Part C: Interacting Particle Systems

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Contents

1 Introduction and Notation 5
  1.1 Introduction ........................................... 5
  1.2 Notation ............................................... 6
  1.3 Brief recap ............................................. 7

2 Electrical networks 11
  2.1 Set-up .................................................. 11
  2.2 Electrickery ............................................ 12
  2.3 Effective Resistance and Energy .......................... 14
  2.4 Some further probabilistic interpretations ............... 16

3 Pólya's Theorem 18
  3.1 Network reduction ...................................... 18
  3.2 Pólya's Theorem and proof ................................ 19

4 Uniform Spanning Tree and Wilson's algorithm 22
  4.1 Wilson's algorithm ..................................... 22
  4.2 Electrical interpretations and negative association .... 25
  4.3 Weak limits ............................................. 26
    4.3.1 A Topological Aside ................................ 26
    4.3.2 Back to the UST measure ............................. 27

5 Percolation 28
  5.1 Model definition ....................................... 28
  5.2 The phase transition .................................... 29

6 Percolation toolbox 32
  6.1 Harris-FKG inequality .................................. 33
  6.2 BK inequality .......................................... 35
  6.3 Margulis-Russo formula .................................. 36

7 Further percolation 38
  7.1 Exponential decay in the subcritical regime .............. 38
  7.2 Kesten's theorem ....................................... 40
  7.3 The Russo-Seymour-Welsh theory .......................... 42

8 Exclusion, contact and voter models 43
  8.1 Semigroups and generators .............................. 43
  8.2 Contact process and graphical construction .............. 47
    8.2.1 Monotonicity ..................................... 48
    8.2.2 Duality ............................................ 49
    8.2.3 Stationary measures and percolation ................ 51
PART C
INTERACTING PARTICLE SYSTEMS
16 lectures MT 2018

Aims
The aims is to introduce fundamental probabilistic and combinatorial tools, as well as key models, in the theory of discrete disordered systems. We will examine the large-scale behavior of systems containing many interacting components, subject to some random noise. Models of this type have a wealth of applications in statistical physics, biology and beyond, and we will see several key examples in the course. Many of the tools we will discuss are also of independent theoretical interest, and have far reaching applications. For example, we will study the amount of time it takes for these process to reach the stationary distribution (mixing time). This concept is also important in many statistical applications, such as studying the run time of MCMC methods.

Prerequisites:
Discrete and continuous time Markov process on countable state space, as covered for example in Part A A8 Probability and Part B SB3a Applied Probability.

Synopsis:
• Percolation and phase transitions.
• Uniform spanning trees, loop-erased random walks, and Wilson's algorithm.
• Random walks on graphs and electrical networks (discrete potential theory).
• Important models: Stochastic Ising model (Glauber dynamics), Random-cluster model, Contact process, Exclusion process, Hard-core model.
• Important tools: Monotonicity, coupling, duality, FKG inequality.
• Gibbs measures and a discussion of phase transitions in this context.
• Mixing times and the spectral gap.

Reading
Classes and problem sheets

There will be 16 hours of lectures; $L1-L16$. There will be 4 problem classes and assignment sheets. The 1st class will be shorter than usual covering material from the first week of lectures, it will run in Week 2. The 2nd and 3rd will be 'standard' and run in Weeks 4-5 and Weeks 6-8 respectively. Class 4 will take place in Week 1 HT. The final sheet will be longer, and include more stretching material, the sort of problems that would be nice to ask but no time during term. Full solutions will be provided to the students.
Chapter 1

Introduction and Notation

1.1 Introduction

Interacting particle systems (IPS) are Markov processes, in continuous or discrete time, which describe 'particles' moving in some underlying discrete space, subject to some random noise and interactions. Such models often arise naturally in theoretical physics (statistical mechanics), and there are now numerous applications of these systems across the natural and social sciences, including in; population genetics, epidemiology, computer science, electrical engineering, and economics. A few examples are given below and on the associated slides. Interacting particle systems also provide a natural framework to study fundamental phenomena which occur in these applications, such as phase transitions, metastability and relaxation to equilibrium. One of the main goals in this field is to understand and predict emergent behaviour on macroscopic (large) scales, as a result of the microscopic dynamics and interactions of the individual components. In the past decades this field has grown in importance and established itself as one of the most active branches of probability theory.

The microscopic dynamics of interacting particle systems are often very simple to describe, however they are frequently very challenging to analyse, in particular on large or infinite scales. On the other-hand, they are often simple enough to be open to detailed mathematical (rigorous) analysis while still capturing the main features of interest in applications. Qualitative changes in large scale macroscopic behavior depending on the system parameters are often known as 'collective phenomena' or phase transitions, and are of particular interest. A couple of real world examples [Figures on lecture slides]:

1. Figure 1 shows colour-coded data for car speed on the M25 as a function of space and time, as recorded by sensors under the road. The striped patterns of low-speed (red) correspond to stop-and-go waves during rush hour. Often there is no obvious external cause such as an accident, so the pattern is interpreted as an intrinsic collective phenomenon emerging from the interactions of cars on a busy road. A minimal mathematical description of this situation in terms of IPS would be to take a one-dimensional lattice (or a subset thereof), and at each site denote the presence or absence of a car with an occupation number 1 or 0 respectively. In terms of dynamics, we simplify the system so that we can determine if the observed phenomena of interest is a result of relatively simple interactions. We only want to model normal traffic on a single lane road without car crashes or overtaking. So cars are allowed to proceed one lattice site to the right, say, with a given rate $1$, provided that the site in front is not occupied by another car. The rate may depend on the surrounding configuration of cars (e.g. number of empty sites ahead), and relatively simple choices depending only on three or four neighboring sites can already lead to interesting patterns and the emergence of stop-and-go waves. The defining features of this process in terms of IPS are that no particles are created or destroyed (conservation of the number of particles) and that there is at most one particle per site (exclusion rule). We will see the most basic example of a model in this class during the course, called the simple exclusion processes, which not only has man applications (including in biological transport systems) but has attracted a great deal of attention in the mathematical literature and is at the heart of many very important results.
2. Figure 2 shows segregation patterns of a microbial species (E. coli) when grown in a nutrient rich medium from a mixed initial population of circular shape. Each individual appears either red or green as a result of a neutral genetic marker that only affects the colour. A possible IPS model of this system might evolve on a two dimensional lattice (say $\mathbb{Z}^2$ for simplicity), where a state is described by assigning to each vertex one of three colours. The dynamics can be modeled by letting each individual split into two with a given rate, and then place the offspring on an empty (white) neighboring site. If there is no empty neighboring site the reproduction rate is zero (or equivalently the offspring is immediately killed). Therefore we have two equivalent species competing for the same resource (empty sites), and spatial segregation is a result of the fact that once the red particles died out in a certain region due to fluctuations, all the offspring is descending from green ancestors. Note that in contrast to the first example, the number of particles in this model is not conserved. The Eden model is a well known model of pure growth in this way. The simplest such process to model extinction or survival of a single species is called contact process.

3. Figure 3 shows a snapshot of a simplified model of magnetic materials, called the Ising model. This is one of the main prototypical models in statistical physics for exploring phase transitions and one of the most studied lattice spin models. Many important developments have been driven by studying this model (and slight generalisations), including the interplay between static (equilibrium properties) and dynamical features. In the Ising model each vertex of $\mathbb{Z}^2$ (or some subset) is assigned a value of either $+1$ (associated with spin-up) or $-1$ (associated with spin-down). The model is then defined by a probability measure on this space, which is parametrized by the system temperature and an external magnetic field. Natural Markovian dynamics can be associated with the model, an a particular special class called the Glauber dynamics have been extensively studied. The figure shows the system switching from one metastable state (mostly $-1$) to a stable one (mostly $+1$) by droplet growth. Metastable dynamics of stochastic systems is currently a rich and diverse research area in its own right.

4. Figure 4 shows a snapshot from a kinetically constrained model under Glauber type dynamics, relaxing to equilibrium. Kinetically constrained models where introduced in the physics literature as minimal models for the dynamics of glassy (or amorphous) materials - such as supercooled liquids. These models have recently attracted significant attention in the mathematics community. Developments here are occurring simultaneously in the more applied and theoretical communities and there is a great deal of interdisciplinary research in the area of glassy-dynamics.

In these notes we will cover the basic definitions, concepts, and tools, that are used in the study of interacting particle systems. Along the way we will introduce some classical examples of these systems systems. We will then focus mainly on the large scale behavior, such as phase transitions, and dynamical properties such as mixing and relaxation. Many of the tools we will discuss are also of independent theoretical interest, and have far reaching applications. For example, we will study the amount of time it takes for these process to reach the stationary distribution (mixing time). This concept is also important in many statistical applications, such as studying the run time of MCMC methods.

1.2 Notation

I will attempt to stick to the following notation and terminology throughout the notes. There maybe some need to stray from this now and again to improve clarity.
1.3. BRIEF RECAP

We will typically denote the state space (i.e. the set of all possible configurations of the system) by \( \Omega \), and configurations will typically be denoted by small Greek letters \( \eta, \sigma, \ldots \in \Omega \). For interacting particle systems \( \Omega \) will be of the form \( S^\Lambda \), where \( S \subset \mathbb{Z} \) is called the local state space, which will typically be finite (e.g. \( S = \{0, 1\} \) corresponding to the presence or absence of a particle at a give site for exclusion type models and \( S = \{-1, +1\} \) for the Ising model). \( \Lambda \) will be a countable set which represents the lattice on which the ‘particles’ live. When \( S \) is finite \( \Omega \) is compact and metrizable with respect to the product topology. We will try to always denote vertices (or sites) in the lattice \( \Lambda \) by small Latin letters, \( x, y, z \in \Lambda \). For \( \eta \in \Omega \) and \( x \in \Lambda \) the local occupation number is denoted by \( \eta(x) \in S \). So a configuration \( \eta \in \Omega \) can be thought of as a vector \((\eta(x))_{x \in \Lambda}\) indexed by the lattice \( \Lambda \), or as a function \( \eta : \Lambda \rightarrow S \). Clearly if \( S \) and \( \Lambda \) are both finite then so is \( \Omega \), if \( S \) is countable and \( \Lambda \) is finite then \( \Omega \) is countable, we will typically focus on these two cases. Often when discussing the ‘large-scale’ behavior we will be considering a sequence of increasingly large systems \( \Omega_n = S^{\Lambda_n} \), where \( |\Lambda_n| \rightarrow \infty \) as \( n \rightarrow \infty \). As soon as \( |S| > 1 \) and \( \Lambda \) is infinite the state space \( \Omega \) is uncountable. This is often the case in models in statistical mechanics which are defined on the entire lattice \( \mathbb{Z}^d \). For a measurable structure in this case we will take the Borel \( \sigma \)-algebra generate by open cylinders, denoted by \( B(\Omega) \). We will denote the set of probability measures on \( \Omega \) by \( P(\Omega) \). Probability measures on \( \Omega \) will typically be called \( \mu, \nu \) or \( \pi \in P(\Omega) \).

When the state space is relatively low dimensional, e.g \( \Omega \subset \mathbb{Z} \) we may use capital letters to denote the process e.g. \((X_t)_{t \geq 0}\).

We will consider chains in continuous time,

\[ t \in \mathbb{R}^+ = [0, \infty) \]

and also sometimes in discrete time

\[ n \in \mathbb{N}_0 = \{0, 1, \ldots\} \]

The letters \( n, m, k \) will typically denote integers, while we reserve \( t \) and \( s \) for real numbers (times).

We write \((\eta_t)_{t \geq 0}\) for a continuous time process and \((\eta_n)_{n \geq 0}\) for a discrete time process, where it is hopefully clear from context which one we are using. We will typically denote the probability measure associated with the process (on path space) by \( P \). In continuous time the canonical path space is given by

\[ D[0, \infty) = \{ \eta : [0, \infty) \rightarrow \Omega \text{ càdlàg} \} , \]

we will be more specific about the probability space etc. when we look at specific models. The associated expectation of a random variable \( f \) will be written as \( \mathbb{E}[f] \).

For a graph \( G = (V, E) \) we will write \( \sigma \sim \eta \iff \{\sigma, \eta\} \in E \).

1.3 Brief recap

This section gives a very brief recap of discrete and continuous time Markov processes on countable state spaces, and some important concepts such as reversibility. For more detail see for example Part A Probability and Part B Applied probability notes, or the book ‘Markov Chains’ by J.R. Norris.

Let \( \Omega \) be a countable set. Let \( P = (P(u, v))_{u,v \in \Omega} \) be a stochastic matrix on \( \Omega \), ie.

\[ P(\eta, \sigma) \geq 0, \forall \eta, \sigma \in \Omega \quad \text{and} \quad \sum_{\sigma \in \Omega} P(\eta, \sigma) = 1 \forall \eta \in \Omega. \]
We say that the $\Omega$-valued random variable $(X_n)_{n \geq 0}$ is a Markov chain with transition matrix $P$ and initial distribution $\nu$ if its law, denoted by $\mathbb{P}_\nu$, is determined by the following: For all $n \geq 1$, $\xi, \zeta, \sigma_1, \ldots, \sigma_{n-1} \in \Omega$, and all events $H_{n-1} = \bigcap_{i=0}^{n-1} \{X_i = \sigma_i\}$ satisfying $\mathbb{P}_\nu(H_{n-1} \cap \{X_n = \xi\}) \neq 0$, we have

(i) $\mathbb{P}_\nu(X_0 = \zeta) = \nu(\zeta)$;

(ii) $\mathbb{P}_\nu(X_{n+1} = \zeta \mid H_{n-1} \cap \{X_n = \xi\}) = P(\xi, \zeta)$.

Equation (ii), often called the Markov property, means that the conditional probability of proceeding from state $\zeta$ to state $\sigma$ is the same no matter what the sequence of states preceding the current state $\zeta$, i.e. the past and future are conditionally independent given the present. If $\nu = \delta_\zeta$ then we write $\mathbb{P}_\zeta$.

**Lemma 1.1.** A discrete-time random process $(X_n)_{n \geq 0}$ is a Markov chain with initial distribution $\nu$ and transition metric $P$ if and only if for all $n \geq 0$ and $\sigma_0, \ldots, \sigma_n \in \Omega$

$$\mathbb{P}(X_0 = \sigma_0, X_1 = \sigma_1, \ldots, X_n = \sigma_n) = \nu(\sigma_0)P(\sigma_0, \sigma_1)P(\sigma_1, \sigma_2)\ldots P(\sigma_{n-1}, \sigma_n).$$

**Proof.** Exercise. \(\square\)

We denote the distribution at time $n$ started from $\nu$ by $\nu_n$, i.e.

$$\nu_n(\sigma) = \mathbb{P}_\nu(X_n = \sigma) = (\nu_{n-1}P)(\sigma) = \sum_{\xi \in \Omega} \nu_{n-1}(\xi)P(\xi, \sigma),$$

or $\nu_n = \nu_{n-1}P$, so $\nu_n = \nu_0P^n \ \forall n \geq 0$.

