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Insurance decisions

Actuarial Science, HT 2012
Daniel Clarke

Indemnity insurance and loading

Definition. An *indemnity insurance contract* is a pair $(P, I(\cdot))$ where the premium $P \geq 0$ is paid to the insurer and claim payment is $I(x) \geq 0$ when loss is x.

Definition. For a random nonnegative loss $\tilde{x}$ and an indemnity insurance contract which pays $I(x)$ in each state x and costs a premium of P, the *loading* λ is defined such that

$$P = (1 + \lambda)\mathbb{E}I(\tilde{x})$$

The product is termed

- *Actuarially fair* if $\lambda = 0$
- *Actuarially unfair* if $\lambda > 0$

Note. Most real life insurance products are actuarially unfair, since the insurer must cover expected claim outgo as well as expenses and profits.

General environment

We will consider the following environment

- Decision maker is strictly nonsatiated and risk averse with twice differentiable utility function u
  - $u'(x) > 0, u''(x) < 0 \forall x.$
- Initial wealth of w_0 but subject to risky loss of $\tilde{x}$
  - Final net wealth is $w_0 - \tilde{x}$
  - Assume that $\tilde{x}$ is nondegenerate (i.e. it really is risky) and is nonnegative
- Indemnity insurance can be purchased which pays $I(x) \geq 0$ when loss is x, and is priced with a loading of $\lambda \geq 0$.

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Insurance decisions: optimal coinsurance

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Mossin's Theorem

Suppose that the decision maker must choose an optimal level of *coinsurance* $\beta \in [0,1]$:

- $I_\beta(x) = \beta x$ when the loss is x
- Premium $P(\beta) = (1+\lambda)\mathbb{E}[\beta\tilde{x}] = \beta P_0$ where $P_0 \coloneqq (1+\lambda)\mathbb{E}[\tilde{x}]$
- Denote realised wealth $\tilde{y} \coloneqq w_0 - \beta P_0 - \tilde{x} + \beta\tilde{x}$

Proposition (Mossin's Theorem). Full insurance ($\beta^* = 1$) is optimal at an actuarially fair price ($\lambda = 0$), while partial coverage ($\beta^* < 1$) is optimal at an actuarially unfair price ($\lambda > 0$)

Proof of Mossin's Theorem

Proof. The policy maker will choose β to maximise expected utility of final wealth, denoted

$$H(\beta) \coloneqq \mathbb{E}u(\tilde{y})$$

The optimisation problem is therefore

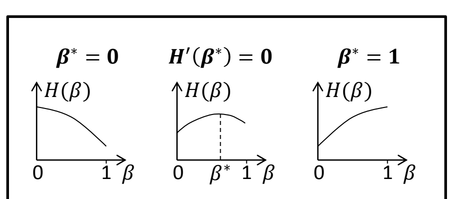
$$\max_\beta H(\beta) \;\; s.t. \;\beta \in [0,1]$$

This optimisation program is continuous and strictly concave ($H''(\beta) < 0$):

$$H(\beta) = \mathbb{E}u(\tilde{y}) = \mathbb{E}[u(w_0 - \beta P_0 - (1-\beta)\tilde{x})]$$
$$H'(\beta) = \mathbb{E}[(\tilde{x} - P_0)u'(\tilde{y})]$$
$$H''(\beta) = \mathbb{E}[(\tilde{x} - P_0)^2 u''(\tilde{y})]$$

and so there is a unique maximum β^* such that

- Interior solution: $H'(\beta^*) = 0$
- Corner solution: $\beta^* = 0$ or $\beta^* = 1$



Proof of Mossin's Theorem

First, consider

$$H'(1) = \mathbb{E}[(\tilde{x} - P_0)u'(w_0 - P_0)]$$
$$= u'(w_0 - P_0)(\mathbb{E}[\tilde{x}] - P_0)$$
$$= u'(w_0 - P_0)(\mathbb{E}[\tilde{x}] - (1 + \lambda)\mathbb{E}[\tilde{x}])$$
$$= -\lambda u'(w_0 - P_0)\mathbb{E}[\tilde{x}]$$

So we have

$$\lambda \leq 0 \Rightarrow H'(1) \geq 0$$

So if there is no (or negative) insurance loading, the optimal contract is full insurance ($\beta^* = 1$).

