Analysis 28, 319–331 (2008) / **DOI** 10.1524/anly.2008.0917

Infinitesimal resistance metrics on Sierpinski gasket type fractals

Mihai Cucuringu, Robert S. Strichartz*

Received: June 29, 2006

Summary: We prove the existence of an infinitesimal resistance metric on the Sierpinski gasket (SG) at boundary points, junction points and periodic points. This is a renormalized limit of the effective resistance metric as we zoom in on the point, and satisfies a self-similar identity. We obtain similar results on PCF fractals with three boundary points.

1 Introduction

On the Sierpinski gasket (SG) and related fractals, it is possible to construct a self-similar energy form $\mathcal{E}$ that has the property that points have positive capacity ([Ki1], [S]). This means it is possible to define an effective resistance metric $R(x, y)$ by

$$R(x,y)^{-1} = \min\{\mathcal{E}(u) : u(x) = 0 \text{ and } u(y) = 1\}. \tag{1.1}$$

Equivalently, $R(x, y)$ is the resistance between x and y when we restrict the energy form to the two point set $\{x, y\}$ ([Ki2]). There is considerable evidence that this metric is the natural metric to use in analytic problems on the fractals. This metric is approximately self-similar, but not strictly self-similar.

To be specific, we consider fractals K that are characterized by a self-similar identity

$$K = \cup F_i K \tag{1.2}$$

for a finite iterated function system (IFS) $\{F_i\}$ of contractive similarities on some Euclidean space. In the case $K = SG$ we take the three homotheties with contraction ratio $1/2$ and fixed points $\{q_1, q_2, q_3\}$, the vertices of a triangle. We assume K is connected and postcritically finite (PCF) (see [Ki1] for the exact definition), and $\mathcal{E}$ satisfies a self-similar identity

$$\mathcal{E}(u) = \sum_i r_i^{-1} \mathcal{E}(u \circ F_i) \tag{1.3}$$

*Research supported in part by the National Science Foundation, Grant DMS–0140194.
AMS 2000 subject classification: Primary: 28A80
Key words and phrases: Sierpinski gasket, analysis on fractals, effective resistance metric

for certain resistance renormalization factors r_i satisfying $0 < r_i < 1$. In the case of SG, the standard energy form has all $r_i = 3/5$, but see [Sa] and [CS] for other choices. What this means is that K consists of copies of itself where all resistances are reduced by a factor of r_i, and these are joined together at a finite set of junction points. In this setting, the minimum is attained (and is nonzero) in (1.1), and this is equivalent to a cluster of conditions (including the continuity of the Green's function) that is described by saying *points have positive capacity*. This is valid for the unit interval (a special case of PCF fractals, although not really fractal) but not for the standard energy on the square (or domains in Euclidean spaces of dimension at least two).

The approximate self-similarity of R is given by

$$c_1 r_i R(x, y) \leq R(F_i x, F_i y) \leq c_2 r_i R(x, y) \tag{1.4}$$

for fixed constants $c_1 < c_2$. Easy computations show that we cannot take $c_1 = c_2$. In this paper we will look at an infinitesimal form of the metric. One motivation for this is that we will recover strict self-similarity. Another motivation is the analog with Riemannian geometry, where the infinitesimal version of the metric plays a fundamental role. We will not carry this analogy very far in this paper, but leave it for future developments.

To be specific, suppose q_i is the fixed point of F_i, we want to define

$$R'(x, q_i) = \lim_{m\to\infty} r_i^{-m} R(F_i^m x, q_i). \tag{1.5}$$

In other words, we zoom in on a sequence of neighborhoods $F_i^m K$ of q_i and blow up the metric there by an appropriate factor, and take the limit. Of course we need to prove that the limit exists, but once that is shown, it follows immediately that the infinitesimal metric R' satisfies the self-similar identity

$$R'(F_i x, q_i) = r_i R'(x, q_i). \tag{1.6}$$

We will not only show that the limit exists (for SG and some other examples), but we will show how to compute it as the effective resistance for an energy obtained from the original one by adding on a finite set of resistors connecting the points $\{q_i\}$. In Section 2 we carry out the computation in detail for SG, not only at boundary points but also at junction points and periodic points (see [BSSY] and [AS] for a detailed study of local analysis at periodic points). In Section 3 we show how to generalize some of these results to PCF fractals with three boundary points. In Section 4 we display results of numerical calculations of the infinitesimal resistance metric on SG, including the shape of balls and their volumes. More data is available at the website www.math.cornell.edu/˜cucuringu.

