SC5 Advanced Simulation Methods

Problem Sheet 0, with answers

1 Exercise 1

1. Let Z be a positive random variable with finite expectation $\mathbb{E}[Z] < \infty$, and let $a > 0$. Prove Markov's inequality, that is show that

$$\mathbb{P}\left(Z > a\right) \leq \frac{\mathbb{E}[Z]}{a}.$$

Hint: You can write

$$\mathbb{P}[Z > a] = \mathbb{E}\left[\mathbb{1}\{Z > a\}\right].$$

Answer. Starting from the hint we have

$$\begin{aligned}\mathbb{P}[Z > a] &= \mathbb{E}\left[\mathbb{1}\{Z > a\}\right] \\ &\leq \mathbb{E}\left[\frac{Z}{a}\mathbb{1}\{Z > a\}\right] \\ &= \frac{1}{a}\mathbb{E}\left[Z\mathbb{1}\{Z > a\}\right] \\ &\leq \frac{1}{a}\mathbb{E}\left[Z\right],\end{aligned}$$

where the last part follows since for $x, a > 0$ we have $x\mathbb{1}\{x > a\} \leq x$.

2. Let X be a random variable such that $\mathbb{E}[X^2] < \infty$ and define

$$\mu := \mathbb{E}[X], \qquad \sigma^2 := \mathrm{Var}(X).$$

Use Markov's inequality to prove Chebyshev's inequality, namely that for any $t > 0$ we have

$$\mathbb{P}\left[|X - \mu| \geq t\right] \leq \frac{\sigma^2}{t}.$$

Answer. Applying Markov's inequality to $Z := (X - \mu)^2$ we have

$$\begin{aligned}\mathbb{P}\left[|X - \mu| \geq t\right] &= \mathbb{P}\left[(X - \mu)^2 \geq t^2\right] \\ &\leq \frac{1}{t^2}\mathbb{E}\left[(X - \mu)^2\right] = \frac{\sigma^2}{t^2}.\end{aligned}$$

3. Let $X_1, X_2, \ldots$ be independent and identically distributed (i.i.d.) random variables, that is the X_i are mutually independent and have the same distribution. Let $\mu := \mathbb{E}[X_1]$ and $\sigma^2 := \mathrm{Var}(X_1)$, and define

$$S_n := \frac{1}{n}\sum_{i=1}^{n} X_i, \quad n \geq 1.$$

Using Chebyshev's inequality, prove the *weak law of large numbers*, that is for any $\epsilon > 0$ we have

$$\mathbb{E}\left[|S_n - \mu| \geq \epsilon\right] \to 0,$$

as $n \to \infty$. In words the weak law of large numbers states that the sample mean of i.i.d. random variables *converges in probability* to the expectation. **Answer**. We immediately get using Chebyshev's inequality that

$$\mathbb{E}\left[|S_n - \mu| \geq \epsilon\right] = \frac{1}{\epsilon^2}\,\mathrm{Var}\left[(S_n - \mu)^2\right],$$

and to proceed we need to compute the expectation on the right hand side. Since $S_n - \mu = \sum_{i=1}^n (X_i - \mu)/n$ we can write $Y_i := X_i - \mu$, which have zero mean and are i.i.d. and we therefore have

$$\begin{aligned}
\operatorname{Var}\left[(S_n - \mu)^2\right] &= \mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^n Y_i\right)^2\right] \\
&= \frac{1}{n^2}\sum_{i,j=1}^n \mathbb{E}\left[Y_i Y_j\right] \\
&= \frac{1}{n^2}\sum_{i=1}^n \mathbb{E}\left[Y_i^2\right] + \frac{1}{n^2}\sum_{i\neq j}^n \mathbb{E}\left[Y_i\right]\left[Y_j\right] \\
&= \frac{1}{n^2}\sum_{i=1}^n \mathbb{E}\left[Y_i^2\right] + 0 \\
&= \frac{n\sigma^2}{n^2},
\end{aligned}$$

and therefore

$$\mathbb{E}\left[|S_n - \mu| \geq \epsilon\right] \leq \frac{\sigma^2}{n\epsilon^2} \to 0$$

as $n \to \infty$.

2 Exercise 2

Consider the quadrant of the unit disk in $\mathbb{R}^2$, given by

$$A := \{x, y \geq 0 : x^2 + y^2 \leq 1\}.$$

Of course its area is given by $\pi/4$. Let U be the unit square with its bottom left corner at the origin, that is

$$U := \{(x, y) : 0 \leq x, y \leq 1\}.$$

Let $X, X_1, X_2 \ldots$ be independent and identically distributed random variables, distributed uniformly on U, that is the probability density function of X_i is $f(x) := 1$ for $x \in U$ and 0 otherwise.

