A specimen practical write-up

The following is given as an example of what a practical write-up might look like. The practical example is NOT part of this year’s assessment. You will, in fact, meet this data set later in the course.

The marker is given the following advice.

Do not insist on close adherence to the layout suggested in the mark scheme. Any sensible layout will suffice and marks should be given for the points made and conclusions drawn, irrespective of where they actually appear in the report. The practicals should be marked generously because the final mark is averaged over 4 assignments.

A typical Practical Session sheet is given first, followed by a specimen write-up.
The data contained in the file coins.txt are the silver content (\% Ag) of a number of Byzantine coins discovered in Cyprus. Nine of the coins came from the first coinage of the reign of King Manuel I, Comnenus (1143 – 1180); there were seven from the second coinage minted several years later and four from the third coinage, minted later still; another seven were from a fourth coinage. Historians are interested in whether there are significant differences in silver content in the coinages. The data are taken from Hendy, M.F. and Charles, J.A. (1970): The production techniques, silver content and circulation history of the twelfth-century Byzantine Trachy. *Archaeometry, 12*, 13 – 21.

<table>
<thead>
<tr>
<th>First</th>
<th>Second</th>
<th>Third</th>
<th>Fourth</th>
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<td>6.9</td>
<td>4.9</td>
<td>5.3</td>
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<td>6.2</td>
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</tbody>
</table>

1. Set up a suitable data array for comparing silver content using a linear regression technique.

2. Use suitable boxplots to obtain an initial comparison of the coinages. Can you make an informal preliminary statement about the comparison? Does log transformation appear to have any merit?

3. Fit a regression model to either the untransformed or the log transformed (whichever you in your judgement prefer) data. Test the assumptions of your model.

4. If you deemed your model to be satisfactory, use it to draw conclusions about comparisons of the silver content in the different mintings. If you deemed it to be unsatisfactory, look for a better transformation. Is there any reason to try a Box-Cox transformation?

5. Whatever your final model you should carry out thorough diagnostic checks of its suitability. What are your conclusions?

Produce a written report on the silver content in the different coinages. Your report should include any graphical displays (e.g. histograms, boxplots, probability plots, scatterplots) which support both your methodology and your conclusions.
Scientific summary

The main objective is to determine whether there were significant differences in the silver content of coins belonging to the different mintings which took place during the reign of Manuel I. There is an important second objective, which is to decide whether differences, if they did indeed exist, were systematic and show an increasing tendency to devalue the coinage.

It was found that differences exist which are highly significant. However, the differences were not systematic and the relative silver content between the mintings can be summarised in the form

Third < Fourth < First < Second.
1. Introduction

The problem is to determine whether there are differences in silver content of coins in the different coinages and to decide whether or not any differences there might be are systematic.

2. The data

The data are measurements of silver content over four coinages: the mintings are in chronological order. Since we are comparing four groups we can use boxplots to take an initial look at the data.

![Boxplots of silver content](image.png)

**Figure 1:** Boxplots of silver content

The data show that the homogeneity of variance assumption involved in using regression to compare these samples is going to be very dubious. Since the variance seems to increase systematically with increasing silver content, it is worth trying a log transformation.
Figure 2: Boxplots of log(silver content)

Figure 2 suggests that, from the point of view of the analysis we have in mind, it is advisable to use log-transformed data.

3. Statistical methodology

We shall re-arrange the data into a single vector (call it Silver) with the groups being identified by indicators and fit a suitable function of Silver, say $f(Silver)$. This involves creating the vectors I, II, III, IV which comprise zeros and ones in the corresponding entries to identify the groups: then we can use a regression model. Analysis of variance can be used to test for equality of the four groups. If the Fourth Coinage is used as a reference group, then fitting a regression model of the form

$$f(Silver) = \alpha + \beta_1 I + \beta_2 II + \beta_3 III$$

will produce contrasts with the Fourth Coinage directly. Other contrasts can be obtained by using

$$\frac{\hat{\beta}_i - \hat{\beta}_j}{\text{Est.var} \left( \hat{\beta}_i - \hat{\beta}_j \right)} \sim t(n-r)$$

where, for this data set, $n = 27$ and $r = 4$.

$$V \left( \hat{\beta}_i - \hat{\beta}_j \right) = V \left( \hat{\beta}_i \right) + V \left( \hat{\beta}_j \right) - 2C \left( \hat{\beta}_i, \hat{\beta}_j \right) = \sigma^2 \left[ (X^T X)^{-1} \right]_{ii} + (X^T X)^{-1} {jj} - 2 (X^T X)^{-1} {ij}$$

where $X$ is the design matrix. The estimated variance is obtained by using the estimate $RSS/(n-r)$ for $\sigma^2$.

