Part A Combinatorial Optimisation HT 2007
Problem set 6 Network Flows Week 8

Hand in solutions by Wednesday Week 8 at 10am.

1 Illustrate the algorithm for finding the maximum flow in a network by finding the maximum flow between the vertices 1 and 5 in the network whose vertex set is $\{1, 2, ..., 5\}$, and where the capacity c_{ij} of the directed edge joining vertex i to vertex j is given by the (i, j)-entry in the matrix

$$\begin{pmatrix} 0 & 7 & 9 & 8 & 0 \\ 0 & 0 & 6 & 8 & 4 \\ 0 & 9 & 0 & 2 & 10 \\ 0 & 3 & 7 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Start from the initial flow obtained by routing 4 units via node 2, 9 via node 3, and 6 via node 4.

2 Let N be a network consisting of a digraph (V, A) with a source s and a sink t, and such that each arc a in A has a capacity $c(a)$. Without justification describe how to find, given a flow f of maximum value, a cut C_f of minimum capacity.

Suppose that $M = (m_{ij})$ is an $m \times n$ matrix, every entry of which is 0 or 1. Let the network $N(M)$ have vertices $s, x_1, x_2, \ldots, x_m, y_1, \ldots, y_n, t$ and arcs (s, x_i) for $i = 1, \ldots, m$, (y_j, t) for $j = 1, \ldots, n$, and (x_i, y_j) for all i, j such that $m_{ij} = 1$. Every arc in $N(M)$ is given capacity 1.

(a) Find a cut of minimum capacity in $N(M)$ when M is the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

(b) In the general case, suppose that f is an integer-valued flow of maximum value in $N(M)$, and C_f is a cut of minimum capacity found by the labelling method. Show that if $m_{ij} = 1$ then $x_i \notin C_f$ or $y_j \in C_f$.

The *term rank* of M is the largest number of 1s in M no two of which lie in the same row or column. Prove that the term rank of M is equal to the smallest number of rows and columns which together contain all the 1s of M.

[You may assume that there exists an integer-valued flow of maximum value if all the capacities are integers.]

[This is from an old examination question, paper B9 1994.]

3 An $n \times n$ matrix (X_{ij}) is *doubly stochastic* if each $X_{ij} \geq 0$, each row sum is 1, and each column sum is 1 (i.e. if it is a feasible solution to the $n \times n$ assignment problem). It is a *permutation matrix* if also each $X_{ij} = 0$ or 1 (i.e. if it corresponds to a perfect matching).

(i) Use the integrality theorem to show that if (X_{ij}) is doubly stochastic then there is a permutation matrix (M_{ij}) such that

$$X_{ij} = 0 \Rightarrow M_{ij} = 0.$$

(ii) Deduce that if $\mathbf{X} = (X_{ij})$ is doubly stochastic then for some k there exist permutation matrices $\mathbf{M_1} \ldots, \mathbf{M_k}$, and real numbers $t_1 \ldots, t_k > 0$ with $\Sigma_i t_i = 1$, such that

$$\mathbf{X} = \sum_{\mathbf{i}}^{\mathbf{k}} \mathbf{t_i M_i}.$$

(iii) Finally, deduce the result (already known from the Hungarian algorithm) that the LP relaxation of the assignment problem always has an optimal solution (X^*_{ij}) where each $X_{ij} = 0$ or 1.

4 (a) Let D be a finite directed graph with at least one arc, such that for each node j, if there is an arc ij into j then there is an arc jk out of j. Show that D has a directed cycle.

(b) Recall that a *circulation* is a flow $\mathbf{x}$ such that there is conservation at **all** nodes. Let $\mathbf{y}$ be a non-zero non-negative circulation in a finite directed graph D. Show that there is a directed cycle C such that $y_{ij} > 0$ for each arc ij in C. Hence show that $\mathbf{y}$ can be written as a sum of positive flows around directed cycles. Further, show that if $\mathbf{y}$ is also integral then it can be written as a sum of unit flows around directed cycles.

5 The following matrix sets out the times taken by three different workers to carry out three different tasks. An assignment is required which minimises the total time needed.

	Tasks		
	11	2	4
Workers	5	6	7
	4	4	10

Starting with the assignment represented by the leading diagonal, use a method based on a general iterative *minimum-cost-flow* algorithm to determine all the optimal assignments.

[*You may identify any non-positive cycles by inspection.*]

[This is part of an old examination question, paper B10 1998.]

finished!