2 SG with added resistors

Let $\mathcal{E}$ denote the standard energy on SG and define

$$\mathcal{E}_s(u) = \mathcal{E}(u) + s(u(q_2) - u(q_3))^2 \tag{2.1}$$

for any fixed $s \geq 0$. We interpret this as adding a resistor of resistance $1/s$ between q_2 and q_3. Let R_s denote the effective resistance metric for $\mathcal{E}_s$. Note that $R = R_0$.

Lemma 2.1 *We have*

$$R_s(F_1x, q_1) = \frac{3}{5} R_{h(s)}(x, q_1) \tag{2.2}$$

for

$$h(s) = \frac{15 + 12s}{20 + 12s}. \tag{2.3}$$

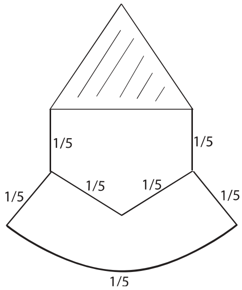

Figure 2.1

Proof: Since $F_1x \in F_1K$ we may reduce the contributions from F_2K and F_3K to two Y-networks with resistors 1/5. Thus we need to analyze the network shown in Figure 2.1, where the top triangle represents F_1K, and edges are marked with resistances. We then simplify as shown in Figure 2.2. Since we reduce energy by 3/5 when we blow up Figure 2.2 (c) to the full SG, we have an added resistor of resistance $\frac{4(5+3s)}{3(5+4s)}$, and hence conductance $h(s)$. □

Lemma 2.2 $h(s)$ *has a unique attracting fixed point* $s = 5/6$ *on* $[0, \infty)$.

Proof: $h(s) - \dfrac{5}{6} = \dfrac{s - \frac{5}{6}}{15 + 6(s - \frac{5}{6})}$. □

Theorem 2.3 *The limit*

$$\lim_{m\to\infty} \left(\frac{5}{3}\right)^m R(F_1^m x, q_1) \tag{2.4}$$

exists uniformly on K *and equals*

$$R_{5/6}(x, q_1). \tag{2.5}$$

Proof: By iterating (2.2) we obtain

$$\left(\frac{5}{3}\right)^m R(F_1^m x, q_1) = R_{h^{(m)}(0)}(x, q_1) \tag{2.6}$$

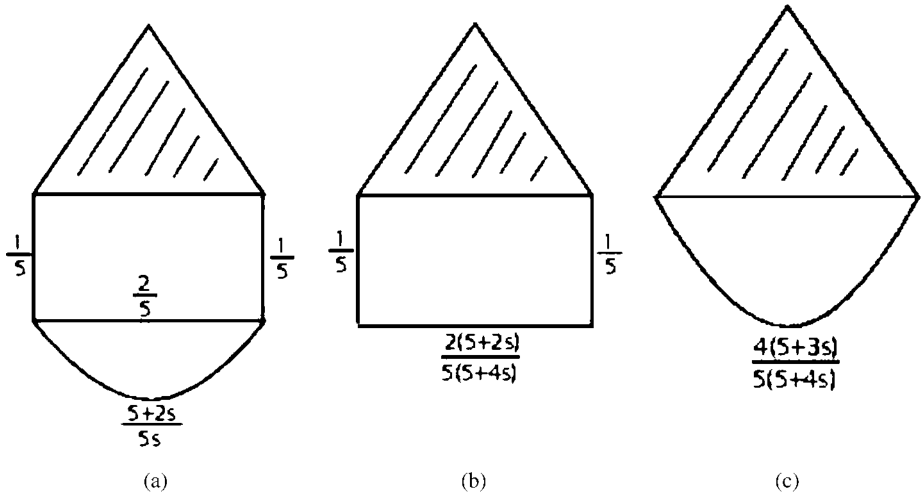

Figure 2.2

where $h^{(m)}$ denotes the m-fold composition of h. By Lemma 2.2 we have $h^{(m)}(0) \to \frac{5}{6}$ at a rate $O\big(\big(\frac{1}{15}\big)^m\big)$. □