The transition matrix acts on functions (observables), which by convention we think of as column vectors, as

$$Pf(\sigma) = \sum_{\xi \in \Omega} P(\sigma, \xi)f(\xi) = \sum_{\xi \in \Omega} f(\xi)\mathbb{P}_\sigma(X_1 = \xi) = \mathbb{E}_\sigma[f(X_1)]$$

in general $P^n f(\sigma) = \mathbb{E}_\sigma[f(X_n)]$ for $n \geq 1$.

That is the $\sigma$th entry of $Pf$ tells us the expected value of the function $f$ after one time step, given the initial distribution was $\delta_\sigma$.

**Definition 1.2** (Stationary distribution). A distribution probability $\pi$ on $\Omega$ is called stationary for the chain if $\pi P = \pi$, equivalently

$$\pi(\sigma) = \sum_{\xi \in \Omega} \pi(\xi)P(\xi, \sigma) \ \forall \sigma \in \Omega.$$ 

Clearly if $\pi$ is stationary and $\nu_0 = \pi$ then $\nu_n = \pi$ for all $n \geq 0$.

The Markov chain is said to be irreducible if it is possible to go from any state to any other using only transitions of positive probability.

**Definition 1.3** (Irreducible). A chain $P$ is irreducible if for any $\sigma, \xi \in \Omega$ there exists an $n \in \mathbb{N}$ such that $P^n(\sigma, \xi) > 0$

Recall the definition of recurrence and positive recurrence.

**Theorem 1.4.** Let $P$ be irreducible Markov transition matrix. Then

(i) $P$ has a stationary distribution if and only if $P$ is positive recurrent.
(ii) In that case, the stationary distribution is unique and is given by 
\[ \pi(\sigma) = \frac{1}{\mathbb{E}[n+1]} \] 
where \( \tau_\sigma^+ := \min\{n \geq 1 \mid X_n = \sigma \} \) is the first return time to state \( \sigma \).

Proof. Seen previously, see e.g. Prop 1.14 and Corollary 1.17 in [2].

We will call a chain ergodic if it has a unique stationary distribution.

**Definition 1.5** (Aperiodic). A discrete time Markov chain is called aperiodic if for all \( \sigma \in \Omega \), \( P^n(\sigma, \sigma) \) is eventually positive, i.e. there exists \( N_\sigma \in \mathbb{N} \) such that \( P^n(\sigma, \sigma) > 0 \) for all \( n \geq N_\sigma \).

**Theorem 1.6** (Convergence to equilibrium). An irreducible, aperiodic, Markov chain with finite state space is ergodic, i.e. it has a unique stationary distribution \( \pi \) and
\[ P^n(\sigma, \xi) = \mathbb{P}_\sigma(X_n = \xi) \to \pi(\xi) \quad \text{as} \quad n \to \infty, \quad \forall \sigma, \xi \in \Omega. \]

**Theorem 1.7** (Ergodic Theorem). Let \( f \) be a real valued function on \( \Omega \) and \( (X_n)_{n \geq 0} \) an irreducible, positive recurrent, Markov chain with stationary distribution \( \pi \), then for any initial distribution \( \nu \)
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n-1} f(X_m) \to \mathbb{E}_\pi[f] \quad \text{a.s.} \quad \text{as} \quad n \to \infty. \]

In words “time averages converge to space averages.”

Proof. See e.g. [2] Section 4.3

A Markov chains past and future are independent given the present. However, consider an ergodic chain (irreducible and aperiodic) starting from a point mass on a single state, convergence to equilibrium demonstrates an asymmetrical behavior, i.e. increasing entropy (or uncertainty). On the other hand, a Markov chain in equilibrium (started from a stationary distribution) run backward is again a Markov chain, although possibly not the same one.

**Theorem 1.8.** Suppose \( (X_n)_{n \geq 0} \) is a Markov chain with irreducible (positive recurrent) transition matrix \( P \) and initial distribution \( \pi \) the unique stationary distribution. Fix \( N > 0 \), then \( (\hat{X}_n)_{n=0}^N \) given by \( \hat{X}_n = X_{N-n} \) is a Markov chain with initial distribution \( \pi \) and transition matrix
\[ \hat{P}(\sigma, \xi) = \frac{\pi(\xi)P(\xi, \sigma)}{\pi(\sigma)}. \]

Proof. See [Thm 1.9.1 in Markov Chains, Norris].

**Definition 1.9.** A Markov chain is called reversible with respect to \( \pi \) if the forward and reversed chains have the same distribution, i.e. \( P = \hat{P} \).

**Definition 1.10.** A transition matrix \( P \) satisfies the detailed balance condition with respect to measure \( \pi \in \mathbb{P}(\Omega) \) if
\[ \pi(\sigma)P(\sigma, \xi) = \pi(\xi)P(\xi, \sigma) \quad \forall \sigma, \xi \in \Omega. \]

**Proposition 1.11.** If \( P \) satisfies the detailed balance condition with respect to measure \( \pi \), then \( \pi \) is a stationary distribution for the Markov chain generated by \( P \). A Markov chain is reversible with respect to \( \pi \) if and only if it’s transition matrix satisfies the detailed balance condition with respect to \( \pi \).

Proof. Exercises.
Remark 1.12. A Markov chain with transition matrix $P$ is reversible with respect to $\pi$ if and only if $P$ is self adjoint in $\ell^2(\pi)$, with inner product $\langle f, g \rangle_\pi = \sum_{\sigma \in \Omega} f(\sigma)g(\sigma)\pi(\sigma)$. That is, for all square integrable random variables $f, g \in \ell^2(\pi)$

$$\langle Pf, g \rangle_\pi := \sum_{\sigma \in \Omega} \pi(\sigma)Pf(\sigma)g(\sigma) = \sum_{\sigma \in \Omega} \pi(\sigma)g(\sigma)\sum_{\xi \in \Omega} P(\sigma, \xi)f(\xi)$$

$$= \sum_{\sigma, \xi \in \Omega} \pi(\xi)P(\xi, \sigma)f(\xi) = \langle f, Pg \rangle_\pi$$

You should revise/be familiar with the analogous results for continuous time chains on countable state space. In particular in terms of the ‘Q-matrix’ or (infinitesimal) generator of the process, and the correspondence with the jump-chain/holding time description (see Fig. 1.1).

We will often denote the generator (Q-matrix) of a continuous time process on countable state space as either $Q = (q(\sigma, \eta))_{\sigma, \eta \in \Omega}$ or by $L$. Recall that if the state space is finite then the continuous time transition matrix (Markov semi-group) can be expressed as $P_t = e^{tL}$. These satisfy the backward and forward equations

$$P_t' = L P_t = P_t L \quad \text{with} \quad P_0 = I. \quad (1.3.1)$$
Chapter 2

Electrical networks

Any Markov process on a countable state space can be thought of as a random walk on a (directed) graph, possibly with self loops, where the vertices’s are the states of the process and edges represent possible transitions. It turns out that there is an intimate connection between reversible random walks on graphs and electrical networks. This connection gives the discrete analogue of the deep connection between continuous potential theory and Brownian motion.

The purpose of this chapter is to give an introduction to this discrete potential theory. In the next lecture we will see a nice application of this theory to transience and recurrence of random walks in $\mathbb{Z}^d$, summarised in the beautiful theorem of Pólya.

In countable or finite state spaces the electrical network analogy looks essentially the same in continuous and discrete time (as a consequence of the jump chain/holding time construction). We will write things predominantly in terms of a discrete time chain, but try to make it clear how things would work similarly in continuous time as we go. When the state space is uncountable there are some extra subtleties that we will not discuss here. A nice and much more detailed account of this subject is given by Doyle and Snell, Random Walks and Electric Networks (available on arXiv).

2.1 Set-up

To any reversible Markov chain on finite state space $\Omega$, with transition matrix $P$ and reversible distribution $\pi$, we can associate an (undirected) graph $G = (\Omega, E)$ and weights $c: E \to (0, \infty)$. We will drop the usual assumption that $G$ is irreflexive, that is $E$ may contain ‘loops’ i.e. edges of the form $\{\eta, \eta\}$ for $\eta \in \Omega$.

The edge set is given by $\{\eta, \sigma\} \in E$ if and only if $P(\eta, \sigma) > 0$ (or analogously in continuous time $q(\eta, \sigma) > 0$). Note that the definition of the edge set makes sense since $P(\eta, \sigma) > 0$ if and only if $P(\sigma, \eta) > 0$ by reversibility (the detailed balance condition). We put a weight function on the edges, $c: E \to (0, \infty)$, specified by $c(\{\eta, \sigma\}) = \pi(\eta) P(\eta, \sigma)$. This is symmetric with respect to switching $\eta$ and $\sigma$ again by detailed balance. We think of $c(\{\eta, \sigma\})$ as the conductance of the edge $\{\eta, \sigma\}$, or equivalently the edge has a resistor on it with resistance $r(\{\eta, \sigma\}) = 1/c(\{\eta, \sigma\})$.

With a slight abuse of notation we will write $c(\sigma, \eta)$. It is straightforward to check that $\pi(\eta) = c(\eta) := \sum_{\sigma \sim \eta} c(\eta, \sigma)$. We will call a graph together with a weight function of this kind a network.

We could also start with a finite graph and associated conductances $c: E \to (0, \infty)$, and use these to construct a reversible Markov process via the relation $P(\eta, \sigma) = c(\eta, \sigma)/c(\eta)$. In this case it turns out that the chain is reversible with respect to the probability distribution given by $\pi(\eta) = c(\eta)/c_\Omega$ where $c_\Omega = \sum_{\eta \in \Omega} c(\eta)$.

For $A \subset \Omega$ define the hitting time and first return time as

$$\tau_A = \min\{n \geq 0 : X_n \in A\}$$

$$\tau_A^+ = \min\{n \geq 1 : X_n \in A\}.$$ 

In problems such as the gambler’s ruin we are often interested in the probability of reaching some point of the state space before another. Consider the probability that the chain started from $\eta \in \Omega$ hits a set $A$ before the set $B$ where $A \cap B = \emptyset$, and let

$$f(\eta) = \mathbb{P}_\eta(\tau_A < \tau_B).$$
Ohm’s Law:
states that the current is the conductance times the gradient of the potential.

We may associate an induced current to be very important and

In words this says that the value of the function at a point is equal to its local average,

Let

Definition 2.4.

Theorem 2.2
(Existence Principle)

Theorem 2.3
(Uniqueness Principle)

The proof follows from the Maximum Principle, exercise.

The following are stated for finite graphs but with week conditions can be easily extended to countable graphs.

**Theorem 2.1 (Maximum Principle).** Let \( A \subset \Omega \) be a connected component of \( G \), if \( f : \Omega \rightarrow \mathbb{R} \) is harmonic on \( A \) and attains its supremum at \( \eta_0 \in A \), then \( f \) is constant on \( A \cup \partial A \).

**Proof.** Let \( K := \{ \sigma \in \Omega : f(\sigma) = f(\eta_0) \} \). If \( \eta \in A \cap K \) and \( P(\eta, \sigma) > 0 \) then \( \sigma \in K \), since \( f \) is harmonic at \( \eta \) and takes its largest value at \( \eta \). We can proceed by iteration. \( \square \)

**Theorem 2.2 (Uniqueness Principle).** Let \( G \) be connected, and \( A \) a proper subset of \( \Omega \). If \( f, g : \Omega \rightarrow \mathbb{R} \) are harmonic on \( A \) and \( f(\eta) = g(\eta) \) for all \( \eta \in A^c \) then \( f \equiv g \).

**Proof.** The proof follows from the Maximum Principle, exercise. \( \square \)

**Theorem 2.3 (Existence Principle).** Let \( A \) be a proper subset of \( \Omega \), if \( f_0 : \Omega \setminus A \rightarrow \mathbb{R} \) then there exists an \( f : \Omega \rightarrow \mathbb{R} \) such that \( f \big|_{A^c} \equiv f_0 \) and \( f \) is harmonic on \( A \).

**Proof.** Check that \( f(\eta) = \mathbb{E}_\eta[f_0(X_{\tau_{A^c}})] \) gives such a function by using the usual one-step analysis. \( \square \)

### 2.2. Electrickery

We will now define potential differences (voltages) and current flows on the network described in the previous section, and give some probabilistic interpretations. Several nice results then follow from physical principles of electrical networks. We will denote the set of oriented edges

\[ E = \{ (\eta, \sigma) \in \Omega^2 : P(\eta, \sigma) > 0 \}. \]

We may think of real valued functions on \( \Omega \) as potentials. To any potential function \( \varphi : \Omega \rightarrow \mathbb{R} \) we may associate an induced current \( j : E \rightarrow \mathbb{R} \) (and vice-versa) according to Ohm’s Law, which states that the current is the conductance times the gradient of the potential.

**Ohm’s Law:** If \( \eta \sim \sigma \) then the current \( j(\eta, \sigma) \) from \( \eta \) to \( \sigma \) is given by

\[ j(\eta, \sigma) = c(\eta, \sigma)(\varphi(\eta) - \varphi(\sigma)). \]

Note that \( j \) is always anti-symmetric, i.e. \( j(\eta, \sigma) = -j(\sigma, \eta) \), since the conductances are symmetric. We will find the following definition very useful in this context.

**Definition 2.4.** Let \( A \) and \( B \) be disjoint subsets of \( \Omega \). We call an anti-symmetric function \( \theta : E \rightarrow \mathbb{R} \) a flow from \( A \) to \( B \) if

- \( \text{div } \theta(\eta) := \sum_{\sigma \sim \eta} \theta(\eta, \sigma) = 0 \) for all \( \eta \notin A \cup B \) (“flow in is equal to flow-out”).
• \( \text{div} \, \theta(\sigma) \geq 0 \) for all \( \sigma \in A \).
• \( \text{div} \, \theta(\sigma) \leq 0 \) for all \( \sigma \in B \).

The strength of a flow \( |\theta| = \sum_{\sigma \in A} \text{div} \, \theta(\sigma) \), and if \( |\theta| = 1 \) we call \( \theta \) a unit flow.

**Exercises (uniqueness):** Check that if \( i \) and \( j \) are both flows from \( A = \{ a \} \) to \( B = \{ b \} \) with strength \( |i| = |j| \), then \( i \equiv j \).

