2 Risk aversion and changes in risk

1. (a) Denoting the risk premium of the lottery by π,

$$u(4-\pi) = (4-\pi)^2 = \frac{1}{2}2^2 + \frac{1}{2}6^2 = 20$$

$\therefore \pi = 4 - \sqrt{20} = -0.472$. The risk premium is negative because the utility function is convex over the range of interest ($u'' = 2 > 0$), and therefore the decision maker is willing to pay to take on a zero mean risk ('risk loving').

(b) Denoting the new risk premium of the lottery by π',

$$v(4-\pi') = (4-\pi')^4 = \frac{1}{2}2^4 + \frac{1}{2}6^4 = 656$$

$\therefore \pi' = 4 - \sqrt[4]{656} = -1.061$, i.e. the risk premium decreases (or equivalently the decision maker is willing to pay a higher price to take on the zero mean risk). $v(w) = f(u(w))$ where $f(u) = u^2$, a convex function ($f''(u) > 0$).

2. $A(w) := -\frac{u''(w)}{u'(w)} = -\frac{d}{dw}\ln u'(w)$. Integrating over $[z_0, z]$ gives $\int_{z_0}^{z} A(w)dw := -\ln(u'(z)) + \ln(u'(z_0))$, and so $u'(z) = u'(z_0)\exp\left\{-\int_{z_0}^{z} A(w)dw\right\}$, which can be rewritten in the desired form.

3. If the decision maker has preferences that satisfy constant relative risk aversion then preferences can be represented by a utility function $u(w) = \frac{w^{1-\gamma}}{1-\gamma}$ for some $\gamma \in \mathbb{R}$ (or $\ln(w)$ for the case of $\gamma = 1$).

Utility of certain wealth w is $\frac{w^{1-\gamma}}{1-\gamma}$ and expected utility of lottery $w(1+\tilde{x})$ is

$$\begin{aligned}\mathbb{E}\left[u(w(1+\tilde{x}))\right] &= \mathbb{E}_{\tilde{x}}\left[\frac{[w(1+\tilde{x})]^{1-\gamma}}{1-\gamma}\right] \\ &= \frac{w^{1-\gamma}}{1-\gamma} \times \mathbb{E}_{\tilde{x}}\left[\frac{(1+\tilde{x})^{1-\gamma}}{1-\gamma}\right]\end{aligned}$$

and so the lottery will be (strictly) preferred to certain wealth iff $\mathbb{E}\left[\frac{(1+\tilde{x})^{1-\gamma}}{1-\gamma}\right] \geq (>)1$. This condition does not depend on w.

Similarly, for logarithmic utility function $u(w) = \ln(w)$, $\mathbb{E}[u(w(1+\tilde{x}))] = \ln(w) + \mathbb{E}[\ln(1+\tilde{x})]$ which is (strictly) greater than $\ln(w)$ iff $\mathbb{E}[\ln(1+\tilde{x})] \geq (>)0$. Again, this condition does not depend on w.

4. (a) u is piecewise linear with gradient $u'(z) = 1$ for $z < z_0$ and $u'(z) = a < 1$ for $z > z_0$. We must show that for any $z, z' \in \mathbb{R}$, $\lambda \in [0,1]$,

$$\lambda u(z) + (1-\lambda)u(z') \leq u(\lambda z + (1-\lambda)z').$$

If $z, z' \geq z_0$ or $z, z' \leq z_0$ then this weak inequality is satisfied with equality since u is linear between z and z'. If $z \leq z_0$ and $z' \geq z_0$ then from the definition of u,

$$\lambda u(z) + (1-\lambda)u(z') = \lambda z + (1-\lambda)az' + (1-\lambda)(1-a)z_0$$

There are two cases. For $\lambda z + (1-\lambda)z' \le z_0$,

$$\begin{aligned} u(\lambda z + (1-\lambda)z') &= \lambda z + (1-\lambda)z' \\ \therefore \lambda u(z) + (1-\lambda)u(z') - u(\lambda z + (1-\lambda)z') &= (1-a)(1-\lambda)(z_0 - z') \le 0 \end{aligned}$$

since $z' \ge z_0$ by assumption, and in the second case, $\lambda z + (1-\lambda)z' \ge z_0$,

$$\begin{aligned} u(\lambda z + (1-\lambda)z') &= (1-a)z_0 + a(\lambda z + (1-\lambda)z') \\ \therefore \lambda u(z) + (1-\lambda)u(z') - u(\lambda z + (1-\lambda)z') &= (1-a)\lambda(z - z_0) \le 0 \end{aligned}$$

since $z \le z_0$ by assumption.

