Part A Linear Programming: Problem Sheet 3 TT 2011

1. 2009 AS1

T1. (a) When the simplex algorithm is used to solve the linear programming problem

$$\begin{array}{lrcrcrcrcrcl} \text{maximise} & 2x_1 & + & x_2 & - & x_3 & & & & & & \\ \text{subject to} & x_1 & + & x_2 & + & 2x_3 & + & x_4 & & & = & 6 \\ & x_1 & + & 4x_2 & - & x_3 & & & + & x_5 & = & 4 \\ & \multicolumn{11}{l}{x_1, \ldots, x_5 \geqslant 0} \end{array}$$

the final tableau obtained, with some entries omitted, is

$$\begin{array}{ccccc|c} \cdot & -1 & \cdot & \frac{1}{3} & -\frac{1}{3} & \cdot \\ \cdot & 3 & \cdot & \frac{1}{3} & \frac{2}{3} & \cdot \\ \hline \cdot & -6 & \cdot & -\frac{1}{3} & -\frac{5}{3} & \cdot \end{array}$$

(i) Find the missing entries in the final tableau.

(ii) What is the optimal solution of the problem, and what is the optimal value of the objective function?

(b) Suppose the value 4 in the second constraint is changed to $4 + \alpha$.

(i) Assuming $|\alpha|$ is small, find (in terms of α) the new optimal solution and the new optimal value of the objective function.

(ii) Find the range of values of α for which this new solution is optimal.

(c) Suppose the linear programming problem

$$\text{maximise } \mathbf{c}^T\mathbf{x} \quad \text{subject to } A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geqslant \mathbf{0}$$

has a unique optimal solution. Show that this optimal solution cannot be an internal point on a line segment joining two feasible solutions.

[*In (b), it's the* 4 *on the right hand side of the second constraint that changes to* $4 + \alpha$.]

2. 2009 AS2

T2. (a) (i) Write down the dual D of the linear programming problem

$$P: \text{ maximise } \mathbf{c}^T\mathbf{x} \quad \text{subject to } A\mathbf{x} \leqslant \mathbf{b},\, \mathbf{x} \geqslant \mathbf{0}.$$

(ii) Suppose P and D both have feasible solutions and that the optimal values of the objective functions of P and D are equal. Show that any optimal solutions $\mathbf{x}$, $\mathbf{y}$ satisfy

$$(\mathbf{y}^T A - \mathbf{c}^T)\mathbf{x} = 0 \quad \text{and} \quad \mathbf{y}^T(A\mathbf{x} - \mathbf{b}) = 0.$$

(iii) If P is given by

$$\begin{array}{lrrrrl} \text{maximise} & 5x_1 + & 12x_2 + & 2x_3 + & 5x_4 & \\ \text{subject to} & 5x_1 + & 3x_2 + & x_3 + & x_4 & \leqslant 2 \\ & x_1 + & 4x_2 + & x_3 + & 5x_4 & \leqslant 1 \\ & \multicolumn{5}{l}{x_1, \ldots, x_4 \geqslant 0} \end{array}$$

find the optimal solutions of both P and D.

(b) Now consider the problem P^* obtained from P by removing the constraints $\mathbf{x} \geqslant \mathbf{0}$.

(i) By introducing new variables, show how to write P^* in a similar form to P. Hence determine the dual D^* of P^*.

(ii) Find the solution of the problem

$$\begin{array}{lrrl} \text{maximise} & x_1 + & x_2 & \\ \text{subject to} & x_1 + & 2x_2 & \leqslant 4 \\ & 3x_1 + & 2x_2 & \leqslant 6 \\ & 4x_1 + & x_2 & \leqslant 6 \end{array}$$

and hence find the solution of its dual.

3. 2007 AS2

T2. A company uses two resources R_1 and R_2 to make three products P_1, P_2 and P_3. The table below gives the number of units of each resource required to produce one unit of the product.

	P_1	P_2	P_3
R_1	2	3	5
R_2	1	2	3

The company has 33 and 18 units of R_1 and R_2 respectively. The profits per unit of product P_1, P_2 and P_3 are 3, 4 and 8 respectively.

(a) Formulate the problem of maximising the profit made by the company using their current resources as a linear programming problem.