- This is not surprising: risk averter will prefer to swap uncertain wealth for certain wealth with same expected value.

Proof of Mossin's Theorem

We also have

$$H'(0) = \mathbb{E}[(\tilde{x} - P_0)u'(w_0 - \tilde{x})]$$
$$= -\mathbb{E}[(P_0 - \mathbb{E}[\tilde{x}])u'(w_0 - \tilde{x})] + \mathbb{E}[(\tilde{x} - \mathbb{E}[\tilde{x}])u'(w_0 - \tilde{x})]$$
$$= -\lambda\mathbb{E}[\tilde{x}]\mathbb{E}[u'(w_0 - \tilde{x})] + cov\big(\tilde{x}, u'(w_0 - \tilde{x})\big)$$

Now if $H'(0) \leq 0$

$$\therefore \lambda \geq \lambda^* \coloneqq \frac{cov\big(\tilde{x}, u'(w_0 - \tilde{x})\big)}{\mathbb{E}[\tilde{x}]\mathbb{E}[u'(w_0 - \tilde{x})]}$$

then corner solution $\beta^* = 0$ is optimal.

- So if the insurance is too expensive ($\lambda \geq \lambda^*$), then zero purchase will be optimal.

For $0 < \lambda < \lambda^*$, $H'(1) < 0 < H'(0)$ and positive, partial insurance is optimal ($0 < \beta^* < 1$)

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Comparative statics: more risk averse

Proposition. Consider two utility functions u and v that are increasing and concave, and suppose that u is more risk averse than v (in the sense of Arrow Pratt). Then, the optimal coinsurance rate β^* is higher for u than v: $\beta_u^* \geq \beta_v^*$

Proof. If $\lambda = 0$ then $\beta_u^* = \beta_v^* = 1$ and we are done.

Otherwise, without loss of generality suppose that

$$u'(w_0 - P_0) = v'(w_0 - P_0)$$

(We can do this since expected utility is cardinal.)

Then it must be that

$$u'(y) \geq v'(y) \forall y < w_0 - P_0$$
$$u'(y) \leq v'(y) \forall y > w_0 - P_0$$

and so

$$(x - P_0)v'(w_0 - (1 - \beta_u^*)x - \beta_u^* \mathrm{P}_0) \leq (x - P_0)u'(w_0 - (1 - \beta_u^*)x - \beta_u^* \mathrm{P}_0) \; \forall x$$

Since for $x > P_0$ both sides are positive and the RHS is larger, and for $x < P_0$ both sides are negative and the RHS is less negative.

Comparative statics: more risk averse

$$(x - P_0)v'(w_0 - (1 - \beta_u^*)x - \beta_u^* \mathrm{P}_0) \leq (x - P_0)u'(w_0 - (1 - \beta_u^*)x - \beta_u^* \mathrm{P}_0) \; \forall x$$

Taking expectations of both sides gives

$$H_v'(\beta_u^*) = \mathbb{E}[(x - P_0)v'(w_0 - (1 - \beta_u^*)x - \beta_u^* \mathrm{P}_0)] \leq \mathbb{E}[(x - P_0)u'(w_0 - (1 - \beta_u^*)x - \beta_u^* \mathrm{P}_0)] = H_u'(\beta_u^*) = 0$$

Where the last equality follows from the first order condition for β_u^*.

Hence, since H_v' is concave, it follows that $\beta_v^* \leq \beta_u^*$.