(b) Denoting the cdf of $\tilde{x}$ as $F(x)$, where $\mathbb{E}[\tilde{x}] = 0$, the risk premium $\pi(z_0, u, k\tilde{x})$ is nonnegative since u is concave, and therefore satisfies

$$\begin{aligned} u(z_0 - \pi) &= z_0 - \pi = \mathbb{E}[u(z_0 + k\tilde{x})] \\ &= \int_{-\infty}^{0} (z_0 + kx)dF(x) + \int_{0}^{\infty} ((1-a)z_0 + a(z_0 + kx))dF(x) \\ &= z_0 + k\left[\int_{-\infty}^{0} xdF(x) + a\int_{0}^{\infty} xdF(x)\right] \\ &= z_0 + k\left[\mathbb{E}[\tilde{x}] - (1-a)\int_{0}^{\infty} xdF(x)\right] \\ &= z_0 - k(1-a)\int_{0}^{\infty} xdF(x) \\ \therefore \pi(z_0, u, k\tilde{x}) &= k(1-a)\int_{0}^{\infty} xdF(x) \end{aligned}$$

For $\tilde{x}$ non-degenerate, $\mathbb{E}[\tilde{x}] = 0$ implies $\int_0^\infty xdF(x)$ is positive and finite and so $\pi(z_0, u, k\tilde{x})$ is linear in k and $\lim_{k \to 0} \pi(z_0, u, k\tilde{x}) = 0$.

(c) Consider

$$\phi(z) := \begin{cases} z & \text{if } z \le z_0 \\ z_0 + b(z - z_0) & \text{if } z > z_0 \end{cases}.$$

where $0 < b < 1$. We therefore have

$$\phi(u(z)) := \begin{cases} z & \text{if } z \le z_{)} \\ z_0 + ab(z - z_0) & \text{if } z > z_0 \end{cases}.$$

ϕ is an increasing concave transformation from the first part of this exercise. Now consider a decrease in parameter a of function u to a'. The new utility function is the same as $\phi \circ u$ where ϕ is defined with parameter $b = a'/a \in [0, 1]$. A reduction in a to a' therefore increases the degree of risk aversion because it can result from transforming u with an increasing, concave function ϕ.

(d) We prove by finding a pure risk with zero risk at wealth w_0 and strictly positive risk premium at wealth $w_0' > w_0$. This is sufficient to show that u does not exhibit decreasing absolute risk aversion.

Let the pure risk to consider be the lottery $\tilde{x} := \left(+x, \frac{1}{2}; -x, \frac{1}{2}\right)$ and let $w_0, w_0 + x \leq z_0$. Then the risk premium is zero at wealth level w_0 since from the definition of the risk premium $w_0 - \pi(w_0, u, \tilde{x}) = 0.5(w_0 - x) + 0.5(w_0 + x) = w_0$. Now consider a level of wealth $w_0' > w_0$ such that $z_0 - x < w_0' < z_0$. The risk premium $\pi(w_0', u, \tilde{x})$ is defined by

$$\begin{aligned} 0.5(w_0' - x) + 0.5((1-a)z_0 + a(w_0' + x)) &= w_0' - \pi(w_0', u, \tilde{x}) \\ \therefore \pi(w_0', u, \tilde{x}) &= 0.5(1-a)(w_0' + x - z_0) > 0 \end{aligned}$$

since $w_0' + x > z_0$ by construction and $a \in (0, 1)$

5. (a) Bill has initial wealth $w_0 = 22$ and the project has outcome $\tilde{x} := \left(-12, \frac{1}{2}; +18, \frac{1}{2}\right)$. $\mathbb{E}\tilde{x} \neq 0$ and so the risk premium is undefined. The certainty equivalent θ is the solution to the following equation

$$\begin{aligned} &\ln(22 + \theta) = \mathbb{E}u(w_0 + \tilde{x}) = \frac{1}{2}\ln(40) + \frac{1}{2}\ln(10) = \frac{1}{2}\ln(400) = \ln(20) \\ &\therefore \theta = 20 - 22 = -2 \end{aligned}$$

He should reject the project since this maximises expected utility (ln(22) instead of ln(20)).

(b) The new choice is between w_0 and $w_0 + \tilde{y}$ where $\tilde{y} := \frac{\tilde{x}}{2} = \left(-6, \frac{1}{2}; +9, \frac{1}{2}\right)$. Bill's expected utility from $w_0 + \tilde{y}$ is

$$\mathbb{E}u(w_0 + \tilde{y}) = \frac{1}{2}\ln(16) + \frac{1}{2}\ln(31) = \ln(\sqrt{496}) = \ln(22.27\ldots) > \ln(22)$$

Bill should accept this offer since this increases expected utility.