If we attach a battery to the graph such that the vertices in set \( A \) are at a positive voltage, \( \phi \), and those in \( B \) at zero, then an equilibrium current will flow which satisfies Kirchhoff’s Node (or current) Law.

**Kirchhoff’s Current Law:** The equilibrium current, \( i_{A,B}^\phi : E \to \mathbb{R} \), associated with a fixed voltage \( \phi \) across \( A \) and \( B \), is a flow from \( A \) to \( B \).

There are several interpretations of these laws. Firstly you could take Ohm’s law as a definition of the current given a potential. Then Kirchhoff’s law is a property of any current given by a potential which is fixed on \( A \) and \( B \) and harmonic on \( (A \cup B)^c \). Alternatively you could take Kirchhoff’s law together with Ohm’s law as a definition of the equilibrium current and voltage. We described this more precisely next.

Suppose we attach a battery with voltage \( \phi \) on \( A \) and 0 on \( B \). Then by Kirchhoff’s current law this induces some current which is a flow called \( i_{A,B}^\phi : E \to \mathbb{R} \). By Ohm’s law this current must be the gradient of some potential, we assume that this potential must be \( \phi \) on \( A \) and 0 on \( B \), then it turns out this uniquely determines the potential and the current flow. We denote by \( \phi_{A,B} : \Omega \to \mathbb{R} \) the potential induced by this voltage \( \phi \), i.e., by its restriction to the sets \( A \) and \( B \), together with Kirchhoff’s current law and Ohm’s law. Ohm’s law together with Kirchhoff’s current law imply that the voltage must be a harmonic function on \( (A \cup B)^c \), and then its restriction to \( A \) and \( B \) determine it uniquely. This immediately gives us a probabilistic interpretation of the voltage.

**Theorem 2.5.** The voltage (potential function) \( \phi_{A,B} : \Omega \to \mathbb{R} \) induced by a voltage \( \phi \) applied to sets \( A \) and \( B \), that is \( \phi_{A,B} |_A = \phi \), \( \phi_{A,B} |_B = 0 \) and \( \phi_{A,B} \) satisfies Ohm’s law with respect to the current flow \( i_{A,B}^\phi \), is harmonic on \( (A \cup B)^c \). Furthermore \( \phi_{A,B}(\eta) = \phi \mathbb{P}_\eta(\tau_A < \tau_B) \).

**Proof.** Let \( U = (A \cup B)^c \), and suppose \( v \) satisfies the conditions of the theorem, then by Kirchhoff’s current law and Ohm’s law

\[
\text{div} \, i(\eta) = \sum_{\sigma \sim \eta} i(\eta, \sigma) = \sum_{\sigma \sim \eta} c(\eta, \sigma)(v(\eta) - v(\sigma)) = 0 \quad \text{for } \eta \in U.
\]

Rearranging the final equality and using the fact that the conductances are symmetric we find

\[
v(\eta) = \frac{1}{c(\eta)} \sum_{\sigma \sim \eta} c(\eta, \sigma)v(\sigma)
\]
i.e. \( v \) is harmonic on \( U \). So, by the uniqueness theorem, \( \phi_{A,B} \) is the unique harmonic function on \( U \) satisfying the boundary conditions \( v |_A = \phi \) and \( v |_B = 0 \). Since \( v'(-) = \phi_{A,B}(-)/\phi \) is harmonic on \( U \) and satisfies \( v' |_A = 1 \) and \( v' |_B = 0 \) (simply check), then by uniqueness \( v'(\eta) = \mathbb{P}_\eta(\tau_A < \tau_B) \), i.e. \( v(\eta) = \phi \mathbb{P}_\eta(\tau_A < \tau_B) \).

There are also other interpretations of the voltage as well as probabilistic and combinatorial interpretations for the current flow. We will see some more of these in the next section. We now
give a connection between unit flows and spanning trees on $G$. We will see spanning trees again alter. Note that flow associated with the voltage $v$ above is not necessarily of unit strength (we will also return to this point).

**Theorem 2.6.** Let $T$ be a tree on $G$ chosen uniformly from the set $\mathcal{T}$ of spanning trees. Then

$$j(\eta, \sigma) = \mathbb{P}(\text{The unique path from } \xi \text{ to } \zeta \text{ in } T \text{ contains } (\eta, \sigma)) - \mathbb{P}(\text{The unique path from } \xi \text{ to } \zeta \text{ in } T \text{ contains } (\sigma, \eta))$$

is a unit flow from $\xi$ to $\zeta$.

**Proof.** See Grimmett, Probability on Graphs Theorem 1.16.

\[\square\]

### 2.3 Effective Resistance and Energy

The strength of the current, $i$, that flows when we apply a voltage $\phi$ on $A$ and 0 on $B$, as in Theorem 2.5, can be calculated as follows,

$$\text{div } i_{A,B}^\phi(a) = \sum_{\sigma \sim a} i_{A,B}^\phi(a, \sigma) = \sum_{\sigma \sim a} c(a, \sigma) [\phi - \phi \mathbb{P}_\sigma(\tau_A < \tau_B)]$$

$$= \phi \sum_{\sigma \sim a} c(a, \sigma) [1 - \mathbb{P}_\sigma(\tau_A < \tau_B)] = \phi \sum_{\sigma \sim a} c(a, \sigma) \mathbb{P}_\sigma(\tau_A \geq \tau_B)$$

$$= \phi \pi(a) \sum_{\sigma \sim a} \mathbb{P}_\sigma(\tau_A \geq \tau_B) = \phi \pi(a) \mathbb{P}_a(\tau_A^+ > \tau_B).$$

It follows that

$$|i_{A,B}^\phi| = \sum_{a \in A} \text{div } i_{A,B}^\phi(a) = \phi \sum_{a \in A} \pi(a) \mathbb{P}_a(\tau_A^+ > \tau_B), \quad (2.3.1)$$

Now in analogy with Ohm’s law we define the effective conductance as

$$C(A, B) := \sum_{a \in A} \pi(a) \mathbb{P}_a(\tau_A^+ > \tau_B),$$

which is independent of the applied voltage. The effective resistance $R(A, B) = 1/C(A, B)$.

Notice that for a given voltage $\phi$ over the sets $A$ and $B$, the strength of the associated current flow is $\phi C(A, B)$, equivalently an applied voltage equal to $R(A, B)$ induces a unit current flow. This holds for the original network and the reduced network $G' = (\{A, B\}, \{\{A, B\}\})$ with $c(A, B) = C(A, B)$. The associated voltage that induces this unit flow is denoted $v_{A,B}(\cdot) := R(A, B)v_{A,B}^\tau(\cdot)$. We will call this (equilibrium) unit current flow $i_{A,B}$, furthermore it is simple to check that $i_{A,B}(\cdot) \equiv i_{A,B}^\phi(\cdot) R(A, B)/\phi$.

We now define the energy dissipated in a flow, and observe that the energy dissipated in the flow associated with potential $\phi$ over $A$ and $B$ is the same on $G$ as on a single resistor with resistance $R_{A,B}$ (further motivation for calling it effective resistance).

**Definition 2.7.** The energy of a flow $\theta$ on $G$ is given by

$$\mathcal{D}(\theta) = \sum_{e \in E} \theta(e)^2 r(e) = \frac{1}{2} \sum_{e \in E} \theta(e)^2 r(e).$$

Note the first sum is over unoriented edges (which makes sense because we square an antisymmetric) and the second is over all oriented edges in the graph. Consider the energy dissipated by a current $I_{A,B}$ flowing across a single edge with resistance $R(A, B)$ when a voltage
of \( \phi \) is applied, given by \( \mathcal{D}(I_{A,B}) = \phi^2 C(A,B)^2 R(A,B) = \phi |I_{A,B}| \). We observe that this is the same as the energy of the equilibrium flow \( i^\phi_{A,B} \) on \( G \) described above (suppressing the sub and superscripts for clarity),

\[
\mathcal{D}(i^\phi_{A,B}) = \frac{1}{2} \sum_{(\eta,\sigma) \in \mathcal{E}} i^\phi_{A,B}(\eta,\sigma)(v(\eta) - v(\sigma)) = \frac{1}{2} \left( \sum_{\eta \in \Omega} v(\eta) \text{div} i(\eta) + \sum_{\sigma \in \Omega} v(\sigma) \text{div} i(\sigma) \right)
\]

(since the potential (voltage) on \( B \) is zero) = \( \phi \sum_{\eta \in A} \text{div} i(\eta) = \phi |i| = \phi |I_{A,B}| \),

where in the final equality we have used the fact that the strength of the induced flow is equal to \( \phi C(A, B) \) on both networks.

We may give an equivalent definition of the effective resistance in terms of the energy.

**Theorem 2.8** (Thomson’s principle). Let \( A \) and \( B \) be disjoint subsets of a finite connected (weighted) graph \( G \), then

\[
R(A, B) = \frac{1}{C(A, B)} = \inf \{ \mathcal{D}(\theta) : \theta \text{ is a unit flow from } A \text{ to } B \},
\]

and the unique minimum is attained by \( i_{A,B} \) (the unit strength equilibrium flow defined above).

**Proof.** It is clear from the calculation above that \( \mathcal{D}(i_{A,B}) = R(A, B) \). Let \( j \) be a unit flow from \( A \) to \( B \) and define \( \delta = j - i_{A,B} \). It is straightforward to check that \( \delta \) is a strength zero flow from \( A \) to \( B \), i.e.

\[
\sum_{a \in A} \text{div} \delta(a) = \sum_{b \in B} \text{div} \delta(b) = \text{div} \delta(\eta) = 0 \quad \text{for each } \eta \notin A \cup B.
\]

Also,

\[
\mathcal{D}(j) = \mathcal{D}(i) + \mathcal{D}(\delta) + \sum_{e \in \mathcal{E}} \delta(e)i(e)r(e)
\]

\[
= \mathcal{D}(i) + \mathcal{D}(\delta) + \sum_{(\eta,\sigma) \in \mathcal{E}} (v(\eta) - v(\sigma))\delta(\eta,\sigma)
\]

\[
= \mathcal{D}(i) + \mathcal{D}(\delta).
\]

It follows that \( \mathcal{D}(j) > \mathcal{D}(i) \) as soon as \( \delta \neq 0 \). \( \square \)

We can also define the energy dissipated by a function \( f : \Omega \to \mathbb{R} \) acting as a potential (voltage). With a slight abuse of notation we use the same letter since it will be clear from context if the argument is a ‘voltage’ or a current flow.

**Definition 2.9.** The energy of a function \( f : \Omega \to \mathbb{R} \) on the weighted graph \( G \), also called the Dirichlet form of \( f \), is given by

\[
\mathcal{D}(f) = \frac{1}{2} \sum_{(\eta,\sigma) \in \mathcal{E}} c(\eta,\sigma)(f(\eta) - f(\sigma))^2.
\]

The Dirichlet form is often expressed in terms of the inner product on \( l^2(\pi) \), this will be explored a little on the first assignment sheet. Note that, from Ohm’s law, \( \mathcal{D}(i^\phi_{A,B}) = \mathcal{D}(v^\phi_{A,B}) \), so it makes sense to use related notation.
Theorem 2.10 (Dirichlet’s principle). Let $A$ and $B$ be disjoint subsets of a finite connect (weighted) graph $G$, then

$$C(A, B) = \inf \{D(f) : f : \Omega \rightarrow \mathbb{R}, f \upharpoonright_A \equiv 1 \text{ and } f \upharpoonright_B \equiv 0\},$$

and the minimum is attained at $v^1_{A,B}$.

Note that the effective conductance $C(A, B)$ is often also called the capacity. In our ‘dimensionless’ electrical network setting it turns out that these two physical concepts coincided.

Proof. The proof is essentially the same as for Thomson’s principle.

To see that $C(A, B) = D(v_{A,B})$ we appeal to Thomson’s principle. By Eq. 2.3.1 we have $|i^1_{A,B}| = C(A, B)$, i.e. $i^1_{A,B} = C(A, B)i_{A,B}$, so $D(v_{A,B}) = D(i^1_{A,B}) = C(A, B)^2/C(A, B)$ by Thomson’s principle.

Fix a function $f : \Omega \rightarrow \mathbb{R}$ such that $f \upharpoonright_A \equiv 1$ and $f \upharpoonright_B \equiv 0$, then define $h := f - v_{A,B}$. A simple calculation gives

$$D(f) = D(v_{A,B}) + D(h) + \sum_{(\eta, \sigma) \in E} c(\eta, \sigma)(v_{A,B}(\eta) - v_{A,B}(\sigma))(h(\eta) - h(\sigma))$$

$$= D(v_{A,B}) + D(h) + \sum_{(\eta, \sigma) \in E} (h(\eta) - h(\sigma))i^1_{A,B}(\eta, \sigma)$$

$$= D(v_{A,B}) + D(h),$$

where the last equality follows since $h \upharpoonright_A \equiv h \upharpoonright_B \equiv 0$ and $\text{div}i^1_{A,B}(\eta) = 0$ for $\eta \notin A \cup B$. It follows that $D(f) > D(v_{A,B})$ whenever $h \not\equiv 0$.

Together Thomson’s principle and Dirichlet’s principle give two variational principles for the effective conductance, the former can be used to get lower bounds on the effective conductance by considering ‘test flows’ on $G$, and the latter can give upper bounds by consider ‘test functions’ on $\Omega$.

2.4 Some further probabilistic interpretations

The Green’s function, $G_\tau(a, \eta)$, for a random walk started from $a$ and stopped at a stopping time $\tau$ is defined by the expected number of visits to $\eta$ before time $\tau$, i.e.

$$G_\tau(a, \eta) := \mathbb{E}_a \left[ \sum_{n=0}^{\tau} 1(X_n = \eta, n < \tau) \right].$$

The following interpretation of the effective resistance is a simple consequence of the calculations in the previous section.

Lemma 2.11. Let $(X_n)$ be the random walk on weighted graph $G$, fix $B \subset \Omega$ and $a \notin B$, then

$$G_{\tau_B}(a, a) = \pi(a)R(\{a\}, B),$$

and more generally (recall $\bar{v}_{(a), B}$ is the voltage the induces a unit current flow from $\{a\}$ to $B$)

$$G_{\tau_B}(a, \eta) = \pi(\eta)\bar{v}_{(a), B}(\eta),$$

Sketch proof. The number of visits to $a$ before visiting $B$, started from $a$, has a geometric distribution with parameter $\mathbb{P}_a(\tau_B < \tau_a^+)$, by definition of the effective resistance this completes the proof by the first part.