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Recall from week 2

For our next proposition we'll be applying the following result form week 2 (proposition demonstrating equivalence between DARA and $A'(w) \leq 0$). DARA implies:

- You can think of $v = -u'$ as a risk-averse utility function (since $v' = -u'' > 0$ and $v'' = -u''' < 0$)
- $v = -u'$ is more risk averse than u

Comparative statics: an increase in initial wealth

Proposition. If u exhibits decreasing absolute risk aversion then an increase in initial wealth will decrease the optimal rate of coinsurance β^*.

Proof. Let β^* be optimal for $w = w_0$. Now consider

$$\frac{\partial H'(\beta)}{\partial w} = \frac{\partial \mathbb{E}[(\tilde{x} - P_0)u'(\tilde{y})]}{\partial w}$$

$$\therefore \frac{\partial H'(\beta)}{\partial w} = \mathbb{E}[(\tilde{x} - P_0)u''(\tilde{y})] \quad (*)$$

Since H is strictly concave in β, we need only show that $(*)$ is negative when evaluated at β^*. (This implies that as wealth increases from w_0, the gradient of $H'(\beta)$ evaluated at β^* become negative, which from the strict concavity of $H(\beta)$ will imply that the new optimal β at a higher level of wealth will be lower than β^*.)

To show this, recall that DARA implies that $(-u')$ has the properties of a risk-averse utility function and is more risk averse than u. (We proved this in proposition demonstrating equivalence between DARA and $A'(w) \leq 0$ in week 2.)

Comparative statics: an increase in initial wealth

$$\frac{\partial H'(\beta)}{\partial w} = \mathbb{E}[(\tilde{x} - P_0)u''(\tilde{y})] \quad (*)$$

Now, reversing the order of differentiation in $(*)$ and recalling that $H(\beta) \coloneqq \mathbb{E}u(\tilde{y})$ note that

$$\frac{\partial H'(\beta)}{\partial w} = \frac{\partial^2 \mathbb{E}u(\tilde{y})}{\partial \beta \partial w} = \frac{\partial \mathbb{E}[u'(\tilde{y})]}{\partial \beta}$$

$$\therefore \frac{\partial \mathbb{E}[-u'(\tilde{y})]}{\partial \beta} = -\mathbb{E}[(\tilde{x} - P_0)u''(\tilde{y})] \quad (\#)$$

By the previous proposition we know that the optimal level of coinsurance for $v = -u'$ is higher than the level for u, so the gradient of $H_v(\beta)$, evaluated at β^* must be strictly greater than zero. Since the LHS of $(\#)$ is > 0 the RHS of $(\#)$ must be > 0, that is $-\mathbb{E}[(\tilde{x} - P_0)u''(\tilde{y})] > 0$ when evaluated at $\beta = \beta^*$. In turn it follows that $(*)$ is negative when evaluated at $\beta = \beta^*$.

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Insurance decisions: optimality of deductibles

Actuarial Science, HT 2012

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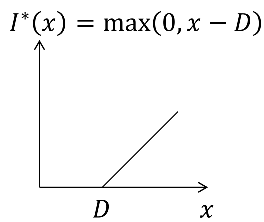
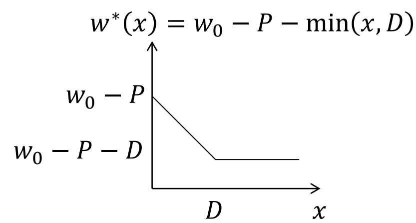
Optimality of deductibles

Proposition. Suppose a risk averse policyholder selects an insurance contract $\left(P, I(\cdot)\right)$ with $P = (1+\lambda)\mathbb{E}[I(\tilde{x})]$ and with $I(x)$ nondecreasing and $I(x) \geq 0$ for all x. Then the optimal contract contains a straight deductible D; that is $I^*(x) = \max(0, x - D)$ is optimal for some D.