To obtain a similar result at junction points we need to understand what happens when we add three resistors connecting all boundary points. By the $\Delta - Y$ transform, this is equivalent to adding a Y-circuit with resistors joining each q_j $(j = 1, 2, 3)$ to a new junction point q_0. So now let $\mathbf{s} = (s_1, s_2, s_3)$ be a vector of conductances, and define $\mathcal{E}_\mathbf{s}$ on $SG \cup \{q_0\}$ by

$$\mathcal{E}_\mathbf{s}(u) = \mathcal{E}(u) + \sum_{j=1}^{3} s_j (u(q_j) - u(q_0))^2 \tag{2.7}$$

and let $R_\mathbf{s}$ be the associated effective resistance. For computing $R_\mathbf{s}(x, y)$ with $x, y \in SG$ we easily find that

$$u(q_0) = \frac{1}{s_1 + s_2 + s_3} \sum_{j=1}^{3} s_j u(q_j) \tag{2.8}$$

so we could substitute (2.8) in (2.7) to remove the value $u(q_0)$ from consideration.

Lemma 2.4 *We have*

$$R_\mathbf{s}(F_1 x, q_1) = \frac{3}{5} R_{H_1(\mathbf{s})}(x, q_1), \tag{2.9}$$

or more generally

$$R_\mathbf{s}(F_1 x, F_1 y) = \frac{3}{5} R_{H_1(\mathbf{s})}(x, y), \tag{2.9'}$$

for $H_1(\mathbf{s}) = (s_1', s_2', s_3')$ *given by*

$$\begin{cases} s_1' = \dfrac{3s_1}{5}\left(\dfrac{4s_2s_3 + 5(s_2+s_3)}{\frac{1}{5}s_1s_2s_3 + 4s_2s_3 + s_1(s_2+s_3) + 5(s_1+s_2+s_3)}\right) \\ s_2' = 2 + \dfrac{s_2 - 3s_3}{\frac{6}{5}s_2s_3 + s_2 + 3s_3} \\ s_3' = 2 + \dfrac{s_3 - 3s_2}{\frac{6}{5}s_2s_3 + s_3 + 3s_2} \end{cases} \tag{2.10}$$

(also $H_1(s_1, 0, 0) = (0, \frac{3}{2}, \frac{3}{2})$*).*

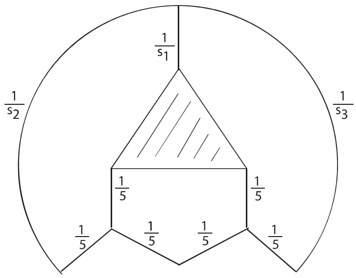

Figure 2.3

Proof: In place of Figure 2.1 we need to analyze the network shown in Figure 2.3 (the point at the top is q_0). We then simplify as shown in Figure 2.4 (note that we go from (a) to (b) using a $\Delta - Y$ transform and introducing a new vertex on top). Taking the reciprocals of the resistances in Figure 2.4 (c) and multiplying by $\frac{3}{5}$ we obtain the conductances given by (2.10). □

Of course we also have

$$R_{\mathbf{s}}(F_j x, q_j) = \frac{3}{5} R_{H_j(\mathbf{s})}(x, q_j) \text{ for } j = 2, 3 \tag{2.11}$$

with H_j obtained from H_1 by cyclic permutation of indices.

Lemma 2.5 $H_1(\mathbf{s})$ *has a unique attracting fixed point* $(0, \frac{5}{3}, \frac{5}{3})$ *on the positive octant in* $\mathbb{R}^3$*, and similarly for* $H_j(\mathbf{s})$*,* $j = 2, 3$*.*

Proof: It is clear that $s_1' \leq \frac{3}{5}s_1$. We can rewrite the other two equations in (2.10) as

$$s_2' - \frac{5}{3} = \frac{\frac{2}{5}(s_2 - \frac{5}{3})(s_3 - \frac{5}{3}) + 2(s_2 - \frac{5}{3}) - \frac{4}{3}(s_3 - \frac{5}{3})}{\frac{6}{5}(s_2 - \frac{5}{3})(s_3 - \frac{5}{3}) + 3(s_2 - \frac{5}{3}) + 5(s_3 - \frac{5}{3}) + 10}$$

