Advanced Simulation

Problem Sheet 1, with answers

Exercise 1

1. Let $Y \sim \mathcal{E}xp(\lambda)$ and let $a > 0$. We consider the variable after restricting its support to be $[a, +\infty)$. That is, let $X = Y|Y \geq a$, i.e. X has the law of Y conditionally on being in $[a, +\infty)$. Calculate $F_X(x)$, the cumulative distribution function of X, and $F_X^{-1}(u)$, the quantile function of X. Describe an algorithm to simulate X from $U \sim \mathcal{U}_{[0,1]}$.

Answer. $F_X(x)$ is zero for $x < a$. For $x \geq a$,

$$\begin{aligned} F_X(x) &= \mathbb{P}(X \leq x) \\ &= \mathbb{P}(Y \leq x | Y \geq a) \\ &= \mathbb{P}(a \leq Y \leq x)/\mathbb{P}(Y \geq a) \\ &= \frac{\int_a^x \lambda e^{-\lambda y} dy}{\int_a^\infty \lambda e^{-\lambda y} dy} \\ &= \frac{e^{-\lambda a} - e^{-\lambda x}}{e^{-\lambda a}} \\ &= 1 - e^{-\lambda(x-a)} \end{aligned}$$

To simulate X, set $X = F_X^{-1}(U)$, so solve $U = F_X(X)$ for X. We get

$$X = a - \lambda^{-1} \log(1 - U).$$

Note that if U is uniformly distributed in $[0, 1]$, then $1 - U$ as well. Hence to simulate from X, one can

- Simulate $U \sim \mathcal{U}_{[0,1]}$,
- Return $X = a - \lambda^{-1} \log(U)$.

Note that X is just a standard exponential variable $\mathcal{E}xp(\lambda)$ shifted by a. You may have simply written this down, since it is the memoryless property of the exponential distribution.

2. Let a and b be given, with $a < b$. Show that we can simulate $X = Y|a \leq Y \leq b$ from $U \sim \mathcal{U}_{[0,1]}$ using

$$X = F_Y^{-1}(F_Y(a)(1 - U) + F_Y(b)U),$$

i.e. show that if X is given by the formula above, then $\Pr(X \leq x) = \Pr(Y \leq x | a \leq Y \leq b)$. Apply the formula to simulate an exponential random variable conditioned to be greater than a, as in the previous question.

Answer. $F_X(x) = \Pr(X \le x)$ is zero unless $a \le x \le b$. In that case,

$$\begin{aligned}
\mathbb{P}(X \le x) &= \mathbb{P}(F_Y^{-1}(F_Y(a)(1-U) + F_Y(b)U) \le x) \\
&= \mathbb{P}(F_Y(a)(1-U) + F_Y(b)U \le F_Y(x)) \\
&= \mathbb{P}\left(U \le \frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)}\right) \\
&= \frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)}.
\end{aligned}$$

For the exponential conditioned to exceed a, take $F_Y(y) = 1 - \exp(-\lambda y)$, and $b = \infty$ so $F_Y(b) = 1$ to get

$$\begin{aligned}
F_Y(a)(1-U) + F_Y(b)U &= (1 - \exp(-\lambda a))(1-U) + U \\
F_Y^{-1}(v) &= -\lambda^{-1}\log(1-v) \\
X &= -\lambda^{-1}\log(1 - (1-\exp(-\lambda a))(1-U) + U) \\
&= -\lambda^{-1}\log(\exp(-\lambda a)(1-U)) \\
&= a - \lambda^{-1}\log(1-U)
\end{aligned}$$

as before.

3. Here is a simple algorithm to simulate $X = Y|Y > a$ for $Y \sim \mathcal{E}xp(\lambda)$:

   (a) Let $Y \sim \mathcal{E}xp(\lambda)$. Simulate $Y = y$.

   (b) If $Y > a$ then stop and return $X = y$, and otherwise, start again at step (a).

   Show that this is just a rejection algorithm, by writing the proposal and target densities π and q, as well as the bound $M = \max_x \pi(x)/q(x)$. Calculate the expected number of trials to the first acceptance. Why is inversion to be preferred for $a \gg 1/\lambda$?