Optimality of deductibles

Proof. (We prove for discrete loss distribution where p_x denotes $\mathbb{P}[\tilde{x} = x]$)

We will show that any other $I(x)$ is second order stochastically dominated by $I^*(x) = \max(0, x - D)$.

Consider another insurance policy $\left(P, I(\cdot)\right)$ with the same premium P. Since the loading is the same we must have

$$\mathbb{E}[I(x)] = \frac{P}{1+\lambda} = \mathbb{E}[I^*(x)]$$

Consider some x_i such that $I(x_i) = I^*(x_i) + \varepsilon_i$ for $\varepsilon_i > 0$. For $\mathbb{E}[I(x)] = \mathbb{E}[I^*(x)]$ there must be some other loss level/s x_j such that $I\left(x_j\right) = I^*\left(x_j\right) - \varepsilon_j$ for $\varepsilon_j > 0$ and $\sum p_i x_i = \sum p_j x_j$.

(#) Since each $I\left(x_j\right) \geq 0$ it must be that each $I^*\left(x_j\right) > 0$, or equivalently $x_j > D$.

Optimality of deductibles

Now, consider the realised net wealth distribution under indemnity function $I^*(x)$,

$$w^*(x) = w_0 - P - \min(x, D)$$

Which attains its minimum of $w_0 - P - D$ for all $x \geq D$.

Now the realised net wealth distribution under $I(x)$

$$w(x) = w_0 - P - x + I(x)$$

increases net wealth in states with $w^*(x) \geq w_0 - P - D$ and *decreases* net wealth in states with $w^*(x) = w_0 - P - D$ (from observation (#) on previous slide), such that $\mathbb{E}[w(x)] = \mathbb{E}[w^*(x)]$.

Therefore $w(x)$ can be obtained from $w^*(x)$ by a mean preserving spread around net wealth of $w_0 - P - D$.

Therefore $w(x)$ is second order stochastically dominated by $w^*(x)$.

■

Interpreting Arrow's Theorem on the optimality of deductibles

- For any given insurance budget (if the insurance loading is constant) the optimal insurance contract concentrates indemnification on the worst outcomes, where the marginal utility of additional wealth is highest.
  - where $u'(w)$ is largest
  - if $u'' > 0$ this is when w is lowest.
- Small risks 'should not' be insured
- Large risk 'should' insured:
  - Premature death of parent/guardian (life insurance)
  - Destruction of house (homeowners insurance)
  - Damage to your expensive car (vehicle insurance)
  - Damage to someone else's expensive car (Third party motor vehicle insurance)

Financial markets don't yet offer catastrophe insurance to some of the most vulnerable

Table 9

The 40 worst catastrophes in terms of victims 1970–2010

Victims[9]	**Insured loss**[10] (in USD m, indexed to 2009)	**Date** (start)	**Event**	**Country**
300000	–	14.11.1970	Storm and flood catastrophe	Bangladesh, Bay of Bengal
255000	–	28.07.1976	Earthquake (M 7.5)	China
[illegible]	[illegible]	[illegible]	Earthquake (M_W [illegible])	[illegible]
220000	2309	26.12.2004	Earthquake (M_W 9), tsunami in Indian Ocean	Indonesia, Thailand et al
[illegible]	–	[illegible]	Tropical cyclone Nargis; Irrawaddy Delta flooded	Myanmar (Burma), Bay of Bengal
138000	3	29.04.1991	Tropical cyclone Gorky	Bangladesh
87449	371	12.05.2008	Earthquake (M_W 7.9) in Sichuan, aftershocks	China
73300	–	08.10.2005	Earthquake (M_W 7.6); aftershocks, landslides	Pakistan, India, Afghanistan
66000	–	31.05.1970	Earthquake (M 7.7); rock slides	Peru
55630	–	15.06.2010	Heat wave in Russia	Russia
40000	192	21.06.1990	Earthquake (M 7.7); landslides	Iran
35000	–	01.06.2003	Heat wave and drought in Europe	France, Italy, Germany et al
26271	–	26.12.2003	Earthquake (M 6.5) destroys 85% of Bam	Iran
25000	–	07.12.1988	Earthquake (M 6.9)	Armenia, ex-USSR
25000	–	16.09.1978	Earthquake (M 7.7) in Tabas	Iran
23000	–	13.11.1985	Volcanic eruption on Nevado del Ruiz	Colombia
22084	287	04.02.1976	Earthquake (M 7.5)	Guatemala
19737	123	26.01.2001	Earthquake (M_W 7.6) in Gujarat	India, Pakistan, Nepal et al
19118	1309	17.08.1999	Earthquake (M_L 7) in Izmit	Turkey
15000	–	11.08.1979	Macchu dam burst in Morvi	India
15000	–	01.09.1978	Floods following monsoon rains in the north	India, Bangladesh
15000	131	29.10.1999	Cyclone 05B devastates Orissa state	India, Bangladesh
11069	–	25.05.1985	Tropical cyclone in Bay of Bengal	Bangladesh
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Source: Swiss Re sigma No 1/2011