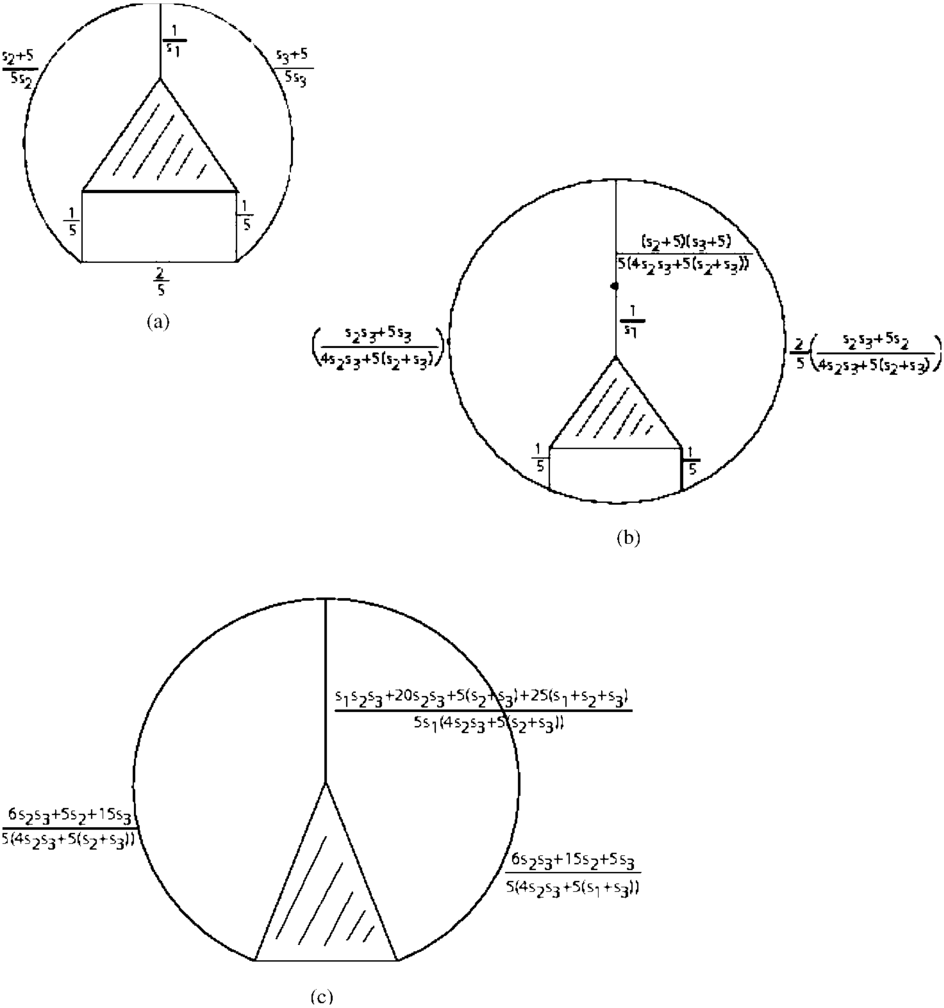

Figure 2.4

$$s_3' - \frac{5}{3} = \frac{\frac{2}{5}(s_2 - \frac{5}{3})(s_3 - \frac{5}{3}) + 2(s_3 - \frac{5}{3}) - \frac{4}{3}(s_2 - \frac{5}{3})}{\frac{6}{5}(s_2 - \frac{5}{3})(s_3 - \frac{5}{3}) + 3(s_3 - \frac{5}{3}) + 5(s_2 - \frac{5}{3}) + 10}$$

and the result follows easily. □

Of course it is easy to verify that $R_{(0,5/3,5/3)} = R_{5/6}$.