Let $g(x) = \mathbb{1}\{x \in A\}$ be the indicator of the set A, that is g takes the value 1 on A and 0 elsewhere.

1. Compute $\mathbb{E}[g(X)]$ and $\operatorname{Var}[g(X)]$.

   Answer. This is straightforward.

$$\begin{aligned}
\mathbb{E}[g(X)] &= \iint_U \mathbb{1}\{(x, y) \in A\} \mathrm{d}x\mathrm{d}y \\
&= \iint_A \mathrm{d}x\mathrm{d}y = \frac{\pi}{4},
\end{aligned}$$

   the area of A. For the variance, since $g(X)^2 = g(X)$ we have

$$\operatorname{Var}[g(X)] = \mathbb{E}[g(X)^2] - \mathbb{E}[g(X)]^2 = \frac{\pi}{4}\left(1 - \frac{\pi}{4}\right) = \frac{3\pi^2}{16}.$$

2. Using the law of large numbers construct a consistent estimator $\hat{\pi}_n$ of π. That is, construct a sequence of random variables $\hat{\pi}_n$ such that $\pi_n \to \pi$ in probability and use Chebyshev's inequality to construct a $(1 - \alpha)$-confidence interval, that is an interval such that for n large enough

$$\mathbb{P}\left(\hat{\pi}_n - A_n \leq \pi \leq \hat{\pi}_n + B_n\right) \geq 1 - \alpha.$$

   Answer. With X_i as in the statement, let $Z_i := 4g(X_i)$. Then from the previous calculation we know that

$$\mathbb{E}[Z_i] = 4\frac{\pi}{4} = \pi, \qquad \operatorname{Var}[Z_i] = 3\pi^2.$$

Therefore by the law of large numbers we can set

$$\hat{\pi}_n := \frac{1}{n}\sum_{i=1}^{n} Z_i.$$

To find the confidence interval, by Chebyshev's inequality, since clearly $\mathbb{E}[\hat{\pi}_n] = \pi$ we have

$$\mathbb{P}\left\{|\hat{\pi}_n - \pi| \geq t\right\} \leq \frac{\operatorname{Var}(\hat{\pi}_n)}{t^2} = \frac{3\pi^2}{nt^2}.$$

Choose $t_n^* := \sqrt{3\pi^2/n\alpha}$, such that

$$\frac{3\pi^2}{n(t_n^*)^2} \leq \alpha.$$

Then

$$\mathbb{P}\left\{-t_n^* \leq \pi - \hat{\pi}_n \leq t_n^*\right\} \geq 1 - \alpha$$

so we can conclude that

$$\mathbb{P}\left\{\hat{\pi}_n - \sqrt{\frac{3\pi^2}{n\alpha}} \leq \pi \leq \hat{\pi}_n + \sqrt{\frac{3\pi^2}{n\alpha}}\right\} \geq 1 - \alpha.$$

3. The above confidence interval should be extremely loose since we have only used second moments. Can you think of a way to make it tighter?

   Hint: For example apply Markov's inequality to $|\hat{\pi}_n - \pi|^k$. You can also use the fact that if $Z_1, \ldots, Z_n$ are i.i.d. , with zero mean and $\mathbb{E}[|Z_i|^k] < \infty$ then

   $$\mathbb{E}\left[\left|\sum_{i=1}^{n} Z_i\right|^k\right] \leq Cn^{k/2},$$

   for some constant $C > 0$ independent of n.

   Answer. Indeed we have used Chebyshev's inequality which only requires second moments. This means that our bound is far from sharp. Actually the central limit theorem implies that $\hat{\pi}_n - \pi$ is roughly normal with variance $1/n$ so that its tails must decay roughly like $\exp(-nt^2)$ rather than $1/(nt^2)$. We can improve on this by applying Markov's inequality to the random variable $|\hat{\pi}_n - \pi|^k$, for $k > 2$ which would then give a bound of the form

   $$\begin{aligned}
   \mathbb{P}\left\{|\hat{\pi}_n - \pi| \geq t\right\} &= \mathbb{P}\left\{\frac{1}{n}\left|\sum_{i=1}^{n} Z_k\right| \geq t\right\} \\
   &= \mathbb{P}\left\{\left|\sum_{i=1}^{n} Z_k\right| \geq nt\right\} \\
   &= \mathbb{P}\left\{\left|\sum_{i=1}^{n} Z_k\right|^k \geq (nt)^k\right\} \\
   &\leq \frac{\mathbb{E}\left[\left|\sum_{i=1}^{n} Z_k\right|^k\right]}{n^k t^k} \\
   &\leq \frac{Cn^{k/2}}{n^k t^k} \leq \frac{C}{n^{k/2} t^k},
   \end{aligned}$$

   where $Z_i = 4g(X_i) - \pi$, and we have used the hint in the last line.