Financial markets don't yet offer catastrophe insurance to some of the most vulnerable

Table 8

The 40 most costly insurance losses 1970–2010

Insured loss[7] (in USD m, indexed to 2010)	**Victims**[8]	**Date** (start)	**Event**	**Country**
72302	1836	25.08.2005	Hurricane Katrina; floods, dams burst, damage to oil rigs	US, Gulf of Mexico, Bahamas, North Atlantic
24870	43	23.08.1992	Hurricane Andrew; floods	US, Bahamas
23131	2982	11.09.2001	Terror attack on WTC, Pentagon and other buildings	US
20601	61	17.01.1994	Northridge earthquake (M 6.6)	US
[illegible]	[illegible]	[illegible]	Hurricane Ike; floods, offshore damage	US, Caribbean; Gulf of Mexico et al
14876	124	02.09.2004	Hurricane Ivan; damage to oil rigs	US, Caribbean; Barbados et al
[illegible]	[illegible]	[illegible]	Hurricane Wilma; floods	US, Mexico, Jamaica, Haiti et al
[illegible]	[illegible]	[illegible]	Hurricane Rita; floods, damage to oil rigs	US, Gulf of Mexico, Cuba
9295	24	11.08.2004	Hurricane Charley; floods	US, Cuba, Jamaica et al
[illegible]	[illegible]	[illegible]	[illegible]	Japan
8043	71	15.09.1989	Hurricane Hugo	US, Puerto Rico et al
8000	562	27.02.2010	Earthquake (M_W 8.8) triggers tsunami	Chile
7794	95	25.01.1990	Winter storm Daria	France, UK, Belgium, NL et al
7594	110	25.12.1999	Winter storm Lothar	Switzerland, UK, France et al
6410	54	18.01.2007	Winter storm Kyrill; floods	Germany, UK, NL, Belgium et al
[illegible]	[illegible]	[illegible]	Storm and floods in Europe	France, UK, Netherlands et al
5941	38	26.08.2004	Hurricane Frances	US, Bahamas
[illegible]	[illegible]	[illegible]	Winter storm Vivian	Europe
5290	26	22.09.1999	Typhoon Bart/No 18	Japan
4723	600	20.09.1998	Hurricane Georges; floods	US, Caribbean
4453	–	04.09.2010	Earthquake (M_W 7.0)	New Zealand
[illegible]	41	[illegible]	Tropical storm Allison; floods	US
4390	3034	13.09.2004	Hurricane Jeanne; floods, landslides	US, Caribbean: Haiti et al
4139	45	06.09.2004	Typhoon Songda/No 18	Japan, South Korea
3800	45	02.05.2003	Thunderstorms, tornadoes, hail	US