Now consider a junction point $F_w q_j = F_{w'} q_{j'}$ for words w and w' of the same length n. Then we can zoom in on this point from either side. It is natural to look for infinitesimal resistance metrics

$$R'(F_w q_j, F_w x) = \lim_{m\to\infty} \left(\frac{5}{3}\right)^{m+n} R(F_w q_j, F_w F_j^m x) \tag{2.12}$$

and

$$R'(F_{w'} q_{j'}, F_{w'} x) = \lim_{m\to\infty} \left(\frac{5}{3}\right)^{m+n} R(F_{w'} q_{j'}, F_{w'} F_{j'}^m x). \tag{2.13}$$

For simplicity of notation we take $j = 1$.

Theorem 2.6 *The limit*

$$\lim_{m\to\infty} \left(\frac{5}{3}\right)^{m+n} R(F_w q_1, F_w F_1^m x) \tag{2.14}$$

exists and is also equal to (2.5).

Proof: By the same reasoning as in the proof of Lemma 2.4, we have

$$\left(\frac{5}{3}\right)^{n} R(F_w q_1, F_w x) = R_{\mathbf{s}}(q_1, x) \text{ for all } x, \tag{2.15}$$

for some $\mathbf{s}$ that depends on w. Take $x = F_1^m x$ in (2.15) and apply (2.9) m times to obtain

$$\left(\frac{5}{3}\right)^{m+n} R(F_w q_1, F_w F_1^m x) = R_{H_1^{(m)}(\mathbf{s})}(x, q_1),$$

and the result follows by Lemma 2.5. □

In order to understand what happens near a periodic point we need to study arbitrary compositions of the mappings H_i. Define the region Ω in $\mathbb{R}^3$ as follows:

$$\Omega = [0,3]\times[1,3]\times[1,3] \cup [1,3]\times[0,3]\times[1,3] \cup [1,3]\times[1,3]\times[0,3]. \tag{2.16}$$

In other words, Ω is the subset of $[0,3]^3$ where at most one variable is less than 1.

Lemma 2.7 *Each H_i is a continuous map from Ω to itself, and for any initial vector $\mathbf{s}$ in the nonnegative octant, and any sequence $i_1, i_2, \ldots$, eventually $H_{i_1} \circ \cdots \circ H_{i_n}(\mathbf{s})$ is in Ω.*

Proof: It is clear from (2.10) that $s_1' \le \frac{3}{5} s_1$, $|s_2' - 2| \le 1$, $|s_3' - 2| \le 1$, so H_1 maps Ω to itself. The same is true for H_2 and H_3. It is also clear that $H_i \circ H_j$ for $i \ne j$ maps the nonnegative octant into Ω, and $H_1^n(\mathbf{s})$ is in Ω once $(\frac{3}{5})^n s_1 \le 3$. □

Lemma 2.8 *Any 5-fold composition $H_{i_1} \circ H_{i_2} \circ H_{i_3} \circ H_{i_4} \circ H_{i_5}$ is infinitesimally contractive (the operator norm of the derivative is strictly less than 1) on Ω.*

Proof: (Computer assisted) We bound the operator norm by the Hilbert–Schmidt norm, which is easier to compute. In fact the square of the Hilbert–Schmidt norm is just the sum of the squares of the nine partial derivatives. Each composition is a rational function of the same form as H_1. We had Maple compute the sum of the squares of the partial derivatives for each of the compositions (actually we used symmetry and some factorizations to reduce the number of compositions drastically), and then maximize over each of the three rectangles in (2.16). The maximum of all these values turns out to be .403 for $H_1 \circ H_2 \circ H_2 \circ H_2 \circ H_2$. □

Since Ω is not convex, it does not follow that the compositions are strictly contractive in the Euclidean metric, but they are strictly contractive in the geodesic metric, which is equivalent to the Euclidean metric.

Let z be a periodic point, so $z = F_w z$ for some word w of length m. Define

$$R'(x, z) = \lim_{n\to\infty} \left(\frac{5}{3}\right)^{mn} R(F_w^n x, z) \tag{2.17}$$

if the limit exists.