For the second part we observe that it is sufficient to show that $G_{\tau_B}(a, \eta)/\pi(\eta)$ is harmonic on $(\{a\} \cup B)^c$. This follows from one-step analysis together with the identity $G_{\tau_B}(a, \eta)\pi(a) = G_{\tau_B}(\eta, a)\pi(\eta)$ from problem Sheet 1.
We give another probabilistic interpretation of the current flow, which is particularly appealing if we take a naive view of electrical current as charged particles moving on the network. If a unit current flows from \(a\) to \(b\) then the current flow on all the edges \((\eta, \sigma)\) is equal to the net number of times that the process started from \(a\) will cross \((\eta, \sigma)\) before it reaches \(b\). As we have already observe the unit current is proportional to \(i^1_{\{a\}, \{b\}}\) and the constant of proportionality is the effective resistance.

**Lemma 2.12.** Let \((X_n)\) be a random walk on \(G\) that is absorbed at \(b\), and for \((\eta, \sigma) \in \mathcal{E}\) let \(S_{\eta, \sigma}\) be the number of transitions from \(\eta\) to \(\sigma\) before \((X_n)\) is absorbed, then \(\mathbb{E}_a [S_{\eta, \sigma} - S_{\sigma, \eta}] = i^1_{\{a\}, \{b\}}(\eta, \sigma)\).

**Proof.** The proof follows by a simple calculation and using Lemma 2.11, and is left as an exercises. \(\square\)
Chapter 3

Pólya’s Theorem

Up to this point we have focused on finite networks. In this Chapter we will see how the electrical network analogy can help us to establish if processes on countable graphs are transient or recurrent in term of limits of finite graphs.

3.1 Network reduction

There are several ways of simplifying electrical networks without changing the quantities of interest, which come in very handy when trying to do calculations. This is one reason that the electrical network analogy can be helpful. The simplest, and most familiar of these, will be the Series and Parallel laws. For the following fix a graph \( G = (\Omega, E) \) with conductances (resistances) \( c : E \to \mathbb{R} \) (respectively \( r \)), and \( A, B \subset \Omega \) disjoint with potential difference \( \phi \) applied across \( A \) and \( B \).

Lemma 3.1 (Series Law). Two resistors \( r_1 \) and \( r_2 \) in series are equivalent to a single resistor with resistance \( r_1 + r_2 \). That is if \( \eta \in (A \cup B)^c \) is a vertex with degree 2 with neighbours \( \sigma_1, \sigma_2 \), and we replace the edges \( \{\eta, \sigma_1\}, \{\eta, \sigma_2\} \) with a single edge \( \{\sigma_1, \sigma_2\} \) with resistance \( r(\eta,\sigma_1) + r(\eta,\sigma_2) \) then all the potentials and currents in \( G \setminus \{\eta\} \) are unchanged, and the current flowing from \( \sigma_1 \) to \( \sigma_2 \) equals \( i(\sigma_1, \eta) \).

Note that, since this network reduction does not change the potential or current flows on any of the vertices or edges in the reduced graph, we must have that all the effective resistances calculated between sets that do not contain \( \eta \) must also be the same (we define them in terms of the strength of the currents and potentials). In particular if we can reduce the entire network in this way to a single resistor between the sets we are interested in then the value of the resistor must be equal to the effective resistance.

Example 3.2. Consider a random walk on \( \mathbb{Z} \) that moves right with probability \( p \) and left with probability \( q = 1 - p \). For the usual Gamblers ruin problem we would be interested in \( \mathbb{P}_k(\tau_0 < \tau_N) \). From detailed balance we have \( c(i,i+1) = (p/q)^i \) (e.g. take as reversible measure \( \pi(i) = \frac{1}{q}(\frac{p}{q})^i \) which is actually a probability measure on \( \mathbb{N}_0 \) if \( p < q \), but we don’t mind if it is not normalised). We may now apply the series law ‘on either side’ of the initial state \( k \). The resistance between \( 0 \) and \( k \) is \( R(\{0\}, \{k\}) = \sum_{i=0}^{k-1} (q/p)^i \) and the resistance between \( k \) and \( N \) is \( R(\{k\}, \{0\}) = \sum_{i=k}^{N-1} (q/p)^i \). By our previous observations \( v^1_{0,N}(k) = \mathbb{P}_k(\tau_0 < \tau_N) \). Since potentials and current flows are unchanged by the application of the series law we know the strength of the flow \( v^1_{0,N} \) is equal in both networks, and is \( C(\{0\}, \{N\}) \) by definition. By Ohm’s law on the reduced network \( v^1_{0,N}(0) - v^1_{0,N}(k) = 1 - v^1_{0,N}(k) = C(\{0\}, \{N\})R(\{0\}, \{k\}) \), therefore it follows that the probability of ruin is \( (p/q)^N - k - 1)/(p/q)^N - 1 \).

Lemma 3.3 (Parallel Law). Two conductors \( c_1 \) and \( c_2 \) in parallel are equivalent to a single conductor with conductance \( c_1 + c_2 \). That is if \( e_1 \) and \( e_2 \) are both edges that join \( \eta, \sigma \in \Omega \) they can be replaced with a single edge \( \{\eta, \sigma\} \) with conductance \( c(e_1) + c(e_2) \) and all the potentials and currents in \( G \setminus \{e_1, e_2\} \) are unchanged, and the current flowing from \( \eta \) to \( \sigma \) equals \( i(e_1) + i(e_2) \).
Proof. The proofs of the above lemmas just requires checking Kirchoff’s and Ohm’s laws with the respective current flows, and that this does not change the voltage function. Or, equivalently check that the new voltage function on the reduced network is still harmonic on the appropriate set.

Lemma 3.4 (Gluing). If two neighbouring vertices have the same voltage, then no current flows between them, and we can glue them in to a single vertex while keeping all other edges, without changing the potential or current flow.

Thomson’s principle immediately gives us another useful tool for bounding resistances.

Theorem 3.5 (Rayleigh monotonicity principle). The effective resistance \( R(A, B) \) is a non-decreasing function of the edge resistances \( (r(e))_{e \in E} \).

Proof. Let \( i_{A,B} \) and \( i'_{A,B} \) be the two unit currents associated with a voltage across \( A \) and \( B \), with resistances \( (r(e))_{e \in E} \) and \( (r'(e))_{e \in E} \) respectively (we drop the bar). Suppose \( r(e) \leq r'(e) \) for all \( e \in E \). By Thomson’s principle we have

\[
R(A, B) = \frac{1}{2} \sum_{e \in E} (i_{A,B}(e))^2 r(e) \leq \frac{1}{2} \sum_{e \in E} (i'_{A,B}(e))^2 r(e).
\]

Now by assumption on the edge resistances and Thomson’s principle again

\[
R(A, B) \leq \frac{1}{2} \sum_{e \in E} (i'_{A,B}(e))^2 r(e) \leq \frac{1}{2} \sum_{e \in E} (i'_{A,B}(e))^2 r'(e) = R'(A, B).
\]

\[ \square \]

Corollary 3.6. Adding edges, that are not incident with \( A \) can only increase the probability of escaping \( A \).

3.2 Pólya’s Theorem and proof

We now apply the tools we have developed to the problem of identifying for which dimensions \( d \) the simple random walk on \( \mathbb{Z}^d \) is transient. Slightly more generally we consider a infinite connected graph \( G = (\Omega, E) \) with finite vertex degree and conductances \((c(e))_{e \in E}\), and a distinguished vertex \( 0 \in \Omega \) called the origin. Let \( d(\cdot,\cdot) \) denote the graph distance, i.e. the length of the shortest path between two vertices, then let

\[
\Omega_n = \{ \eta \in \Omega : d(0, \eta) \leq n \} \quad \text{and} \quad \partial \Omega_n = \{ \eta \in \Omega : d(0, \eta) = n \}
\]

and \( G_n = (\Omega_n, E_n) \) be the sub-graph of \( G \) given by all the edges between vertices in \( \Omega_n \). For each \( n \) we construct a graph \( G^*_n = (\Omega^*_n, E^*_n) \) by gluing all the vertices in \( \Omega \setminus \Omega_n \) together, i.e. \( \Omega^*_n = \Omega_n \cup \{\zeta_n\} \) and \( E^*_n = E_n \cup \{\{\eta, \zeta_n\} : \eta \in \partial \Omega_n\} \). The weight function on \( G^*_n \) is equal to \( c \) on all edges present in \( G \) and on the edges that connect with \( \zeta_n \) we put \( c(\eta, \zeta_n) = \sum_{\xi \in \Omega^*_n} c(\eta, \xi) \).

Let \( R_{\text{eff}}(n) = R(\{0\}, \{\zeta_n\}) \), it follows from the Rayleigh’s monotonicity principle that the limit

\[
R_{\text{eff}} = \lim_{n \to \infty} R_{\text{eff}}(n)
\]

exists, and therefore

\[
\mathbb{P}_0(\tau^+_0 = \infty) = \lim_{n \to \infty} \mathbb{P}_0(\tau_{\zeta_n} < \tau^+_0) = \lim_{n \to \infty} \frac{1}{R_{\text{eff}}(n)} = \frac{1}{R_{\text{eff}}}.
\]

We define a flow from \( 0 \) to \( \infty \) on \( G \) as an anti-symmetric function \( \theta \) satisfying \( \text{div} \, \theta(\eta) = 0 \) for all \( \eta \neq 0 \).
Proposition 3.7. The process $(X_n)_{n \geq 0}$ associated with $G = (\Omega, E)$ with finite vertex degree and conductances $(c(e))_{e \in E}$ is;

1. recurrent if and only if $R_{\text{eff}} = \infty$,
2. transient if and only if there exists a non-zero 0 to $\infty$ flow $i$ on $G$ satisfying $D(i) < \infty$.

Proof. The first part follows directly from 3.2.1, since the walk is recurrent if and only if $\mathbb{P}_0(\tau_0^+ = \infty) = 0$, for more details see Grimmett, Probability on graphs.

For 2. we first assume that the chain is recurrent, i.e. $R_{\text{eff}} = \infty$, and let $i$ be a non-zero 0 to $\infty$ flow on $G$, by dividing by its strength we may assume (wlog) that it is a unit flow. By Thomson principle there exists a unit flow $i_n$ on $G_n^*$ such that $D(i_n) = R_{\text{eff}}(n)$. It is simple to check that the obvious restriction of $i$ to $G_n^*$ is a unit flow from 0 to $\zeta_n$, so by Thomson's principle

$$D(i) \geq \sum_{e \in E_n} i(e)^2/c(e) \leq D(i_n).$$

Hence $D(i) \geq \lim_{n \to \infty} D(i_n) = R_{\text{eff}} = \infty$.

Suppose, conversely, that the chain is transient, i.e. $R_{\text{eff}} < \infty$. There exists a subsequence $n_k$ such that $i_{n_k}(e) \to j(e)$ for every $e \in E$ (sometimes called diagonal selection). Since $i_{n_k}$ is a unit flow from the origin, $j$ must be a unit flow from 0 to $\infty$. Now,

$$D(i_{n_k}) = \sum_{e \in E_{n_k}} i_{n_k}(e)^2/c(e) \geq \sum_{e \in E_m} i_{n_k}(e)^2/c(e)$$

for $m \leq n$, so

$$\infty > R_{\text{eff}} = \lim_{k \to \infty} D(i_{n_k}) \geq \lim_{k \to \infty} \sum_{e \in E_m} i_{n_k}(e)^2/c(e) = \sum_{e \in E_m} j(e)^2/c(e) \to D(j) \quad \text{as} \quad m \to \infty,$$

therefore $j$ is a flow with the required property. \hfill \Box

Corollary 3.8. If $G \subset G'$ and $G$ is transient, then $G'$ is also transient. If $G'$ is recurrent then $G$ is also recurrent.

Proof. This follows immediately from the previous proposition and Rayleigh's monotonicity principle. \hfill \Box

Theorem 3.9 (Pólya’s Theorem). The simple random walk on $\mathbb{Z}^d$ is recurrent if $d = 1, 2$ and transient if $d \geq 3$.

In this case we are applying the tools we have developed so far to $G = (\mathbb{Z}^d, \mathcal{E} = \{(x, y) : |x - y|_1 = 1\})$, with conductances $c(x, y) = 1$ if and only if $(x, y) \in \mathcal{E}$. Note that in this case the reversible measure $\pi$ is uniform on $\mathbb{Z}^d$, with $\pi(x) = 2d$ for all $x$, all our theory so-far applies equally well whether or not $\pi$ is normalised. In general, for a countable graph with finite vertex degree, we call any process with associated with edge conductances which are all constant the random walk on the graph.

Proof. It is clear by the previous corollary that we can restrict to the cases $d = 2$ and $d = 3$. Firstly, for $d = 2$ we aim to show that $R_{\text{eff}} = \infty$, and we therefore only need a lower bound on $R_{\text{eff}}(n)$ that is diverging with $n$. By Rayleigh’s monotonicity principle we reduce the effective resistance by adding edges (formally this corresponds to reducing the resistance between states from $\infty$). In this case we connect all the states in $\partial \mathbb{Z}_n$ with edges that have zero resistance, or equivalently $\infty$ conductance. These vertices must be at the same voltage and all neighbours in
the new graph, so we may glue them. We are left with a one dimensional graph that we can reduce using the series/parallel laws, so

\[
R_{\text{eff}}(n) \geq \sum_{i=1}^{n-1} \frac{1}{4(2i - 1)},
\]

and hence \( R_{\text{eff}}(n) \geq C \log n \to \infty \) as \( n \to \infty \). Alternatively we can construct Nash-Williams cut sets using the same sets (see Peres and Lyons Probability on trees and networks).

Now suppose \( d = 3 \), we wish to construct a non-zero flow from 0 to \( \infty \) such that \( D(j) < \infty \). There are many ways to do this of course, but it makes sense to make the most of the symmetry and take inspiration from the obvious flow field on \( \mathbb{R} \) given by a unit source at the origin. We construct such a flow as follows. Let \( u \sim \text{Uniform}(S^3) \) be a uniform random point on the unit sphere (with associate measure \( \mathbb{P} \)), and let \( L_u \) be the unique straight half-line from 0 to \( \infty \) that passes through the point \( u \). Let \( \gamma^u \) be the (almost surely) unique path in \( G = (\mathbb{Z}^3, E = \{(x,y) : |x - y|_1 = 1\}) \) from 0 to \( \infty \) that is closest to \( L_u \) in the Hausdorff metric, thought of as a sequence of directed edges. Clearly \( \gamma^u \) stays within a distance of 4 from \( L \). Now define the flow by

\[
j(e) = \sum_{n \geq 0} \left[ \mathbb{P}(\gamma^n_{n+1} = e) - \mathbb{P}(\gamma^n_{n+1} = -e) \right].
\]

You should check that this defines a flow (in fact a unit flow) from 0 to \( \infty \), this trick of defining unit flows as the expectation of random unit flows is a more general one. We now show that \( j(e) \) has finite energy. By construction, for any given edge \( e \in E \) whose midpoint is a distance \( R \) from the origin, there is a constant \( A \) (independent of \( R \)) such that \( \mathbb{P}(e \in \gamma^u) \leq \frac{A}{R^2} \). Also, there is a constant \( B \) such that there are at most \( \frac{Bn^2}{n} \) edges a distance between \( n \) and \( n+1 \) from the origin. It follows that \( D(j) \leq \sum_{n=1}^{\infty} A^2n^{-4}Bn^2 \), which is finite, and the result follows from Proposition 3.7 2.. 

\[ \square \]
Chapter 4

Uniform Spanning Tree and Wilson’s algorithm

We have already seen the idea of uniform spanning trees (USTs) appear briefly in the previous chapter (see Theorem 2.6). It turns out that connection between USTs and random walks on graphs (and the electrical network analogy) goes much deeper. Also these objects turn out to have connections with our next topic - percolation.