Source: Swiss Re sigma No 1/2011

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Static portfolio choices

Actuarial Science, HT 2012

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General environment

We will consider the following environment

- Decision maker is strictly nonsatiated and risk averse with twice differentiable utility function u
  - $u'(x) > 0, u''(x) < 0 \forall x.$
- Initial wealth of w_0 which can be invested in one risk-free asset and in one risky asset.
  - Can invest α in a risky asset (e.g. stock) with uncertain return $\tilde{x}$.
  - With $w_0 - \alpha$ in the risk free asset (e.g. bond) with certain return r.
- Value of the realised portfolio is

$$(w_0 - \alpha)(1 + r) + \alpha(1 + \tilde{x}) = w_0(1 + r) + \alpha(\tilde{x} - r)$$
$$= w + \alpha\tilde{y}$$

  - where $w \coloneqq w_0(1 + r)$ is future wealth under risk-free strategy
  - and $\tilde{y} \coloneqq \tilde{x} - r$ is excess return from risky asset.

Equivalence between static portfolio choice and coinsurance choice

The problem of the investor is to choose α to maximise EU:

$$\alpha^* \in \arg\max_{\alpha} \mathbb{E}u(w + \alpha\tilde{y})$$

This problem is formally equivalent to the coinsurance problem from yesterday. To see this, define

$$w \equiv w' - P_0, \qquad \alpha \equiv (1-\beta)P_0, \qquad \tilde{y} \equiv \frac{P_0 - \tilde{x}}{P_0}$$

Consequently

$$\mathbb{E}u(w + \alpha\tilde{y}) = \mathbb{E}u\left[(w' - P_0) + (1-\beta)P_0\left(\frac{P_0 - \tilde{x}}{P_0}\right)\right]$$
$$= \mathbb{E}u(w' - \beta P_0 - (1-\beta)\tilde{x})$$

- We may interpret $\alpha = 0$ as full insurance coverage (zero risk)
- By increasing α (i.e. decreasing the coinsurance level β) the consumer accepts some of the risk in exchange for a higher expected final wealth.
- Increasing loading λ in the coinsurance model is equivalent to increasing the mean of $\tilde{y}$ in the static portfolio choice model.

Optimal static portfolio choices

Proposition. Consider the static portfolio choice problem, where $\tilde{y}$ is the return of the asset over the risk-free rate, and α^* is the optimal dollar investment in the risky asset. Then the optimal investment in the risky asset is positive iff the expected excess return is positive: $\alpha^* = 0$ if $\mathbb{E}\tilde{y} = 0$ and $\alpha^* > 0$ if $\mathbb{E}\tilde{y} > 0$. Moreover, when the expected excess return is positive,

1. α^* is reduced when the risk aversion of the investor is increased in the sense of Arrow and Pratt;
2. α^* is increasing in wealth if absolute risk aversion is decreasing.

Proof. See proofs of comparative statics results for coinsurance problem.

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Worked example: Static portfolio choices under CRRA

Proposition. Under constant relative risk aversion, the demand for the risky asset is proportional to wealth: $\alpha^*(w) = kw$.

Proof. Suppose that

$$u'(c) = c^{-\gamma}\ \forall c$$

Where γ is the coefficient of relative risk aversion. Under this specification, the first order condition may be written as

$$\mathbb{E}[\tilde{y}u'(w + \alpha^*\tilde{y})] = 0$$
$$\therefore\ \mathbb{E}[\tilde{y}(w + \alpha^*\tilde{y})^{-\gamma}] = 0$$
$$\therefore\ \mathbb{E}[\tilde{y}(1 + k\tilde{y})^{-\gamma}] = 0$$

Where k is a positive constant and $\alpha^* = kw$.