Diagnostics following the fitting of $log(Silver)$ indicated that the fit could be improved by trying another function. Accordingly a Box-Cox transformation of the form

$$f(Silver) = \frac{Silver^\lambda - 1}{\lambda}$$
was tried with maximum likelihood being used to estimate \( \lambda \). Writing \( y_i \) for the \( i \)th entry of \( \text{Silver} \), the log-likelihood is

\[
\ell(\theta, \sigma^2, \lambda; y) = \text{const.} - \frac{n}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \|f(y) - X\theta\|^2 + \sum_i \log y_i^{\lambda - 1}.
\]

The m.l.e. of \( \sigma^2 \) is \( \hat{\sigma}^2 = \frac{\text{RSS}}{n} \); note that, because \( f(y) \) is a function of \( \lambda \), the RSS is a function of \( \lambda \), so

\[
\ell(\hat{\theta}, \hat{\sigma}^2, \lambda; y) = \text{const.} - \frac{n}{2} \log \frac{\text{RSS}(\lambda)}{n} + (\lambda - 1) \sum_i \log y_i \\
= \text{const.} - \frac{n}{2} \log \text{RSS}(\lambda) + n (\lambda - 1) \log \bar{y} \\
= \text{const.} - \frac{n}{2} \log \left( \frac{\text{RSS}(\lambda)}{(\bar{y})^{2(\lambda - 1)}} \right),
\]

where \( \bar{y} \) represents the geometric mean of the response data, i.e.

\[
\bar{y} = \sqrt[n]{\prod y_i}.
\]

An approximate numerical solution for \( \lambda \) was obtained.

4. Analysis of the data

A preliminary analysis began by fitting the model suggested by the boxplots

\[
\log(\text{Silver}) = \alpha + \beta_1 I + \beta_2 II + \beta_3 III,
\]

and looked at a plot of Residuals against Fitted values

![Figure 3: Plot of Residuals against Fitted values](image)

and a normal QQ plot of the Residuals.
Neither of these is satisfactory. Figure 3 clearly shows that the variance increases with fitted value and there is curvature in the QQ plot.

Running the Box-Cox function from R on the untransformed model gave the profile likelihood shown in Figure 5.

From this graph $\lambda$ can be estimated as being approximately $\hat{\lambda} = -1.1$. Silver was there-
fore transformed using the transformation

\[ f(y) = \frac{1 - y^{-1.1}}{1.1}. \]

The ANOVA table for this model is given below.

<table>
<thead>
<tr>
<th></th>
<th>d.f.</th>
<th>S.S.</th>
<th>M.S.</th>
<th>F-value</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression on I, II, III</td>
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<td>0.004963</td>
<td>34.23</td>
<td>1.186960e-08</td>
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<tr>
<td>Residual</td>
<td>23</td>
<td>0.00334</td>
<td>0.000145</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>26</td>
<td>0.01823</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

The evidence for a difference in the silver content of the coinages is overwhelming: given the boxplots this is hardly surprising. Regression produces the following output.

Coefficients:

|             | Estimate | Std. Error | t value | Pr(>|t|) |
|-------------|----------|------------|---------|---------|
| (Intercept) | 0.772265 | 0.004556   | 169.508 | < 2e-16 *** |
| I           | 0.024712 | 0.006075   | 4.068   | 0.000475 *** |
| II          | 0.045769 | 0.006443   | 7.104   | 3.09e-07 *** |
| III         | -0.023453| 0.007555   | -3.104  | 0.004997 **  |

Residual standard error: 0.01205 on 23 degrees of freedom

Multiple R-Squared: 0.8167

R-squared is 0.8167, which is satisfactory. All of the coefficients are significantly different from zero, which means that the silver content of each of the first 3 coinages is significantly different from that of the Fourth coinage. The p-values of the contrasts of coinages I and II with III are both zero to 4 decimal places and the p-value of the contrast between I and II is 0.0021. III has the lowest silver content, followed by IV, I and II in that order.

Figure 6 shows a plot of Studentised residuals against fitted values.

![Figure 6: Plot of Studentised residuals against fitted values](image)
We can see that the variance has been stabilised. A plot of Cook’s distance against index does not reveal any obvious problems.

![Figure 7: Cook’s distances](image)

One Cook’s distance is a little larger than some of the others, but it is not orders of magnitude different and does not give cause for concern: it is no bigger than one might reasonably expect.

The normal QQ plot is very reassuring.

![Figure 8: Normal QQ plot for Studentised residuals](image)

We can conclude from the diagnostic plots in Figures 6, 7 and 8 that the model is satisfactory.
5. Conclusion

The silver content is significantly different for the four mintings. The conclusion may be expressed succinctly in the form

Third < Fourth < First < Second.

It is clear that, whilst differences undoubtedly exist, there was no systematic decrease in silver content over the period of Manuel I's reign.