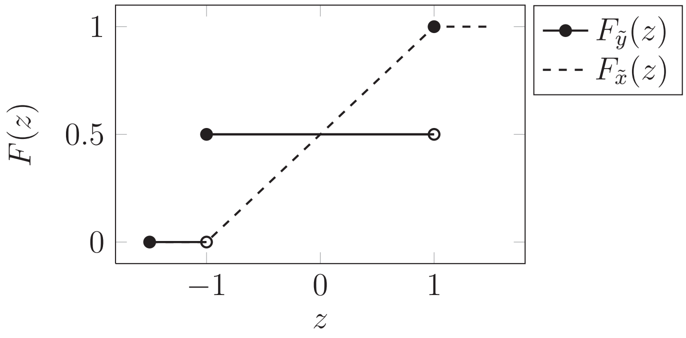
(c) If they take on two statistical independent projects and split the proceeds then Bill's (and Ben's) income distribution would be $w_0 + \tilde{z}$ where $\tilde{z} := (-12, \frac{1}{4}; +3, \frac{1}{2}; +18, \frac{1}{4})$ which yields expected utility of

$$\mathbb{E}u(w_0 + \tilde{z}) = \frac{1}{4}\ln(10) + \frac{1}{2}\ln(25) + \frac{1}{4}\ln(40) = \ln(\sqrt{500}) = \ln(22.36\ldots)$$

This gives the highest of the four options presented so far (no project, one project with Bill bearing all the risk, one project with Bill and Ben sharing the risk, and two projects with Bill and Ben sharing the risks) so they should pool their risks and invest in the projects.

(d) No. Logarithmic utility satisfies constant relative risk aversion and (from question 3) multiplying their wealth and project payoffs by the same factor would not affect choices.

6. (a) See the following figure



(b) Both distributions have support on the interval $[-1, 1]$. We have

$$\int_{-1}^{\theta} F_{\tilde{x}}(z)dz = \int_{-1}^{\theta} \frac{z+1}{2}dz = \left[\frac{z^2}{4} + \frac{z}{2}\right]_{-1}^{\theta} = \frac{\theta^2}{4} + \frac{\theta}{2} + \frac{1}{4}$$

$$\int_{-1}^{\theta} F_{\tilde{y}}(z)dz = \int_{-1}^{\theta} \frac{1}{2}dz = \left[\frac{z}{2}\right]_{-1}^{\theta} = \frac{\theta}{2} + \frac{1}{2}$$

$$\therefore \int_{-1}^{\theta} F_{\tilde{y}}(z) - F_{\tilde{x}}(z)dz = \frac{1-\theta^2}{4} \begin{cases} = 0 \text{ for } \theta = 1 \\ > 0 \text{ for } \theta \in [0, 1) \end{cases}$$

$\tilde{x}$ therefore second order stochastically dominates $\tilde{y}$, and so $\tilde{y}$ is said to be riskier than $\tilde{x}$.

(c) For each outcome x of $\tilde{x}$ add a zero mean noise $\tilde{\varepsilon}_x := \{1-x, \frac{1+x}{2}; -(1+x), \frac{1-x}{2}\}$. Conditional on any x, there are two potential outcomes, $x + 1 - x = +1$ and $x - (1 + x) = -1$, and $\mathbb{E}[\tilde{\varepsilon}_x] = 0$. By symmetry, and over all possible outcomes of x, these two outcomes of $\tilde{x} + \tilde{\varepsilon}$ have equal probability of $\frac{1}{2}$, where conditional on $\tilde{x}$, $\tilde{\varepsilon}$ has distribution $\tilde{\varepsilon}_x$, and so $\tilde{x} + \tilde{\varepsilon} \sim (-1, \frac{1}{2}; +1, \frac{1}{2})$, which is the distribution of $\tilde{y}$.

7. Let $\tilde{A} = (80, \frac{1}{4}; 100, \frac{1}{4}; 120, \frac{1}{4}; 140, \frac{1}{4})$, $\tilde{B} = (90, \frac{1}{2}; 130, \frac{1}{2})$ and $\tilde{\varepsilon} = (+10, \frac{1}{2}; -10, \frac{1}{2})$. Then $\tilde{B} + \tilde{\varepsilon} \sim \tilde{A}$. Conditional on each outcome of $\tilde{B}$, $\tilde{\varepsilon}$ has zero mean. Therefore by by the Rothschild-Stiglitz Theorem, Project B SSD Project A.

8. This questions is discursive in nature and insightful comments, other than those here, would receive credit.

Based on the following reasoning I would suggest the following ordering from highest to lowest σ: σ_b, σ_c, σ_a.

In situation (b) the CEO has very limited exposure to downside risk; the manager earns the same income if the company is marginally bankrupt ($\tilde{v}$ is small and negative) as if the company is spectacularly bankrupt ($\tilde{v}$ is large and negative). Loosely speaking, her incentive is to create a distribution of outcomes with a ‘fat upside tail’, regardless of the mean outcome or downside risk. This is called ‘gambling for resurrection’, where a decision maker takes on additional risk to increase upside potential because she has limited exposure to downside potential. The CEO might even be willing to choose a high σ if $\alpha = 0$ since $\mathbb{E} \max[0, \tilde{v}]$ is likely to be increasing in σ.

In situations (a) and (c) the CEO has (at least partial) exposure to downside risk and so is likely to choose a lower level of risk than in (b). The restrictions on the CEO's utility function mean that her preferences satisfy DARA. Since in she has the same choice of risk to take in (a) and (c) but has higher higher wealth (i.e. higher fixed income) in (c) she will optimally take more risk in (c) than in (a).