Answer. The proposal and target densities are respectively

$$\begin{aligned}
q(y) &= \lambda\exp(-\lambda y), y \ge 0, \\
\pi(x) &= \lambda\exp(-\lambda(x-a))\mathbb{1}_{x \ge a}.
\end{aligned}$$

The bound $M = \max_x \pi(x)/q(x)$ is $M = \exp(\lambda a)$ and so the acceptance probability of the rejection algorithm is

$$\pi(y)/Mq(y) = \begin{cases} 1 & \text{if} \quad y \ge a, \\ 0 & \text{if} \quad y < a. \end{cases}$$

Hence we always accept if $y \ge a$, and reject otherwise. This corresponds to the algorithm above. The number of trials N is geometrically distributed with success probability p, where p is equal to the probability of drawing $Y \ge a$, which is $p = \exp(-\lambda a)$. The expected number is $\mathbb{E}(N) = 1/p = \exp(\lambda a)$. If $a\lambda \gg 1$, then $\mathbb{E}(N)$ is large. Inversion gives a sample with a single function evaluation.

Exercise 2

1. Let X_1, X_2 be two independent random variables with $X_1 \sim \mathcal{G}amma\,(a,1)$ and $X_2 \sim \mathcal{G}amma\,(b,1)$. Show that $R = X_1/(X_1 + X_2)$ and $S = X_1 + X_2$ are independent and that $R \sim \mathcal{B}eta\,(a,b)$, $S \sim \mathcal{G}amma\,(a+b,1)$. Recall that Gamma and Beta densities are

$$f_\Gamma(x;\alpha,\beta) = \frac{\beta^\alpha x^{\alpha-1}e^{-\beta x}}{\Gamma(\alpha)}, \; x \in (0,\infty)$$

and

$$f_B(x;a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, \; x \in (0,1).$$

Answer. We have

$$f_{X_1,X_2}(x_1,x_2) = \frac{x_1^{a-1}e^{-x_1}}{\Gamma(a)} \frac{x_2^{b-1}e^{-x_2}}{\Gamma(b)}.$$

Now the mapping $(r,s) = T(x_1,x_2) = (x_1/(x_1+x_2),(x_1+x_2))$ is bijective $T:(0,\infty)^2 \to (0,1)\times(0,\infty)$ with inverse transformation $(x_1,x_2) = T^{-1}(r,s) = (sr, s(1-r))$ that has Jacobian

$$J = \begin{pmatrix} s & r \\ -s & 1-r \end{pmatrix} \quad \text{and hence} \quad |\det J| = s.$$

Therefore,

$$\begin{aligned} f_{R,S}(r,s) &= f_{X_1,X_2}\left(T^{-1}(r,s)\right)|\det J| \\ &= \frac{(sr)^{a-1}e^{-sr}}{\Gamma(a)} \frac{(s(1-r))^{b-1}e^{-s(1-r)}}{\Gamma(b)} s \\ &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} r^{a-1}(1-r)^{b-1} \frac{s^{a+b-1}e^{-s}}{\Gamma(a+b)} \end{aligned}$$

as required.

2. Let $U \sim \mathcal{U}_{[0,1]}$ and $a > 0$. Show that $X = U^{1/a} \sim \mathcal{B}eta(a,1)$.

Answer. We have $\mathbb{P}(X \le x) = \mathbb{P}\left(U^{1/a} \le x\right) = \mathbb{P}(U \le x^a) = x^a$ for $x \in (0,1)$ and so $f_X(x) = ax^{a-1}$, that is $X \sim \mathcal{B}eta(a,1)$.