In this chapter we will see our first correlation inequality, these turn out to be extremely useful, and we will revisit such concepts in more detail later. In large graphs there are typically a huge number of spanning trees, and it is not obvious how to generate one in an efficient algorithmic way. We will see one very effective way called Wilson’s algorithm which is efficient and theoretically very important, with links to loop erased random walks. At the end of the section we will touch on the idea of weak limits of finite-volume measures, a concept that will also be important later.

A spanning tree of an undirected graph will be composed of undirected edges, since we connect with the previous electrical network picture we will also consider directed graphs and directed spanning trees. It should be clear from context which case we are dealing with.

Definition 4.1. Let $G = (\Omega, E)$ be a finite connected graph. A spanning tree of $G$ is a sub-graph of $G$ that is connected, contains all vertices of $G$, and is a tree (has no cycles). Let $T$ denote the set of all spanning trees of $G$.

We call $T$ a uniform spanning tree, abbreviated UST, if it is distributed according to the uniform measure on $T$, i.e.

$$P(T = \omega) = \frac{1}{|T|}, \quad \omega \in T.$$  

We may think of $T$ as being a random subset of all the edges $E$ (satisfying the constraints to be a spanning tree), in this sense we can identify $T$ with a special subset of $\Omega_E = \{0, 1\}^E$, that is for $w \in T$ we may think of $\omega(e) = 1$ if the edge $e \in E$ is present in the tree and $\omega(e) = 0$ if the edge does not belong to the tree.

4.1 Wilson’s algorithm

There are many ways to generate a UST of $G$. The following algorithm highlights the close connection between USTs and random walks, and exploits some hidden independence in Markov chains. In fact we will present the algorithm in sufficient generality that it can not only sample UST be other weighted random trees.

We will consider the directed, and connect, graph $G$ generated by a finite, irreducible, Markov process with transition matrix $P$. For $e = (\eta, \sigma) \in E$ we call $e^- = \eta$ the tail of the edge, and $e^+ = \sigma$ the head.

Definition 4.2. We call a sub-graph of $G$, with one vertex $r \in \Omega$ marked as a root, a (directed and rooted) spanning tree (sometimes called a spanning arborescence) if it includes every vertex, there is no cycle, and every vertex other than the root is the tail of exactly one edge.
4.1. WILSON’S ALGORITHM

The root \( r \) will be arbitrary but fixed from this point onwards. The edges in the directed spanning tree all point towards the root. It is clear that if we ignore the root and the edge directions then we obtain a spanning tree. Wilson’s algorithm will select spanning trees proportional to the weights

\[
\Psi(T) = \prod_{e \in T} P(e^-, e^+),
\]

by explicitly using the process with transition matrix \( P \). If the process is reversible, recall \( c(e) = \pi(e^-)P(e) \), so that

\[
\Psi(T) = \frac{\prod_{e \in T} c(e)}{\prod_{h \neq r} \pi(\eta)} \propto \prod_{e \in T} c(e),
\]

since the root is fixed. In particular for the simple random walk on \( G \) we have \( P(\eta, \sigma) = 1/\deg(\eta) \) if \( \eta \sim \sigma \), and all other entries are zero, and the reversible stationary measure satisfies \( \pi(\eta) \propto \deg(\eta) \), i.e. in this case we will select a UST (when we ignore the direction and root).

To describe Wilson’s algorithm we must first define the important concept of loop-erasure (meaning cycle erasure since we have already used loop in the context of the graph having self loops). For any path \( \Gamma = (\eta_0, \eta_1, \ldots, \eta_k) \) on the graph \( G \), meaning \( (\eta_i, \eta_{i+1}) \in E \) for \( 0 \leq i < k \), which can be thought of as a connected set of edges, we can construct a non-intersecting sub-path, we call \( \text{LE}(\Gamma) \), by removing cycles in the order they appear. Note that paths can be thought of as a sequence of edges and induce a sub-graph in the obvious way. More precisely let \( J = \min\{j \geq 1 : \eta_j = \eta_i \text{ for some } i < j\} \) be the first time you re-visit a vertex in the path, and let \( I \) be the index of the point in the path it hits, i.e. \( I \) is the unique value of \( i \) satisfying \( i < J \) and \( w_i = w_J \). Let \( \Gamma' = (\eta_0, \ldots, \eta_I, \eta_{I+1}, \ldots, \eta_k) \) be the sub path attained after removing the first cycle. This procedure is iterated until we arrive at a path with no cycles, which is denoted \( \text{LE}(\Gamma) \) (Pictured in lectures).

Wilson’s algorithm: First let \( (\sigma_1, \sigma_2, \ldots, \sigma_n) \) be an arbitrary fixed ordering of \( \Omega \setminus \{r\} \), and let \( T_0 = \{r\} \).

1. If \( T_i \) spans \( G \) we are done and the algorithm stops. Otherwise pick the smallest element of \( \Omega \) that belongs to \( T_i \), call this \( \eta_i \), and run the process with transition matrix \( P \) on \( G \) started from \( \eta_i \) until it hits \( T_i \), call the path it followed \( \Gamma_i \).

2. Let \( T_{i+1} = T_i \cup \text{LE}(\Gamma_i) \) be the union of the two edge disjoint paths.

It is clear that each stage of the algorithm gives a sub-tree of \( G \), directed toward the root, also the final tree must be a spanning tree since it contains every element of \( \Omega \). Marvellously, the random tree grown in this way has the desired distribution.

Theorem 4.3. Wilson’s algorithm yields a random spanning tree rooted at \( r \) with distribution proportional to \( \Psi(\cdot) \).

Note that, in the reversible case, if we forget the root and the orientation then Wilson’s algorithm will construct a weighted spanning tree of \( G \) were each tree has probability proportional to \( \prod_{e \in E} c(e) \).

In order to prove this result we will construct a Markov process with transition matrix \( P \) in a special way and remove cycles in a way that is called ‘cycle popping.’ We therefore have some setting up to do, and first prove a deterministic result at the heart of the algorithm. We think of fixing the randomness of the process in advance, as follows. At each state we assign an infinite sequence of moves (stack of instructions) with the correct distribution. That is, for each \( \eta \in \Omega \setminus \{r\} \) let \( S_i^\eta, i \geq 1 \), be independent random variables laws

\[
P(S_i^\eta = \sigma) = P(\eta, \sigma), \text{ for } \sigma \in \Omega.
\]
4.1. WILSON’S ALGORITHM

Since we only care about the process up to the first time it hits the root we put an empty stack on the root. Then the process which moves according to the next instruction in the stack each time it visits any state, and then discards that instruction, is clearly a Markov process with the given transition matrix (absorbed at the root).

The top of these stacks, or visible moves, always determines a directed graph via the edges \((\eta, S^n_i), \eta \in \Omega\) and \(i \geq 1\), which we call the visible graph. If this visible graph contains no directed cycles then it is a spanning tree rooted at \(r\). Otherwise we will proceed by removing visible cycles in a certain order, called ‘cycle popping’. When a cycle is popped, the moves in that cycle get removed from the top of the stack. It turns out the actual order that visible cycles are popped does not matter.

**Lemma 4.4.** The order in which cycles are popped is irrelevant. Either every order props an infinite number of cycles, or every order pops the same finite set of cycles, thus leaving the same tree on the top of the stacks at the end.

**Proof.** We will track the position of the moves in the stack, we say that the edge \((\eta, S^n_i)\) has colour \(i\). A coloured cycle is a visible cycle all of whose edges have an associated colour. The edges in a coloured cycle do not all have to be of the same colour. Although the same cycle could be popped several times, a coloured cycle can be popped at most once. We will show that if \(C\) is any coloured cycle that can be popped, i.e. there exists a sequence \(C_1, C_2, \ldots, C_n = C\) that may be popped in that order, but we instead chose to pop some other coloured cycle \(C' \neq C\) first, then either; (1) \(C = C'\) or else, (2) \(C\) can still be popped from the stack after \(C'\). Showing this completes the proof, since we either never stop popping or every cycle that can pop will be popped.

If all the vertices of \(C'\) are disjoint from those of \(C_1, \ldots, C_n\) then \(C\) can obviously still be popped. Otherwise, let \(k\) be the first cycle with at least one vertex in common with \(C'\). Fix \(\eta \in C' \cap C_k\), since all edges in \(C'\) have colour 1 by assumption, and \(\eta \notin C_1, \ldots, C_{k-1}\), the edge with head \(\eta\) in \(C_k\) must also have colour 1. It follows that the edge with head \(\eta\) in \(C'\) and \(C_k\) must both point to the same state, let’s call it \(\sigma\). We can now repeat the same argument for the state \(\sigma\), and it follows that \(C' = C_k\). Therefore \(C' = C\) or we can pop \(C\) in the order \(C', C_1, C_2, \ldots, C_k-1, C_{k+1}, \ldots, C_n\).

We are now in a position to prove Theorem 4.3.

**Proof of Theorem 4.3.** It is clear that Wilson’s argument stops with probability 1, it also pops visible cycles in some order, and so by the previous lemma the output is independent of the choice of order in the implementation.

We now show that the final distribution is the desired one. We think of the stacks (up to the point we stop popping) as defining a finite set of coloured cycles (loops) \(O = (C_1, C_2, \ldots, C_n)\) lying over a non coloured spanning tree \(T\), and let \(\Pi\) be the set of all such pairs we can construct. If \((O, T) \in \Pi\) then so is \((O, T')\) for any spanning tree \(T'\) since the final tree is dictated by the values on the bottom of the stacks after popping all the cycles, and the values of the preceding elements in the stack imparts no information on these. So \(\Pi = O \times T_r\) where \(O\) is the set of all sequences \(O\) that can occur, and \(T_r\) is the set of directed spanning trees rooted at \(r\). By construction the probability of seeing a pair \((O, T) \in \Pi\) is given by \(\prod_{C \in O} \prod_{i \in C} P(e^-, e^+)^\Psi(T)\). Since the probability factorises the random sequence of popped cycles and the final tree are independent and the marginal on the tree is the desired one.

The consequences of Wilson’s algorithm are far reaching, we state some consequences of Theorem 4.3 here.

**Corollary 4.5.** Given two states \(\eta\) and \(\sigma\) in \(\Omega\) finite), the distribution of the path in the weighted spanning tree from \(\eta\) to \(\sigma\) is equal to the distribution of the loop-erased path taken by the process from \(\eta\) to \(\sigma\).
The next corollary is somewhat less trivial, and although it can be proved in many ways it also follows from Theorem 4.3.

**Corollary 4.6.** The number of un-rooted trees with $n$ vertices is $n^{n-2}$.

One proof is to examine the USTs of the complete graph on $\{1, 2, \ldots, n\}$, and in particular calculate the probability that a UST is given by a single path $(1, 2, \ldots, n)$, which occurs with probability $1/n^{n-2}$, as required.

# 4.2 Electrical interpretations and negative association

In this section we will discuss some important theoretical consequence of the connection between random walks and spanning trees on finite networks, and in particular the connection again with electrical networks.

It turns out that knowing any given edge is present in the tree can never increase the probability that another edge is also in the tree. The most immediate way to express this property of *negative association* is as follows.

**Theorem 4.7.** For $e_1, e_2 \in E$ such that $e_1 \neq e_2$,

$$\mathbb{P}(e_2 \in T \mid e_1 \in T) \leq \mathbb{P}(e_2 \in T).$$

(4.2.1) The proof makes use of the link between the current flow and the probability of the random walk crossing given edges. In particular the following results which you have already proved part of on the first example sheet.

**Lemma 4.8** (Kirchhoff’s Effective Resistance Formula). Let $T$ be an un-rooted weighted spanning tree (with law $\mathbb{P}$) of a finite network $G$ (associated with transition matrix $P$, i.e. edge weights $c(e) = \pi(e^-)P(e^-, e^+) + \pi(e^+)P(e^+, e^-)$ and $e \in E$, then

$$\mathbb{P}(e \in T) = \mathbb{P}_{e^-}(1\text{st hit } e^+ \text{ via travelling along } e) = i(e) = c(e)R(\{e^+\}, \{e^-\}),$$

where $i$ is the unit current flow from $e^-$ to $e^+$.

**Proof.** Exercise. Hint use the current as edge crossing probabilities result from Sheet 1. Note that this result also follows immediately from Theorem 2.6. \qed

**Proof of Theorem 4.7.** Denote the dependence of the random tree on the graph by $T_G$. The contraction of the graph $G$ along the edge $e$ is defined by removing the edge $e$ and identifying the end points, and is denoted by $G/e$ (gluing the vertices connected via $e$), and leaving the other edges and vertices the same. Note that even if $G$ is a simple graph the contraction may be a multi-graph. There is a one-to-one correspondence between spanning trees of $G$ containing the edge $e$ and spanning trees of $G/e$. Furthermore, the distribution of $T_G/e$ given $e \in T_G$ is the same as that of $T_{G/e}$. It follows from the previous lemma that

$$\mathbb{P}(f \in T \mid e \in T) = \mathbb{P}(f \in T_{G/e}) = R_{G/e}(\{f^+\}, \{f^+\}) \leq R_G(\{f^-\}, \{f^+\}) = \mathbb{P}(f \in T),$$

where the key inequality follows from Rayleigh’s monotonicity principle. \qed

The negative association in Theorem 4.7 was generalised by Feder and Mihail in 1992 to more general `increasing` events and random variables, as follows. Recall we can think if subgraphs of $G$ as elements of $\{0, 1\}^E$, on which there is an obvious partial ordering, $\omega \leq \omega'$ if $\omega(e) \leq \omega'(e)$ for all $e \in E$ (this type of partial ordering will be come up again later on our typical statespace for IPS $S^A$). An event $A \subset \{0, 1\}^E$ is called increasing if for all $\omega, \omega' \in \{0, 1\}^E$ with $\omega \leq \omega'$, we have that $\omega' \in A$ whenever $\omega \in A$, i.e. adding any edge to any configuration (subgraph) in $A$ results in another configuration (subgraph) in $A$. Examples of increasing events include, the edge $e$ is in the sub-graph, and there is a path connecting $x$ to $y$ in the sub-graph. A random variable $X : \{0, 1\}^E \to \mathbb{R}$ is increasing if $X(\omega) \leq X(\omega')$ whenever $\omega \leq \omega'$. 