(b) Solve this problem using the Simplex method step by step in tabular form. You should explain the rationale for each step, in particular, (i) reasons for choosing the entering variable and leaving variable, and (ii) how you know that your solution is optimal.

(c) Now you are told that some of the parameter values are just rough estimates. For each of the following parameters, determine the allowable range of values for which the optimal solutions you got in (b) will remain feasible and optimal:

  (i) the unit of R_2 the company current has (18);

  (ii) the profit per unit of product P_3 (8);

  (iii) profit per unit of product P_2 (4).

(d) The R&D department of the company has developed a new product P_4, each unit of which requires 1 unit of R_1 and 2 units of R_2 and produces an estimated profit of 5 units. Would the company increase its profit by making this new product? If it would, find the new optimal solution and optimal value of this problem.

4. 2010 AS1

T1. Consider a 2-player zero-sum game with $m \times n$ payoff matrix A (so that entry a_{ij} gives the reward payable to Player I by Player II if Player I chooses $i \in \{1, \ldots, m\}$ and Player II chooses $j \in \{1, \ldots, n\}$).

(a) What is a *mixed strategy* for Player II?

(b) Consider the problem

$$\min_{\mathbf{q}} \left\{ \max_{i \in \{1,\ldots,m\}} \sum_{j=1}^{n} a_{ij} q_j \right\} \quad \text{subject to} \quad \sum_{j=1}^{n} q_j = 1, \;\; \mathbf{q} \geqslant \mathbf{0}.$$

Explain carefully how this problem relates to the game, and explain what is meant by the *value* of the game.

(c) Suppose all entries of A are non-negative, and that each row and each column contains at least one strictly positive entry. Show that the value of the game is strictly positive.

(d) Suppose A is given by

$$A = \begin{pmatrix} 1 & 6 & 9 \\ 2 & 5 & 2 \\ 3 & 1 & 4 \end{pmatrix}.$$

Using a graphical method or otherwise, find the optimal strategy for Player II, and the value of the game.

5. 2010 AS2

T2. (a) Let A be an $m \times n$ matrix with $m \leqslant n$ and consider the linear programming problem

$$P: \text{ maximise } \mathbf{c}^T\mathbf{x} \quad \text{subject to } A\mathbf{x} \leqslant \mathbf{b},\ \mathbf{x} \geqslant \mathbf{0}.$$

(i) Write down the dual D of P. State and prove the *weak duality theorem.*

(ii) Suppose that $\mathbf{x}$ and $\mathbf{y}$ are feasible for P and D, and suppose that the optimal values of P and D are equal. Show that $\mathbf{x}$ and $\mathbf{y}$ are optimal for P and D if and only if they satisfy the complementary slackness conditions

$$\begin{aligned} (A^T\mathbf{y} - \mathbf{c})_i x_i &= 0 \quad \text{for } i = 1, \ldots, n \\ \text{and} \quad (A\mathbf{x} - \mathbf{b})_j y_j &= 0 \quad \text{for } j = 1, \ldots, m. \end{aligned}$$

(iii) Using a graphical method or otherwise, find optimal solutions for P and D when

$$A = \begin{pmatrix} 5 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 5 \\ 6 \\ 4 \end{pmatrix}.$$

(b) The problem P' is given by

$$P': \text{ maximise } \mathbf{c}^T\mathbf{x} \quad \text{subject to } A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geqslant \mathbf{0}.$$

(i) By first writing P' in the same form as P, obtain the dual D' of P'.

(ii) Suppose that $\mathbf{x} = (u_1, \ldots, u_m, 0, \ldots, 0)^T$ is optimal for P', where $u_i > 0$ for $i = 1, \ldots, m$, and let G be the $m \times m$ matrix consisting of the first m columns of A.

Consider the modified problem

$$P^*: \text{ maximise } \mathbf{c}^T\mathbf{x} \quad \text{subject to } A\mathbf{x} = \mathbf{b}^*,\ \mathbf{x} \geqslant \mathbf{0}.$$

Let $\mathbf{v} = G^{-1}\mathbf{b}^*$. By considering complementary slackness or otherwise, show that $\mathbf{x}^* = (v_1, \ldots, v_m, 0, \ldots, 0)$ is optimal for P^* provided that $\mathbf{x}^* \geqslant \mathbf{0}$.