Theorem 2.9 *$R'(x, z)$ exists and equals $R_{\mathbf{s}}(x, z)$, where $\mathbf{s}$ is the unique fixed point of $H_{w_m} \circ \cdots \circ H_{w_1}$.*

Proof: By iterating Lemma 2.4 we obtain

$$\left(\frac{5}{3}\right)^{mn} R(F_w^n x, z) = R_{H_w^n(\mathbf{0})}(x, z),$$

where $H_w = H_{w_m} \circ \cdots \circ H_{w_1}$. The result follows by Lemma 2.8 and the contractive mapping principle. □

Of course $R'(x, z)$ satisfies the self-similar relation

$$R'(F_w x, z) = \left(\frac{3}{5}\right)^{m} R'(x, z) \tag{2.18}$$

analogous to (1.6).

Figure 2.5 shows the fixed points of mappings H_w for moderate values of m. The set of all fixed points is dense in the invariant set for the IFS $\{H_1, H_2, H_3\}$.

3 Fractals with three boundary points

Let K be a PCF fractal with boundary $V_0 = \{q_1, q_2, q_3\}$, generated by an IFS of N mappings $\{F_i\}$ such that $F_i q_i = q_i$ for $i = 1, 2, 3$. We assume that K has a self-similar energy $\mathcal{E}$ satisfying

$$\mathcal{E}(u) = \sum_{i=1}^{N} r_i^{-1} \mathcal{E}(u \circ F_i) \text{ for some } \{r_i\} \tag{3.1}$$

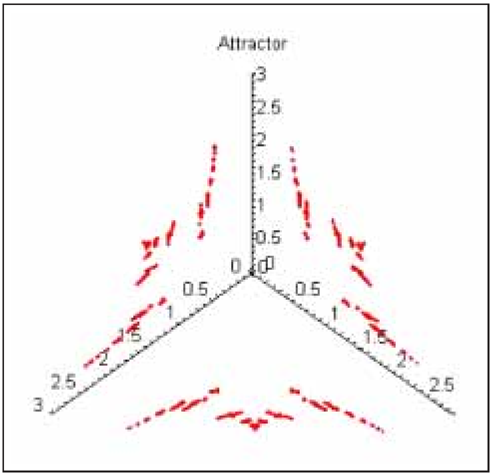

Figure 2.5

with $0 < r_i < 1$. (See [P] for a proof of existence of such energies.) Let R denote the effective resistance metric for $\mathcal{E}$, and R_s the effective resistance metric for

$$\mathcal{E}_s(u) = \mathcal{E}(u) + s(u(q_2) - u(q_3))^2. \tag{3.2}$$

As in the case of SG we would like to establish an identity

$$R_s(q_1, F_1x) = r_1 R_{h(s)}(q_1, x) \tag{3.3}$$

for an explicit map $h(s)$. To compute $R_s(q_1, F_1x)$ we note that the contribution of $\sum_{j=2}^{N} r_j^{-1}\mathcal{E}(u \circ F_j)$, the energy on all the cells except F_1K, can be reduced to

$$\begin{aligned} d(u(q_2) - u(q_3))^2 + e(u(F_1q_2) - u(F_1q_3))^2 \\ + \sum_{i=2}^{3}(a_i(u(q_2) - u(F_1q_i))^2 + b_i(u(q_3) - u(F_1q_i))^2) \end{aligned} \tag{3.4}$$

for a set $\{a_i, b_i, d, e\}$ of six positive numbers that depend on the fractal. Note that when we pass from $\mathcal{E}$ to $\mathcal{E}_s$ we just replace d by $d+s$. Let u be the energy minimizing function for $R_s(q_1, F_1x)$. Then the values $u(q_2)$ and $u(q_3)$ are determined by minimizing (3.4) (with $d+s$ in place of d). This leads to

$$\lambda u(q_2) = (a_2(b_2+b_3)+(d+s)(a_2+b_2))u(F_1q_2)+(a_3(b_2+b_3)+(d+s)(a_3+b_3))u(F_1q_3)$$

$$\lambda u(q_3) = ((a_2+a_3)b_2+(d+s)(a_2+b_2))u(F_1q_2)+((a_2+a_3)b_3+(d+s)(a_3+b_3))u(F_1q_3)$$

for $\lambda = (d+s)(a_2+a_3+b_2+b_3)+(a_2+a_3)(b_2+b_3)$. When we substitute this back in (3.4) we obtain (using MAPLE)

$$\frac{A(s+d)+B}{C(s+d)+D}(u(F_1q_2)-u(F_1q_3))^2 \tag{3.5}$$

for

$$\begin{cases} A = e(a_2+a_3+b_2+b_3)+(a_2+b_2)(a_3+b_3) \\ B = e(a_2+a_3)(b_2+b_3)+a_2a_3(b_2+b_3)+(a_2+a_3)b_2b_3 \\ C = a_2+a_3+b_2+b_3 \\ D = (a_2+a_3)(b_2+b_3). \end{cases} \tag{3.6}$$

Note that all these coefficients are positive.