Theorem 4.9. Let $G$ be a finite network and $A$ an increasing event that ignores the edge $e \in E$, then $\mathbb{P}(A \mid e \in T) \leq \mathbb{P}(A)$. Furthermore, if $X$ and $Y$ are increasing random variables then $E[XY] \leq E[X]E[Y]$.

For proofs see Chapter 3, Section 4.2 in Lyons and Peres Probability on Trees and Networks. Later we will see an extremely useful counterpart to this (with the inequality the other way) called ‘positive association’ or positive correlations, which will be called the FKG inequality.

4.3 Weak limits

It turns out that Wilson’s algorithm can also be run using recurrent, irreducible Markov processes on countable graphs, provided we start the loop-erased walks appropriately so that each vertex belongs to the final tree. Can this object on the countable graph be interpreted in terms of weighted spanning trees? Usually the weighted spanning tree on the $d$-dimensional cubic lattice is not defined directly on the infinite graph, but instead on finite subgraphs $G_n$ which increase to the cubic lattice as $n \to \infty$. Then the probability of observing an event that only depends on finitely many edges converges to the probability of observing this event on the infinite lattice as $G_n \to (\mathbb{Z}^d, E)$. Also, it turns out that these local probabilities determine the distribution of the infinite tree, that is the infinite object locally looks like a weighted spanning tree. To make this discussion more precise we will examine weak limits of probability measures.

4.3.1 A Topological Aside

The state space for random trees as viewed as subsets of $\{0, 1\}^E$ is of the more general form that will be of interest later for IPS and was discussed in the intro, i.e. $S^\Lambda$ for some finite $S$ and countable $\Lambda$. So the discussion here will be relevant also later in the course. We endow this space with the product topology, i.e. the coarsest topology with respect to which all the canonical projections, $p_e : \{0, 1\}^E \to \{0, 1\}$ given by $p_e(\omega) = \omega(e)$, are continuous. Equivalently this is the topology is generate by countable unions and finite intersections of sets of the form

$$p_e^{-1}(U) = \{ \omega \in \{0, 1\}^E : \omega(e) \in U \} \quad \text{for } U \subset \{0, 1\},$$

called open cylinders. Note that since we take the discrete topology on $\{0, 1\}$ (or more generally finite $S$) every subset if clopen. Finite intersections of open cylinders, of the form

$$\{ \omega \in \{0, 1\}^E : \omega(e_i) \in U_1, \ldots, \omega(e_n) \in U_n \} \quad \text{for } U_i \subset \{0, 1\}, \ 1 \leq i \leq n,$$

are called cylinder sets, in words these are exactly the events that depend on only finitely many coordinates. Since $\{0, 1\}$ is compact (more generally if $S$ is finite), then by Tychonoff’s Theorem $\{0, 1\}^E$ is compact. It is also metrisable by

$$d(\omega, \omega') = \sum_{i=1}^{\infty} 2^{-i} |\omega(e_i) - \omega'(e_i)|, \quad \omega, \omega' \in \{0, 1\}^E,$$

moreover this gives us a complete separable metric space.

For a measurable structure on $\{0, 1\}^E$ we take the product $\sigma$-algebra $\mathcal{F}$ of $\{0, 1\}^E$ generated by the cylinder sets. Equivalently this is just the Borel $\sigma$-algebra with respect to the topology discussed above.

We would like to discuss convergence of probability measures, as such we will take the topology of weak convergence on the set $\mathcal{M}_1(\{0, 1\}^E)$ of probability measures on our state space. We say $\mu_n$ converges weakly to $\mu$, written $\mu_n \to \mu$ as $n \to \infty$, if

$$\mu_n(f) \to \mu(f) \quad \text{as } n \to \infty,$$
for all bounded continuous functions \( f : \{0,1\}^E \to \mathbb{R} \). Recall \( \mu(f) \) denotes the expectation of the function \( f \) with respect to \( \mu \). Since \( \Omega \) is compact \( \mathcal{M}_1 \) is also compact with respect to this topology, and every family of probability measures on \((\{0,1\}^E, \mathcal{F})\) is relatively compact (this turns out to be very useful). Since all cylinder events are both open and closed, and they generate \( \mathcal{F} \), we also have \( \mu_n \to \mu \) if and only if \( \lim_{n \to \infty} \mu_n(C) = \mu(C) \) for all cylinder sets \( C \).

### 4.3.2 Back to the UST measure

We make a slight shift in notation in line with the topological discussion previously. Let \( \mu_n \) be the distribution of the UST on the \( d \)-dimensional box \( \Lambda_n = [-n, n]^d \), with edge set given by the usual \( \ell_1 \) neighbours (i.e. the sub-graph of the \( d \)-dimensional lattice). As above, we think of this as a measure on \( \Omega_n = \{0,1\}^{E_n} \) where \( E_n \) are the edges contained in \( \Lambda_n \) (we should be careful with our change of notation since we previously called the vertex set of the graph \( \Omega \)).

**Theorem 4.10.** The weak limit \( \mu = \lim_{n \to \infty} \mu_n \) exists, it is a translation-invariant and (shift)-ergodic probability measure. It is supported in the set of forests of \((\mathbb{Z}^d, E)\) (i.e. no cycles but possibly not connected) with no bounded component.

For \( x \in \mathbb{Z}^d \) let \( \theta_x \) be the shift operator, i.e. \( \theta_x(y) = y + x \) for \( y \in \mathbb{Z}^d \). This induces an obvious action on the edge set and \( \Omega \), by \( \theta_x(e) = (\theta_x(e^-), \theta_x(e^+)) \) and \( \theta_x(\omega) = (\omega(\theta_x(e)) \in E^d \). In this was the translation by \( x \) is a graph automorphism on the \( d \)-dimensional lattice.

An event \( A \in \mathcal{F} \) is called shift-invariant if \( A = \theta_x(A) := \{ \theta_x(\omega) : \omega \in A \} \) for each \( x \in \mathbb{Z}^d \). A probability measure \( \nu \) is called ergodic (with respect to the shift) if every shift invariant event has probability 0 or 1. That is in words, the operator mixes things up with respect to \( \nu \), that is no event with non-trivial measure is left fixed by shifts. A probability measure is said to be supported on an event \( F \in \mathcal{F} \) if \( \nu(F) = 1 \).

**Proof of Theorem 4.10.** In the interest of time we skip this proof, but it may be found for example in Grimmett *Probability on Graphs*.

An important question now is; when is this spanning forest actually a tree? We will not prove the following result, see for example Pemantle (1991).

**Theorem 4.11.** The limit measure \( \mu \) is supported on the set of spanning trees if and only if \( d \leq 4 \).

For proof of a similar result in \( d \leq 2 \) (in fact more generally when the random walk is recurrent), using the current flows see Lyons and Peres, *Probability on Trees and Networks*.

Another important question becomes, does the limit in Theorem 4.10 depend on what we did with the boundary of the set \( \Lambda_n \). As it stands we did not insist on anything special happening on the boundary (a sort of ‘free’ boundary condition), and so \( \mu_n \) is sometimes called the free uniform spanning tree measure. However we could have imposed other boundary conditions, one natural thing would be to glue all the points outside \( \Lambda_n \) into a single vertex connected to the boundary of \( \Lambda_n \) (as we did earlier). This gives rise to the so called wired uniform spanning tree measure. These turn out to give rise to the same weak limit on the \( d \)-dimensional integers lattice, and the key reason why the limits are the same is because these graphs are edge amenable, meaning the the ratio of the size of the edge set and the volume of large boxes goes to zero

\[
\lim_{n \to \infty} \frac{|\partial E \Lambda_n|}{|\Lambda_n|} = 0,
\]

where \( \Lambda_n \neq \mathbb{Z}^d \) and \( \partial E \Lambda_n \) is the set of edges that connect to \( \Lambda_n \) from outside. Then we could ask, when is the limiting object the same thing. It turns out in this case that we do get the same thing, but we will return to this in more generality later when we look at Gibbs sates.
Chapter 5

Percolation

Percolation theory was introduced over 50 years ago as a mathematical framework for studying random spatial processes such as the flow of liquids through disordered porous mediums. It gave rise to a huge amount of work in both applied and pure mathematics. These relatively simple models to define have given rise to a spectacular array of challenging mathematical problems, with many deep results (with surprising connections to other areas of mathematics) now proven, and many more still conjectured. One of the central features of these models, that we will be particularly interested in during this course, is the concept of a phase transition. This is a point in parameter space where the behaviour of the model changes in some abrupt way (we will be more precise later). More generally, these models give rise to very rich large scale random geometries.

5.1 Model definition

We will focus on independent bond percolation on \( \mathbb{Z}^d \), as opposed to site percolation. We fix the underlying graph \( G = (\mathbb{Z}^d, E) \), where \( E = \{ \{x, y\} : |x - y|_1 = 1 \} \), and a parameter \( p \in [0, 1] \).

To each edge \( e \in E \) we assign a state 0 or 1, corresponding to closed or open respectively. Our probability space will therefore be \( \Omega = \{0, 1\}^E \) endowed with the Borel \( \sigma \)-algebra generated by finite dimensional cylinder sets that we discussed at the end of the last chapter. For a probability measure we take the product measure \( P_p \), parametrised by \( p \), under-which each edge is independently open with probability \( p \). We think of an open edge as being open to the passage of some material, such as a liquid through a porous media.

It turns out that the following construction, that simultaneously couples bond percolation on \( G \) for all parameter values \( p \in [0, 1] \) on the same probability space, is often useful. The idea of coupling is an important one, the concept is deceptively simple looking and extremely general, but it is a very handy tool. By coupling, we simply mean that the random variables are defined on a single probability space, with a single measure, such that the marginal on each of the random variables is the desired one. In this way there is a huge amount of freedom in specification of the joint law. Suppose \( (U_e : e \in E) \) is a family of independent uniform random variables on \( [0, 1] \), then define

\[
\omega_p(e) = \begin{cases} 
1 & \text{if } U_e < p, \\
0 & \text{if } U_e \geq p,
\end{cases}
\]

and we call the associated probability measure \( P \) (without the \( p \) subscript). Then the marginal law on \( \omega_p \in \Omega \) for a fixed \( p \in [0, 1] \) is bond percolation on \( G \) with parameter \( p \), i.e. \( (\omega_p(e) : e \in E) \) is independent in its components and \( P(\omega_p(e) = 1) = P_p(\omega(e) = 1) = p \).

We are typically interested in the geometry of the open sub-graph, in particular the connected components, or clusters, of this sub-graph containing a given vertex \( x \), called \( C_x \). For \( x, y \in V \) we write \( x \leftrightarrow y \) if there exists a path of open edges connecting \( x \) and \( y \), and so

\[
C_x = \{ y \in V : x \leftrightarrow y \}.
\]

We also write \( x \leftrightarrow \infty \) if \( |C_x| = \infty \). Since the graph \( G \) is translation invariant (vertex transitive - i.e. for each pair \( x, y \in V \) there exists a graph automorphism that takes \( x \) to \( y \)), it is clear that \( |C_x| \) has the same distribution as \( |C_0| \), and so we will focus on the latter.
5.2. THE PHASE TRANSITION

One of the principle concerns is whether or not there exists an infinite open cluster.

**Definition 5.1** (Percolation probability and the critical probability). *The percolation probability as a function of the parameter* \( p \) *is defined as*

\[
\theta(p) = \mathbb{P}_p(|C_0| = \infty).
\]

*The critical probability is defined by*

\[
p_c(d) = \sup\{ p \in [0, 1] : \theta(p) = 0 \}.
\]

By the coupling given above we immediately get the following lemma.

**Lemma 5.2.** *The percolation probability, \( \theta(p) \), is an increasing function of* \( p \), *and*

\[
\theta(p) = \begin{cases} 
0 & \text{if } p < p_c, \\
> 0 & \text{if } p > p_c.
\end{cases}
\]

5.2 The phase transition

In one dimension it is easy to check that \( p_c = 1 \), and there is no non-trivial phase transition. However it turns out that for all dimensions \( d \geq 2 \) there is a non-trivial phase transition.

**Theorem 5.3.** *For all* \( d \geq 2 \) *we have* \( p_c \in (0, 1) \).

It turns out the for all \( p > p_c \) the percolation probability \( \theta(p) \) is a \( C^\infty \) function of \( p \) for \( p \in (p_c, 1] \), by definition it is 0 for \( p < p_c \), and by the previous coupling it must be increasing (this gives the typical picture for the function \( \theta(p) \) in Fig. given in lecture). However, it is still a significant outstanding problem to show that there is continuity at \( p_c \).

**Conjecture 1.** *The percolation probability* \( \theta(p) \) *is continuous at* \( p_c \) *for each* \( d \geq 3 \).

The result is known to hold for \( d = 2 \), and in high dimensions (currently for \( d \geq 11 \)), but not in between - and in particular it is not known for \( d = 3 \).

**Proof of Theorem 5.3.** By considering the obvious embedding it is clear that \( p_c(d + 1) \leq p_c(d) \). Therefore it is sufficient to show that \( p_c(d) > 0 \) for all \( d \geq 2 \) and \( p_c(2) < 1 \). We do the former by a path counting argument, and the latter using a technique which is often referred to as a Peierls contour argument, after Peierls who used a similar argument applied to the two dimensional Ising model.

We first show that \( \theta(p) = 0 \) for \( p \) sufficiently close to zero, i.e. \( p_c(d) > 0 \) for \( d \geq 2 \). We do this by counting self avoiding paths. A self avoiding walk (SAW) is a path in the graph that visits each site at most once. Let \( N_n \) be the the (random) number of SAWs of length \( n \), starting from the origin, in the percolation sub-graph (using only open edges). Then

\[
\theta(p) = \mathbb{P}_p(|C_0| = \infty) = \mathbb{P}(N_n \geq 1 \text{ for all } n \geq 1) = \lim_{n \to \infty} \mathbb{P}_p(N_n \geq 1) \leq \lim_{n \to \infty} E_p(N_n), \quad (5.2.1)
\]

where the last inequality follows from Markov’s inequality. We upper bound the expectation by a crude upper bound on all possible SAWs in \( \mathbb{Z}^d \) and using the independence structure. Let \( \sigma_n \) denote the total number of SAWs, starting from the origin, in the \( d \)-dimensional lattice (deterministic graph). There are \( 2d \) choices for the first edge, and at most \( 2d - 1 \) choices for each
subsequent edge, since they certainly can’t backtrack. Since each edge is open independently with probability \( p \) it follows that

\[
E_p(N_n) \leq \sigma_n p^n \leq 2d(2d-1)^{n-1}p^n = \frac{2d}{2d-1}((2d-1)p)^n.
\]

Combining with Inequality (5.2.1) we find that \( p_c > 1/(2d-1) \).

