Part A Linear Programming: Problem Sheet 1 — TT 2011

1. A manufacturer wishes to produce an alloy that is, by weight, 30% metal A and 70% metal B. Five alloys are available at various prices indicated below:

Alloy	1	2	3	4	5
% A	10	25	50	75	95
% B	90	75	50	25	5
Price/kg	£5	£4	£3	£2	£1.50

The desired alloy will be produced by combining some of the other alloys. The manufacturer wishes to find the amounts of the various alloys needed and to determine the least expensive combination. Write a linear programming formulation of this problem.

2. Solve the following linear programming problem by drawing a diagram:

$$\begin{array}{ll} \text{maximise} & 3x_1 + 2x_2 \\ \text{subject to} & -x_1 + 3x_2 \leqslant 12 \\ & x_1 + x_2 \leqslant 8 \\ & 2x_1 - x_2 \leqslant 10 \\ & x_1, x_2 \geqslant 0. \end{array}$$

3. Show how to introduce slack variables to put the problem in the previous question into the standard form

$$\text{maximise } \mathbf{c}^T\mathbf{x} \quad \text{subject to } A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geqslant \mathbf{0}$$

and solve it using the simplex method. You should use the obvious initial solution with the slack variables positive and the other variables zero. Try both choices of pivot column on the first step. Compare the various tableau with your diagram—will the number of pivots be minimised by pivoting on the first column or the second column at the first step?

4. Solve the following problem using the simplex algorithm:

$$\begin{array}{ll} \text{maximise} & 3x_1 + x_2 + 3x_3 \\ \text{subject to} & 2x_1 + x_2 + x_3 \leqslant 2 \\ & x_1 + 2x_2 + 3x_3 \leqslant 5 \\ & 2x_1 + 2x_2 + x_3 \leqslant 6 \\ & x_1, x_2, x_3 \geqslant 0. \end{array}$$

5. Consider applying the simplex algorithm to the problem

$$\begin{array}{ll} \text{maximise} & 2x_1 + 3x_2 \\ \text{subject to} & x_1 - x_2 \leqslant 2 \\ & -2x_1 + x_2 \leqslant 1 \\ & x_1, x_2 \geqslant 0. \end{array}$$

Explain what happens using a diagram.

6. Consider the problem

$$\begin{array}{ll} \text{maximise} & 2x_1 + 3x_2 + x_3 \\ \text{subject to} & x_1 + 2x_2 + 3x_3 + x_4 \qquad = 6 \\ & 2x_1 + x_2 + 2x_3 \qquad + x_5 = 4 \\ & x_1, \dots, x_5 \geqslant 0 \end{array}$$

with initial tableau:

1	2	3	1	0	6
2	1	2	0	1	4
2	3	1	0	0	0

and final tableau:

·	·	·	$\frac{2}{3}$	$-\frac{1}{3}$	·
·	·	·	$-\frac{1}{3}$	$\frac{2}{3}$	·
·	·	·	$-\frac{4}{3}$	$-\frac{1}{3}$	·

Without going through the simplex algorithm, fill in the other numbers in the final tableau. Suppose that you now wish to change the problem by adding a term $+2x_6$ to the objective function and a term $+x_6$ to the left-hand side of the first constraint. Show how to add an extra column to the final tableau (again without using the simplex algorithm) to accommodate this change. Complete the solution of the new problem using the simplex algorithm.

7. (a) Introduce artificial variables and use Phase I of the two-phase simplex algorithm to find a basic feasible solution to

$$\begin{aligned} 2x_1 + x_2 + 2x_3 &= 4 \\ 3x_1 + 3x_2 + x_3 &= 3 \\ x_1, x_2, x_3 &\geqslant 0. \end{aligned}$$

(b) Hence *minimise* $4x_1 + x_2 + x_3$ subject to the constraints in (a).

8. Define $P(\varepsilon)$ to be the problem obtained by replacing the vector $\mathbf{b} = (12, 8, 10)^T$ in questions 2 and 3 by the perturbed vector $\mathbf{b}(\varepsilon) = (12 + \varepsilon_1, 8 + \varepsilon_2, 10 + \varepsilon_3)^T$. Give a formula, in terms of $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$, for the optimal value for $P(\varepsilon)$ when the ε_i are small. If $\varepsilon_1 = \varepsilon_2 = 0$, for what range of ε_3 values does the formula hold?

9. Consider the problem

$$P: \text{ maximise } \mathbf{c}^T\mathbf{x} \quad \text{subject to } A\mathbf{x} \leqslant \mathbf{b},\ \mathbf{x} \geqslant \mathbf{0}.$$

If $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are distinct optimal solutions, show that P has infinitely many optimal solutions.

10. The random variable Y takes one of the values $a_1, \dots, a_n$. We know that Y either has distribution p, given by $P(Y = a_j) = p_j$ (with $\sum_j p_j = 1$), or it has distribution q, given by $P(Y = a_j) = q_j$ (with $\sum_j q_j = 1$). Based on a single observation of Y, we wish to say whether Y has distribution p or distribution q: i.e., for each possible outcome a_j, we will assert with probability $1 - x_j$ that the distribution is p and with probability x_j that the distribution is q (where $x_j \in [0, 1]$ for $j = 1, \dots, n$).

We wish to choose $x_1, \dots, x_n$ so that

- the probability that we say the distribution is q when it is actually p is at most α,
- and subject to this constraint, the probability that we say the distribution is p when it is actually q is minimised.

(a) Write the above problem as a linear programming problem and show that it is equivalent to choosing $x_1, \ldots, x_n$ to

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} q_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} p_j x_j \leqslant \alpha \\ & 0 \leqslant x_j \leqslant 1, \quad j = 1, \ldots, n. \end{aligned}$$

(b) For convenience, assume that $\frac{q_1}{p_1} > \frac{q_2}{p_2} > \cdots > \frac{q_n}{p_n}$ and $\sum_{j=1}^{m} p_j = \alpha$, where $m < n$. By first writing the problem in terms of new variables u_j, where $u_j = p_j x_j$, find the solution to the problem in (a).

(c) How does this relate to the Neyman–Pearson lemma?