3. Let $U \sim \mathcal{U}_{[0,1]}$ and $V \sim \mathcal{U}_{[0,1]}$ be independent and $a \in (0,1)$. For $Y = U^{1/a}$ and $Z = V^{1/(1-a)}$, calculate

$$\mathbb{P}\left(\frac{Y}{Y+Z} \le t, Y+Z \le 1\right)$$

for any $t \in (0,1)$ and deduce that the conditional distribution of $W = Y/(Y+Z)$ given $Y+Z \le 1$ is $\mathcal{B}eta(a,1-a)$. *(Hint: writing both inequalities as constraints on Z could ease the calculation).*

Answer. We have

$$\begin{aligned} \mathbb{P}\left(\frac{Y}{Y+Z} \le t, Y+Z \le 1\right) &= \mathbb{P}\left(\frac{Y(1-t)}{t} \le Z \le 1-Y\right) \\ &= \int_0^t \int_{y(1-t)/t}^{1-y} ay^{a-1}(1-a)z^{-a}dzdy \\ &= \int_0^t ay^{a-1}\left\{(1-y)^{1-a} - y^{1-a}(1-t)^{1-a}t^{a-1}\right\}dy \\ &= a\int_0^t y^{a-1}(1-y)^{1-a}dy - a(1-t)^{1-a}t^a \end{aligned}$$

Note that the above integral is from 0 to t because Y has to be less than t in order for $Y(1-t)/t$

to be less than $1-Y$. By differentiating with respect to t, we obtain

$$\begin{aligned} f_{W|Y+Z\leq 1}(t) &= \frac{at^{a-1}(1-t)^{1-a} + a(1-a)(1-t)^{-a}t^{a} - a^{2}(1-t)^{1-a}t^{a-1}}{\mathbb{P}(Y+Z\leq 1)} \\ &= \frac{at^{a-1}(1-t)^{-a}}{\mathbb{P}(Y+Z\leq 1)}\{(1-t)+(1-a)t-a(1-t)\} \\ &\propto t^{a-1}(1-t)^{-a}. \end{aligned}$$

4. In the setting of (3), show that the conditional distribution of TW given $Y+Z\leq 1$, for an independent $T\sim\mathcal{E}xp(1)=\mathcal{G}amma(1,1)$ random variable, is $\mathcal{G}amma(a,1)$.

Answer: Given $Y+Z\leq 1$, W is $\mathcal{B}eta(a,1-a)$ and $T\sim\mathcal{G}amma(1,1)$. Now we can use (1) to show directly that, given $Y+Z\leq 1$, WT is $\mathcal{G}amma(a,1)$ (Here W plays the role of R, T the role of S and $b=1-a$.)

5. Let $a\in(0,1)$.

   (a) Simulate two independent $U,V\sim\mathcal{U}_{[0,1]}$.

   (b) Set $Y=U^{1/a}$ and $Z=V^{1/(1-a)}$.

   (c) If $(Y+Z)\leq 1$ go to (d), else go to (a).

   (d) Simulate an independent $A\sim\mathcal{U}_{[0,1]}$ and set $T=-\log(A)$.

   (e) Return $TY/(Y+Z)$.

   What is this procedure doing? Explain its relevance for simulations.

Answer: This procedure simply returns $\mathcal{G}amma(a,1)$ random variables. It gives an efficient way to simulate Gamma random variables from uniform variables which does not require inverting the cumulative distribution function of the Gamma distribution.

6. Based on (1), explain how you can generate a $\mathcal{B}eta(a,b)$ random variable from a sequence of $\mathcal{U}_{[0,1]}$ random variables, for any $a>0$ and $b>0$. *(Hint: consider $a\in(0,1)$ first and use the additivity of Gamma variables to generate $\mathcal{G}amma(a,1)$ variables, from which the Beta variables can be constructed).*

Answer: We can obtain a $\mathcal{B}eta(a,b)$ random variables using $X_1/(X_1+X_2)$ for independent $X_1\sim\mathcal{G}amma(a,1)$, $X_2\sim\mathcal{G}amma(b,1)$. Now to generate $\mathcal{G}amma(a,1)$ we can use the algorithm in (5) for $a\in(0,1)$. For $a>1$, we simply use $a=[a]+\{a\}$ where a is the integer part and fractional part and then use

$$X_1=-\sum_{k=1}^{[a]}\log(U_k)+X_0$$

where $(U_k)_{1\leq k\leq[a]}$ are independent $\mathcal{U}_{[0,1]}$ random variables and $X_0\sim\mathcal{G}amma(\{a\},1)$ is generated using the algorithm of question (5).

Exercise 3

The R questions are optional and should not be handed back. The solution will not be covered in the classes, but will be directly posted on the course's website.

1. Reproduce the figures on the estimation of the number π in the lecture.