We now use planar duality and a Peierls argument in two dimensions to show that \( \theta(p) > 0 \) for \( p \) sufficiently large. We now define the dual graph (see figure in lectures). The (planer) dual of a planer graph \( G \) (i.e. a graph embedded in \( \mathbb{R}^2 \) without edges crossing) is described as follows. Each face of \( G \) (with respect to the edges of \( G \)), is associated with a vertex of the dual, and there is an edge between each of these vertices which have an edge in common (share a boundary). The dual of the \( \mathbb{Z}^2 \) lattice is isomorphic to the \( \mathbb{Z}^2 \) lattice, i.e. it is self dual, and in the obvious embedding the dual is just shifted by \((1/2,1/2)\). There is a 1-to-1 correspondence between edges of the dual and edges of the original (primal) graph since each edge of the dual crosses a unique edge of the primal graph. The percolation measure on the primal graph induces a percolation on the dual by calling each edge of the dual open if and only if it crosses an open edge.

Suppose that \(|C_0| < \infty\) then there exists a closed cycle in the dual percolation which encloses \( C_0 \). The converse also holds, if the origin lies inside a closed loop in the dual then \(|C_0| < \infty\). The proof of this fact is straightforward by tedious to write out, so we take it as given (see the figure in lectures). It was originally proved by Kesten in ’82. We now count again. Let \( M_n \) be the number of closed loops in the dual of length \( n \), containing the origin, then

\[
1 - \theta(p) = \mathbb{P}_p(|C_0| < \infty) = \mathbb{P}(\sum_{n \geq 1} M_n \geq 1)
\]

\[
\leq \mathbb{E}_p[\sum_{n \geq 1} M_n] = \sum_{n \geq 1} \mathbb{E}_p[M_n] \leq \sum_{n \geq 1} \rho_n(1-p)^n,
\]

where \( \rho_n \) is the (deterministic) number of loops (closed paths) of length \( n \) containing the origin in the dual lattice. Every loop of length \( n \) which contains the origin must use an edge of the form \((k+1/2,1/2)\) for some \( 0 \leq k < n \) (with the obvious embedding in \( \mathbb{R}^2 \)), and so by applying previous bound on the number of SAWs we have \( \rho_n \leq n\sigma(n-1) \leq n4^n \). So by taking \( p \) sufficiently small we can make the sum on the right hand side of (5.2.2) smaller than 1, as required.

From the self duality in two dimensions you might guess that \( p_c(2) = 1/2 \). That is, closed edges in the dual form a percolation model where each edge is present independently with probability \( 1-p \), if the origin belongs to an infinite cluster in the primal graph then, due to the constrained geometry, it seems unlikely that there is simultaneously an infinite closed cluster in the dual. On the other hand, for \( p < p_c \) we may picture many finite ‘islands’ of open edges in the primal graph, surrounded by a see of closed edges in the dual, and presumably this see contains some infinite cluster. Surprisingly this result was not made rigorous until Kesten in 1980, building on the work of Russo, Seymour and Welsh.

**Lemma 5.4.** Let \( A_\infty \) be the event that there exists an infinite cluster. Then the following dichotomy holds:

\( (a) \) If \( \theta(p) = 0 \) then \( \mathbb{P}_p(A_\infty) = 0 \).

\( (b) \) If \( \theta(p) > 0 \) then \( \mathbb{P}_p(A_\infty) = 1 \).

**Proof.** This result also follows from the fact that \( \mathbb{P}_p \) turns out to be shift-ergodic, which we will see later. For now we can deduce the result immediately from Kolmogorov’s zero-one law.

\( (a) \) Suppose \( \theta(p) = 0 \), the result follows by a simple union bound and translation invariance.

\[
\mathbb{P}_p(A_\infty) \leq \sum_{x \in \mathbb{Z}^d} \mathbb{P}_p(|C_x| = \infty) = \sum_{x \in \mathbb{Z}^d} \theta(p) = 0.
\]
(b) Recall Komogorov’s 0-1 law, if \((X_i)_{i \in \mathbb{N}}\) are i.i.d. r.v.s, \(F_n = \sigma(X_k : k \geq n)\) and \(F_\infty = \cap_{n \geq 0} F_n\) is the asymptotic \(\sigma\)-algebra, then \(F_\infty\) is trivial, i.e. for all \(A \in F_\infty\) either \(\mathbb{P}(A) = 1\) or \(\mathbb{P}(A) = 0\). Now suppose \(\theta(p) > 0\). Order the edges of \(\mathbb{Z}^d\) then \((w(e_i))_{i \in \mathbb{N}}\) are i.i.d., and \(A_\infty \in F_\infty\) since changing finitely many edges does not change the occurrence of \(A_\infty\). Furthermore \(\mathbb{P}_p(A_\infty) \geq \theta(p) > 0\), and so \(\mathbb{P}_p(A_\infty) = 1\). \(\square\)
Chapter 6

Percolation toolbox

In this chapter we present some of the fundamental tools that are often applied in the study of percolation, and more broadly in other areas of probability, for example in the study of random graphs and interacting particle systems.

Recall the natural partial order on $\Omega$ and the definition of increasing events and functions. Note that the events \( \{ |C_0| = \infty \} \) and \( \{ x \leftrightarrow y \} \) are increasing. We can take the consequences of the partial order on the state space yet another step, in that it also induces a partial order on probability measures on $\Omega$ (we do not always make explicit reference to the probability space but it is always the one we discussed in Section 4.3).

**Definition 6.1.** Given two probability measures $\mu_1, \mu_2 \in M_1(\Omega)$ we say $\mu_2$ stochastically dominates $\mu_1$, written $\mu_1 \leq \mu_2$, if

$$\mu_1(f) \leq \mu_2(f) \quad \text{for all increasing functions } f.$$  

(Equivalently we could formulate this in terms of increasing events).

The following results, which states that if $\mu_1 \leq \mu_2$ then the two measures can be coupled in a ‘pointwise monotone’ way, is often called Strassen’s theorem.

**Theorem 6.2** (Strassen’s Theorem). Let $\mu_1$ and $\mu_2$ be probability measures on $\Omega$ then a necessary and sufficient condition for $\mu_1 \leq \mu_2$ is that there exists a probability measure $\nu$ on $\Omega^2$ such that

$$\nu(\{(\eta, \sigma) : \eta \leq \sigma\}) = 1,$$

and whose marginals are $\mu_1$ and $\mu_2$.

**Proof.** Non examinable.

This should be compared with the increasing coupling we gave in the previous chapter for bond percolation. The coupling has the following immediate consequences (which we have already used in the context of $\theta(p)$).

**Proposition 6.3.** If $f$ is an increasing random variable then

$$\mathbb{E}_{p_1}(f) \leq \mathbb{E}_{p_2}(f) \quad \text{whenever } p_1 \leq p_2,$$

so long as the expectation exists. If $A$ is an increasing event then

$$\mathbb{P}_{p_1}(A) \leq \mathbb{P}_{p_2}(A) \quad \text{whenever } p_1 \leq p_2.$$

**Proof.** Exercise.
6.1 Harris-FKG inequality

We have previously seen negative association for weighted spanning trees, we will now see an extremely useful converse to this of positive association. Often we which to calculate the probability of certain intersections, or conditional expectations of certain random variables. Unfortunately when the events (random variables) are not independent this is often difficult or impossible to do directly. However the following inequalities can give very useful bounds for a certain class of events (or random variables).

Theorem 6.4 (Harris inequality). Let $A$ and $B$ be two increasing events then,

$\mathbb{P}_p(A \cap B) \geq \mathbb{P}_p(A)\mathbb{P}_p(B)$ \quad i.e. \quad $\mathbb{P}_p(A \mid B) \geq \mathbb{P}(A)$.

More generally if $f$ and $g$ are increasing, bounded, functions then

$\mathbb{E}_p[f \, g] \geq \mathbb{E}_p[f]\mathbb{E}_p[g]$.

Remark 6.5. Intuitively the meaning of the inequality is that conditioning on an increases the chance that edges are open, and therefore increases the probability of observing another increasing event.

Example 6.6. For bond percolation on any infinite graph $G$ with countable edge set, the critical probability is independent of the vertex you pick. Let $p_c^x = \sup\{p : \theta^x(p) = 0\}$ where $\theta^x(p) = \mathbb{P}_p(|C_x| = \infty)$, then $p_c^x = p_c^y$ for all $x, y \in V$, since

$\theta^x(p) \geq \mathbb{P}_p(\{x \leftrightarrow y\} \cap \{y \leftrightarrow \infty\}) \geq \mathbb{P}_p(x \leftrightarrow y)\theta_y(p)$,

so $p_c^x \leq p_c^y$. By the same argument we can reverse the inequality.

Proof of the Harris inequality. The second inequality is more general than the first (take $f = 1_A$ and $g = 1_B$), so we prove the second. Since $E$ is countable we let $E = \{e_i\}_{i \geq 1}$ and let $\omega_1 = \omega(e_1)$. Fix $f$ and $g$ bounded and increasing functions. We first show that it is sufficient to consider functions that depend on only a finite number of the edges. Let $f_n := \mathbb{E}_p[f \mid \omega_1, \ldots, \omega_n]$ and $g_n := \mathbb{E}_p[g \mid \omega_1, \ldots, \omega_n]$, which are functions only of the first $n$ edge variables. If $G_n = \sigma(w_1, \ldots, w_n)$ then $\{G_n\}_{n \geq 1}$ is a filtration, and trivially we have

$\mathbb{E}_p[f_{n+1} \mid G_n] = \mathbb{E}_p[\mathbb{E}_p[f \mid G_{n+1}] \mid G_n] = \mathbb{E}_p[f \mid G_n] = f_n,$

since $G_n \subset G_{n+1}$, i.e. $(f_n)_{n \geq 1}$ is a martingale. So by the martingale convergence theorem we have $f_n \to f$ and $g_n \to g$, $\mathbb{P}_p$-a.s. and in $L^2(\mathbb{P}_p)$. We also have $\mathbb{E}_p[f \, g] = \lim_{n \to \infty} \mathbb{E}_p[f_n g_n]$ since

$\mathbb{E}_p[f_n g_n - f_g] \leq \mathbb{E}_p[\|(f_n - f)g_n + |f(g_n - g)\|]
\leq \left(\mathbb{E}_p[\|(f_n - f)\|^2] \mathbb{E}_p[g_n^2]\right)^{1/2} + \left(\mathbb{E}_p[\|(g_n - g)\|^2] \mathbb{E}_p[f^2]\right)^{1/2} \to 0 \quad \text{as} \quad n \to \infty,$

where we use the triangle inequality in the first line and Cauchy-Schwarz inequality in the second. It therefore suffices to show that $\mathbb{E}_p[f_n g_n] \geq \mathbb{E}_p[f_n]\mathbb{E}_p[g_n]$ for each $n \geq 1$. We do this by induction on $n$.

For $n = 1$ we have $f_1, g_1 : \{0, 1\} \to \mathbb{R}$, and $f_1(\omega_1) - f_2(\omega_2) \geq 0$ if $\omega_1 \geq \omega_2$ and vice versa, and the same for $g_1$, therefore

$(f_1(\omega_1) - f_2(\omega_2))(g_1(\omega_1) - g_2(\omega_2)) \geq 0 \quad \text{for all} \quad \omega_1, \omega_2 \in \{0, 1\}.$

It follows that

$0 \leq \sum_{a,b \in \{0,1\}} \mathbb{P}_p(\omega_1 = a)\mathbb{P}_p(\omega_2 = b)(f_1(a) - f_2(b))(g_1(a) - g_2(b)) = 2(\mathbb{E}_p[f_1 g_1] - \mathbb{E}_p[f_1]\mathbb{E}_p[g_1]),$
i.e. the required inequality holds.

Now assume that for an function depending on $k < n$ edges the required inequality holds. The following gets notationally a little nasty. We firstly use the tower property, i.e. the required inequality holds.

$$\mathbb{E}_p[f_n g_n] = \mathbb{E}_p[\mathbb{E}_p[f_n g_n | \omega_1, \ldots, \omega_{n-1}]]$$

where $\mathbb{E}_\omega[f_n(\omega_1, \ldots, \omega_{n-1}, \cdot)]$ is understood to be the expected value with respect to the Bernoulli random variable on edge $e_n$ with $\omega_1, \ldots, \omega_{n-1}$ fixed, therefore $f_n(\omega_1, \ldots, \omega_{n-1}, \cdot)$ is understood to be a function of a single edge configuration, and we may therefore apply the 'k = 1 case’. It follows that

$$\mathbb{E}_p[f_n g_n] \geq \mathbb{E}_p[\mathbb{E}_\omega[f_n(\omega_1, \ldots, \omega_{n-1}, \cdot)]\mathbb{E}_\omega[g_n(\omega_1, \ldots, \omega_{n-1}, \cdot)]]$$

where the final inequality follows from the 'k = n - 1 case'. This completes the proof.

The Harris inequality has been extended to measure more general than the product measure on $\Omega$, by Fortuin, Kasteleyn and Ginibre (1971). The following result is often called the FKG inequality and is very useful in many interacting particle systems. We first give a precursor, from which the FKG inequality follows. For $\omega_1, \omega_2 \in \Omega$ we define the (pointwise) maximum and minimum by

$$(\omega_1 \lor \omega_2)(e) = \max\{\omega_1(e), \omega_2(e)\},$$

$$(\omega_1 \land \omega_2)(e) = \min\{\omega_1(e), \omega_2(e)\},$$

for $e \in E$. A probability measure $\mu$ on $\Omega$ is called positive if it assigns a strictly positive probability measure to each point $\omega \in \Omega$, i.e. $\mu(\omega) > 0$.

**Theorem 6.7** (Holley inequality). Suppose $\mu_1$ and $\mu_2$ are two positive probability measures on $\Omega$, if

$$\mu_1(\omega_1 \land \omega_2)\mu_2(\omega_1 \lor \omega_2) \geq \mu_1(\omega_1)\mu_2(\omega_2), \quad \text{for all} \ \omega_1, \omega_2 \in \Omega,$$

then $\mu_1 \leq \mu_2$.

