P^{(m)} \cdots P^{(N)}(k) = \sum_{l=0}^{m-1} \mu(l) \mathbb{P}(Y_N^{l,m} = k) + \nu P^{(m)} \cdots P^{(N)}(k). \]
Note that in the Nth step of the Markov chain, the probability to jump to state zero is $1/(N+1)$ for all states in $\{1, \ldots, N-1\}$ and bigger than $1/(N+1)$ for the state 0. Thus one can couple the Markov chains $(Y_{N}^{l,m})$ and $(Y_{N}^{0,1})$ in such a way that

$$\mathbb{P}(Y_{N}^{l,m} = Y_{N+1}^{0,1} = 0 \mid Y_{N}^{l,m} \neq Y_{N}^{0,1}) = \frac{1}{N+1},$$

and that once the processes meet at one site they stay together. Then

$$\mathbb{P}(Y_{N}^{l,m} = Y_{N}^{0,1}) \geq 1 - \prod_{i=m}^{N-1} \frac{i}{i+1} \rightarrow 1.$$

Since, looking at the matrix products, we see $0 \leq \nu P^{(m)} \cdots P^{(N)}(k) \leq \mu([m, \infty))$, we get

$$\limsup_{N \to \infty} \left| \mu(k) - \mathbb{P}(Y_{N}^{0,1} = k) \sum_{l=0}^{m-1} \mu(l) \right| \leq \mu([m, \infty)).$$

As $m \to \infty$ we thus get that

$$\lim_{N \to \infty} \mu_{N}(k) = \mu(k).$$

In the next step we show that the sequence of the empirical indegree distributions $(X_{N}^{\text{in}})_{N \in \mathbb{N}}$ converges almost surely to $\mu$. Note that $NX_{N}^{\text{in}}(k)$ is a sum of $n$ independent Bernoulli random variables. Thus Chernoff’s inequality implies that for any $t > 0$

$$\mathbb{P}(X_{N}^{\text{in}}(k) \leq \mathbb{E}[X_{N}^{\text{in}}(k)] - t) \leq e^{-Nt^{2}/(2\mathbb{E}[X_{N}^{\text{in}}(k)])} = e^{-Nt^{2}/(2\mu_{N}(k))}.$$

Since

$$\sum_{N=1}^{\infty} e^{-Nt^{2}/(2\mu_{N}(k))} < \infty,$$

the Borel-Cantelli lemma implies that almost surely $\liminf_{N \to \infty} X_{N}^{\text{in}}(k) \geq \mu(k)$ for all $k \in \mathbb{N} \cup \{0\}$. If $A \subseteq \mathbb{N} \cup \{0\}$ we thus have by Fatou’s lemma

$$\liminf_{N \to \infty} \sum_{k \in A} X_{N}^{\text{in}}(k) \geq \sum_{k \in A} \liminf_{N \to \infty} X_{N}^{\text{in}}(k) = \mu(A).$$

Noting that $\mu$ is a probability measure and passing to the complementary events, we also get

$$\limsup_{N \to \infty} \sum_{k \in A} X_{N}^{\text{in}}(k) \leq \mu(A).$$

Hence, given $\epsilon > 0$, we can pick $M \in \mathbb{N}$ so large that $\mu((M, \infty)) < \epsilon$, and obtain for the total variation norm

$$\limsup_{N \to \infty} \frac{1}{2} \sum_{k=0}^{\infty} \left| X_{N}^{\text{in}}(k) - \mu(k) \right|$$

$$\leq \limsup_{N \to \infty} \frac{1}{2} \sum_{k=0}^{M} \left| X_{N}^{\text{in}}(k) - \mu(k) \right| + \frac{1}{2} \lim_{N \to \infty} \sum_{k=M+1}^{\infty} X_{N}^{\text{in}}(k) + \frac{1}{2} \mu((M, \infty)) \leq \epsilon.$$