Thus $\mathcal{E}_s(u)$ is the sum of $r_1^{-1}\mathcal{E}(u \circ F_1)$ and (3.5). When we minimize this we obtain (3.3) with

$$h(s) = r_1\Big(\frac{A(s+d)+B}{C(s+d)+D}\Big). \tag{3.7}$$

This is the analog of Lemma 2.1. To find the fixed points of (3.7) we solve a quadratic equation, with one positive root

$$\frac{r_1A-D-Cd+\sqrt{(r_1A-D-Cd)^2+4r_1(B+Ad)C}}{2C}. \tag{3.8}$$

Note that

$$h'(s) = \frac{r_1(AD-BC)}{(C(s+d)+D)^2}$$

is decreasing in s, so

$$h'(s) \le h'(0) = \frac{r_1(AD-BC)}{(Cd+D)^2} \le \frac{AD-BC}{D^2} = \frac{(a_2b_3-a_3b_2)^2}{(a_2+a_3)^2(b_2+b_3)^2} < 1$$

so the fixed point is attracting. This is the analog of Lemma 2.2. The analog of Theorem 2.3 thus holds for the value of s given by (3.8).

It seems plausible that the analogs of Theorems 2.6 and 2.9 are also valid.

4 Numerical results

In this section we show the results of computing the infinitesimal effective resistance $R'(x,q_1)$ on two fractals, SG and SG3 (defined by six homotheties with contraction ratio $1/3$, with all $r_i = 7/15$ [S]). Figures 4.1 and 4.2 display graphs of this function. It is also of interest to compute the balls

$$B_t(q_1) = \{x : R'(x,q_1) \le t\} \tag{4.1}$$

for different values of t. In Figures 4.3 and 4.4 we show images of such balls for t values chosen so that the balls are actually disconnected subsets of the fractal.

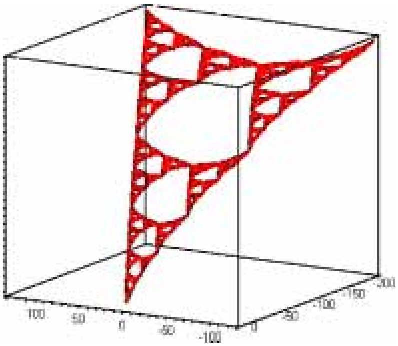

Figure 4.1 The graph of the function $R'(x, q_1)$ on SG.

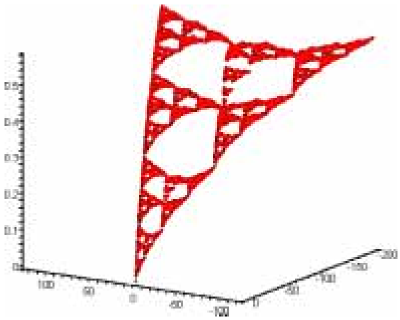

Figure 4.2 The graph of the function $R'(x, q_1)$ on SG3.

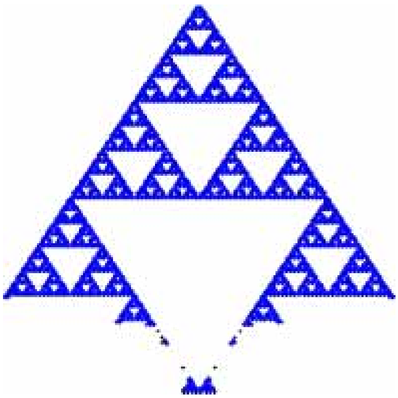

Figure 4.3 The ball $B_t(q_1)$ for $t = .509$ on SG (the point q_1 is on top).