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Static optimisation: some useful results

Actuarial Science, HT 2012

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Preliminaries

Consider

- $f(\boldsymbol{x})$ is a real-valued function of n real variables, $f: \mathbb{R}^n \to \mathbb{R}$.
- S is a subset of $\mathbb{R}^n$, $S \subset \mathbb{R}^n$.
- f is twice continuously differentiable on S
- Df is the Jacobian matrix and $D^2 f$ the Hessian:

$$Df = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_n} & \cdots & \frac{\partial f}{\partial x_n} \end{pmatrix}$$

$$D^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

The general optimisation problem

The general optimisation problem is to maximise (or minimise) f over some subset S:

$$\text{maximise } f(\boldsymbol{x}), \text{ subject to } \boldsymbol{x} \in S$$

$\boldsymbol{x} \in S$ is a **global maximum** of f on S if $f(\boldsymbol{x}) \geq \mathrm{f}(\boldsymbol{y}) \forall \boldsymbol{y} \in S$.

$\boldsymbol{x} \in S$ is a **local maximum** of f on S if $f(\boldsymbol{x}) \geq \mathrm{f}(\boldsymbol{y}) \forall \boldsymbol{y}$ in some neighbourhood of x.

The Weierstrass Theorem. Let $S \in \mathbb{R}^n$ be compact and $f: S \to \mathbb{R}$ be a continuous function on S. Then f attains a global maximum and global minimum on S.

Concavity and the general optimisation problem

Theorem. If f has a local maximum or minimum at an interior point $\boldsymbol{x}^*$ of S, then $Df(\boldsymbol{x}^*) = 0$.

Theorem. Consider the general optimisation problem:

$$\text{maximise } f(\boldsymbol{x}), \text{ subject to } \boldsymbol{x} \in S$$

If f is (strictly) concave then

- If a local maximum exists it is a (unique) global maximum;
- If a stationary point exists, it is a (unique) global maximum.

Lagrange's Theorem

Consider the problem:

$$P\!: \ \max f(\boldsymbol{x}) \text{ such that } h(\boldsymbol{x}) = \boldsymbol{a} \ (h\!: \mathbb{R}^n \to \mathbb{R}^m)$$

Define the *Lagrangian* L as

$$L(\boldsymbol{x}; \boldsymbol{\lambda}) = f(\boldsymbol{x}) + \sum_{j=1}^{m} \lambda_j \big(a_j - h_j(\boldsymbol{x})\big)$$

Lagrange's Theorem. If $\boldsymbol{x}^*$ is a local maximum of f subject to the constraint $h(\boldsymbol{x}) = \boldsymbol{a}$, and the matrix $Dh(\boldsymbol{x}^*)$ is of full rank, m, then $\exists \lambda^* = (\lambda_1^*, \ldots, \lambda_m^*)$ such that $D_x L(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) = 0$ where

$$D_x L(\boldsymbol{x}, \boldsymbol{\lambda}) = Df(\boldsymbol{x}) - \sum_{j=1}^{m} \lambda_j Dh_j(\boldsymbol{x})$$

Notes

- The condition that the matrix $Dh(\boldsymbol{x}^*)$ must be of rank m is called the *constraint qualification*.
- If $h(\boldsymbol{x}) = \boldsymbol{a}$ then $\boldsymbol{x}$ is said to be *feasible* for P.

Concavity and Lagrange's Theorem

Consider the problem:

$$P:\ \max f(\boldsymbol{x}) \text{ such that } h(\boldsymbol{x}) = \boldsymbol{a}$$

Theorem. Let f be concave and let each h_j be convex. Then the Lagrange first order conditions are sufficient for a global maximum.

Solving an equality-constrained maximisation problem

- If the constraint qualification holds at all points then any candidates for local maxima (and hence global maxima) can be found by solving the first-order conditions:

$$\frac{\partial L}{\partial x_i}(x^*;\lambda^*) = 0 \text{ for } i = 1, \dots, n$$
$$h_j(x^*) = a_j \text{ for } j = 1, \dots, m$$

- (If there are points x^* where $Dh(x^*)$ has less than full rank these points need to be checked separately as they may not show up as solutions to the first-order conditions even if they are local maxima)
- If f is concave and each h_j is convex then any local maxima are global maxima.