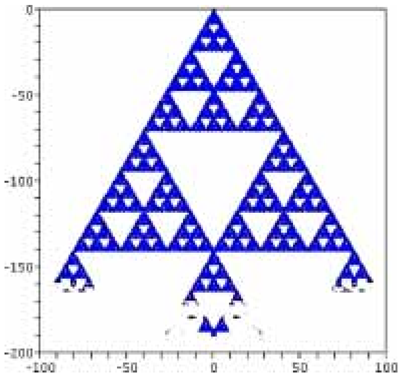

Figure 4.4 The ball $B_t(q_1)$ for $t = .502$ on SG3 (the point q_1 is on top).

To get an overall impression of the size of the balls, we compute their measure. Here we use the balanced self-similar probability measure μ satisfying

$$\mu = \sum_{i=1}^{N} \frac{1}{N} \mu \circ F_i^{-1} \tag{4.2}$$

with $N = 3$ for SG and $N = 6$ for SG3. Let

$$f(t) = \mu(B_t(q_1)). \tag{4.3}$$

It follows from (1.6) that

$$B_{r_1 t}(q_1) = F_1 B_t(q_1) \tag{4.4}$$

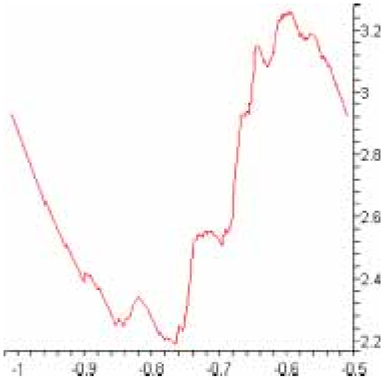

Figure 4.3: The graph of $\varphi(t)$ vs. $\log t$ over a single period on SG (computed to level 6).

provided that t is less than the maximum distance. It follows from (4.2) and (4.4) that

$$f(r_1 t) = \frac{1}{N} f(t). \tag{4.5}$$

This means that

$$\varphi(t) = \frac{f(t)}{t^\alpha} \text{ for } \alpha = \frac{\log N}{\log 1/r_1} \tag{4.6}$$

is multiplicatively periodic of period r_1. In Figure 4.5 we show the graph of φ on SG on a logarithmic scale over one period. Volumes of balls in the resistance metric play an important role in heat kernel estimates [Ki3].

Acknowledgments. We are grateful to Jun Kigami for suggesting the idea of adding resistors.

References

[AS] C. Avenancio-Leon and R. Strichartz, Local behavior of harmonic functions on the Sierpinski gasket, *Illinois J. Math.*, to appear

[BSSY] N. Ben-Gal, A. Shaw-Krauss, R. Strichartz, and C. Young, Calculus on the Sierpinski gasket II: Point singularities, eigenfunctions, and normal derivatives of the heat kernel *Trans. Amer. Math. Soc.*, 358:3883–3936, 2006

[CS] M. Cucuringu and R. Strichartz, Self-similar energy forms on the Sierpinski gasket with twists, *Potential Anal.*, 27:45–60, 2007

[Ki1] J. Kigami, *Analysis on Fractals*, Cambridge University Press, New York, 2001

[Ki2] J. Kigami, Harmonic analysis for resistance forms, *J. Funct. Anal.*, 204:399–444, 2003

[Ki3] J. Kigami, Volume doubling measures and heat kernel estimates on self-similar sets, *Memoirs Amer. Math. Soc.*, to appear

[P] R. Peirone, Existence of eigenforms on fractals with three vertices, *Proc. Royal Soc. Edinburgh, Sec. A*, 137:1073–1080, 2007

[Sa] C. Sabot, Existence and uniqueness of diffusions on finitely ramified self-similar fractals, *Ann. Scient. Ec. Norm. Sup. 4eme série* 30:605–673, 1997

[S] R. Strichartz, *Differential equations on fractals: a tutorial*, Princeton University Press, 2006

Mihai Cucuringu
Program in Applied and Computational Mathematics
Princeton University
Princeton, NJ 08544
USA
mcucurin@math.princeton.edu

Robert S. Strichartz
Mathematics Department
Malott Hall
Cornell University
Ithaca, NY 14853
USA
str@math.cornell.edu