**Proof.** The typical proof is via stochastic monotonicity, which we will come to later, and constructing suitable coupled processes and applying Strassen’s theorem.

The FKG inequality follows as a consequence of the Holley inequality.

**Theorem 6.8** (FKG inequality). If $\mu$ is a positive probability measure on $\Omega$ such that

$$\mu(\omega_1 \land \omega_2)\mu(\omega_1 \lor \omega_2) \geq \mu(\omega_1)\mu(\omega_2), \quad \text{for all} \ \omega_1, \omega_2 \in \Omega,$$

then $\mu$ is positively associated (has positive correlations) i.e.

$$\mu(fg) \geq \mu(f)\mu(g) \quad \text{for all increasing} \ f, g : \Omega \to \mathbb{R}.$$
In this section we will see a partial converse to the FKG inequality which holds in the special case of product measures on $\Omega$. The inequality, typically called the BK-inequality after Berg and Kesten, obviously can’t hold in general for events of the form $A \cap B$ for increasing $A$ and $B$, so we have to replace the intersection with something else (but closely related) called the ‘disjoint occurrence’ or ‘disjoint intersection’. In words, we will define $A \circ B$ to be the event that $A$ and $B$ occur on disjoint edge sets.

Suppose $A$ and $B$ are increasing events that depend on finitely many edges $\{e_i\}_{i=1}^n$, each $\omega = (\omega_1, \ldots, \omega_n)$ is specified uniquely by the set of edges on which it is non-zero, $K(\omega) = \{e_i : \omega(e_i) = 1\}$. We define $A \circ B$ to be the set of all $\omega$ such that there exists a subset $H$ of $K(\omega)$ with $\omega' \in A$ whenever $K(\omega') = H$, and $\omega'' \in B$ whenever $K(\omega'') = K(\omega) \setminus H$. In other-words, if

$$\omega_H(e) = \begin{cases} \omega(e) & \text{for } e \in H, \\ 0 & \text{for } e \notin H, \end{cases}$$

then

$$A \circ B = \{\omega : \text{there exists } H(\omega) \subset E \text{ s.t. } \omega_H \in A, \ \omega_{E \setminus H} \in B\}.$$ 

In words, there are disjoint edge sets (possibly depending on $\omega$) such that the configuration being open on the first is sufficient to belong to $A$ and the configuration being open on the second is sufficient to belong to $B$. Note that $A \circ B$ is both commutative and associative.

Example 6.9. Suppose $G$ is a finite sub graph of the $d$-dimensional lattice, and let $A_G(x,y) = \{\text{there is an open path in } G \text{ connecting } x \text{ and } y\} = \{x \xleftarrow{G} y\}$, then $A_G(u,v) \circ A_G(x,y)$ is the event that there exists two edge-disjoint paths in $G$ connecting $x$ to $y$ and $u$ to $v$. It is then eminently plausible that

$$\mathbb{P}(A_G(u,v) \circ A_G(x,y) \mid A_G(u,v)) \leq \mathbb{P}(A_G(x,y)),$$

since the disjoint intersection can’t “re-use” the edges used for the event “$A_G(u,v)$.”

This conclusion is the assertion of the BK inequality.

Theorem 6.10 (BK inequality). If $A$ and $B$ are increasing events that depend on finitely many edges, then

$$\mathbb{P}_p(A \circ B) \leq \mathbb{P}_p(A)\mathbb{P}_p(B).$$

Proof. Guided exercise on the final sheet. □

The restriction to events that depend on only finitely many edges is important, but it can be dropped in practice for some important applications (with a little extra work).

It was conjectured by Berg and Kesten that the inequality should hold in general without the restriction to increasing events, this was shown later in 1997 by Reimer. The following result actually contains the BK and Harris inequality but has not seen any application so far that was not one of these two special cases.

For $\omega \in \Omega$ and $F \subset E$ we define the cylinder event $C(\omega, F)$ generate by $\omega$ on $F$ as

$$C(\omega, F) = \{\omega' \in \Omega : \omega'(e) = \omega(e) \text{ for all } e \in F\}.$$ 

We define $A \square B$ as the set of all $\omega$ such that there exists a $K(\omega) \subset E$ with $C(\omega, K) \subset A$ and $C(\omega, K') \subset B$. Alternatively, if we write $\omega|_I$ for the restriction of $\omega$ to $I \subset E$, then if $\omega \in A$ we say $I$ is a witness to the event $A$ for $\omega$ if $\omega' \in A$ whenever $\omega'|_I = \omega|_I$. Then in words $A \square B$ is the set of configurations $\omega$ such that there exists two disjoint witnesses $I(\omega)$ and $J(\omega)$ of $A$ and $B$ for $\omega$. 

Remark 6.11. The disjoint occurrence satisfies:

- \( A \cap B \subset A \cap B \).
- If \( A \) and \( B \) are increasing then so is \( A \cap B \) and \( A \cap B = A \circ B \).
- If \( A \) is increasing and \( B \) is decreasing then \( A \cap B = A \cap B \).

Theorem 6.12 (Reimer's inequality). If \( A \) and \( B \) are two events that depend on finitely many edges then

\[
\mathbb{P}_p(A \cap B) \leq \mathbb{P}_p(A) \mathbb{P}_p(B).
\]

Corollary 6.13. Consider a finite set \( S \subset \mathbb{Z}^d \) containing the origin and \( x \notin S \), then,

\[
\mathbb{P}_p(o \leftrightarrow x) \leq \sum_{y \in \partial^o S} \mathbb{P}_p(0 \leftrightarrow y) \mathbb{P}_p(y \leftrightarrow x),
\]

where \( \partial^o S = \{ x \in S : \text{there exists a } y \in S^c \text{ s.t. } \{ x, y \} \in E \} \) is the internal boundary of \( S \).

Proof. For \( n \geq 1 \) let \( \Lambda_n = [-n, n]^d \). Fix \( n \) large enough such that \( S \subset \Lambda_n \). If \( \omega \in \{ 0 \not\leftrightarrow \Lambda_n \ x \} \), then there is a SAW of open edges in \( \omega \) from 0 to \( x \) that must cross \( S \), let \( y \) be the first vertex in \( \partial^o S \) that this path visits, then necessarily \( \omega \in \{ 0 \not\leftrightarrow \Lambda_n \ y \} \circ \{ y \not\leftrightarrow \Lambda_n \ x \} \) since the set of edges in the path up to \( y \) is witness to \( \{ 0 \not\leftrightarrow \Lambda_n \ y \} \) for \( \omega \) and the remainder of the path is witness to \( \{ y \not\leftrightarrow \Lambda_n \ x \} \). It follows that \( \{ 0 \not\leftrightarrow \Lambda_n \ x \} \subset \bigcup_{y \in \partial^o S} \{ 0 \not\leftrightarrow \Lambda_n \ y \} \circ \{ y \not\leftrightarrow \Lambda_n \ x \} \), and so by a union bound we have

\[
\mathbb{P}_p(o \not\leftrightarrow \Lambda_n \ x) \leq \sum_{y \in \partial^o S} \mathbb{P}_p(\{ 0 \not\leftrightarrow \Lambda_n \ y \} \circ \{ y \not\leftrightarrow \Lambda_n \ x \}).
\]

Then the BK inequality implies that

\[
\mathbb{P}_p(o \not\leftrightarrow \Lambda_n \ x) \leq \sum_{y \in \partial^o S} \mathbb{P}_p(0 \not\leftrightarrow \Lambda_n \ y) \mathbb{P}_p(\Lambda_n \not\leftrightarrow \Lambda_n \ x),
\]

and the claim follows by letting \( n \) tend to infinity. \( \square \)

6.3 Margulis-Russo formula

We have already seen that \( \mathbb{P}_p(A) \) is non decreasing in \( p \) if \( A \) is an increasing event. The final tool we will examine in this chapter will help to control the rate at which this probability changes as a function of \( p \).

Consider our increasing coupling using the i.i.d. uniform r.v.s \( \{ U_e \}_{e \in E} \) (recall we denote the measure by \( \mathbb{P} \). For an increasing event \( A \) that depends on only finitely many edges, using this coupling, we have

\[
\mathbb{P}_{p+\delta}(A) - \mathbb{P}_p(A) = \mathbb{P}(\eta_{p+\delta} \in A, \eta_p \notin A). \tag{6.3.1}
\]

If \( \eta_{p+\delta} \in A \) and \( \eta_p \notin A \) we know there must exist at least one edge \( e \in E \) with \( p \leq U_e < p + \delta \), denote the set of all such edges by \( E_{p,\delta} \). Since the edges have independent weights it is clear that \( \mathbb{P}(|E_{p,\delta}| \geq 2) = o(\delta) \), i.e. with high probability there is a single edge with \( U_e \in [p, p + \delta] \). If \( e \) is the unique edge satisfying \( U_e \in [p, p + \delta] \) then this edge must be ‘essential’ for the event \( A \) in the sense that \( \eta_p \notin A \) but \( \eta_{p+\delta} \in A \), where in general for \( \omega \in \Omega \) and \( e \in E \) we define

\[
\omega^e(f) = \begin{cases} 
\omega(f) & \text{if } f \neq e, \\
1 - \omega(f) & \text{if } f = e.
\end{cases}
\]
to be the configuration ‘flipped’ at edge \( e \). Note that this ‘essentialness’ of this particular edge could depend on \( \eta_p \in A \) but not the specific value of \( \eta_p(e) \). It follows that this specific edge contributes to (6.3.1) an amount \( \mathbb{P}(U_e \in [p, p+\delta]) \mathbb{P}(\text{is ‘essential’ for } A) = \delta \mathbb{P}(\text{is ‘essential’ for } A) \). Now summing over all such contributions and taking \( \delta \to 0 \) we have

\[
\frac{d}{dp} \mathbb{P}_p(A) = \sum_{e \in E} \mathbb{P}_p(e \text{ is ‘essential’ for } A).
\]

It turns out that this heuristic is essentially valid. Now we make the argument precise.

Fix \( A \) an increasing even depending only finitely many edges and \( \omega \in \Omega \), we call the edge \( e \in E \) pivotal for the pair \((A, \omega)\) if \( \mathbb{1}_A(\omega) \neq \mathbb{1}_A(\omega^e) \), and define the event

\[
\text{piv}_e(A) = \{ e \text{ is pivotal for } A \} = \{ \omega \in \Omega : e \text{ is pivotal for the pair } (A, \omega) \}.
\]

**Example 6.14.** If \( A = \{0 \leftrightarrow \partial^o \Lambda_n\} \) then

\[
\text{piv}_e(A) = \{ \omega : e \text{ is open then } 0 \leftrightarrow \partial^o \Lambda_n \text{ and if } e \text{ is closed then } 0 \not\leftrightarrow \partial^o \Lambda_n \}
\]

which is all the \( \omega \in \Omega \) such that \( \omega^- \leftrightarrow 0 \), \( \omega^+ \leftrightarrow \partial^o \Lambda_n \), and there exists a closed loop in the dual surrounding 0 (except for the edge crossing \( e \)).

**Theorem 6.15** (Russo’s formula). If \( A \) is an increasing event, that depends on only finitely many edges, then

\[
\frac{d}{dp} \mathbb{P}_p(A) = \sum_{e \in E} \mathbb{P}_p(\text{piv}_e(A)) = \mathbb{E}_p[ N(A) ] ,
\]

where \( N(A)(\omega) = |\{e \in E : e \text{ is pivotal for } (A, \omega)\}| \).

Again, it is important that \( A \) depends on only finitely many edges, consider for example \( \theta(p) \) - we expect that this is not differentiable at the critical point.

**Proof.** We start by showing that if \( A \) depends on only finitely many edges then \( f(p) = \mathbb{P}_p(A) \) is differentiable. In fact (with slight abuse of notation since really we have average with respect to the variables on \( E \setminus \{e_1, \ldots, e_n\} \)),

\[
f(p) = \mathbb{E}_p[\mathbb{1}_A] = \sum_{\omega_1, \ldots, \omega_n} \mathbb{1}_A(\omega)p^{\omega}(1-p)^{n-\omega},
\]

where \( |\omega| = \sum_{i=1}^n \omega_i \). So in particular \( f(p) \) is a polynomial in \( p \). We can simply calculate the derivative and collect terms to find (check),

\[
f'(p) = \frac{1}{p(1-p)} \sum_{i=1}^n \sum_{\omega_1, \ldots, \omega_n} (\omega_i - p) \mathbb{1}_A(\omega)p^{\omega}(1-p)^{n-\omega} = \frac{1}{p(1-p)} \sum_{i=1}^n \mathbb{E}_p[\mathbb{1}_A(\omega_i)p - \mathbb{1}_A(\omega)] = \frac{1}{p(1-p)} \sum_{i=1}^n \mathbb{C}ov_p[\mathbb{1}_A(\omega_i), \omega_i]. \tag{6.3.2}
\]

This is already in a form which may come in handy later, but to express it in the form stated we make a couple more observations. Firstly, if \( \omega \in A \cap \text{piv}_e(A)^c \) then \( \omega^e \in A \) by definition of \( \text{piv}_e(A) \) and \( \omega^e \in \text{piv}_e(A)^c \) since \( \text{piv}_e(A) \) does not depend on \( \omega(e) \), therefore

\[
\mathbb{E}_p[\mathbb{1}_A(\omega_i)p - \mathbb{1}_{\text{piv}_e(A)^c}(\omega_i)] = \mathbb{E}_p[(\omega_i - p)] \mathbb{E}_p[\mathbb{1}_A(\omega_i)p] - \mathbb{E}_p[\mathbb{1}_A(\omega_i)p] \mathbb{E}_p[\mathbb{1}_{\text{piv}_e(A)^c}(\omega_i)] = 0.
\]

It follows by the total law that

\[
\mathbb{E}_p[\mathbb{1}_A(\omega_i)p - \mathbb{1}_A(\omega_i)p] = \mathbb{E}_p[(\omega_i - p)p] = (1-p) \mathbb{E}_p(p|\text{piv}_e(A)^c) \mathbb{1}_A(\omega_i),
\]

and since \( A \) is increasing, if \( \omega \in \text{piv}_e(A) \), then \( \mathbb{1}_A(\omega) = 0 \) if \( \omega_i = 0 \) and 1 if \( \omega_i = 1 \). Therefore

\[
\mathbb{E}_p[\mathbb{1}_A(\omega_i)p - \mathbb{1}_A(\omega_i)p] = (1-p) \mathbb{P}_p(\text{piv}_e(A) \cap \{w_i = 1\}) = (1-p) p \mathbb{P}_p(\text{piv}_e(A)) ,
\]

which completes the proof. \( \square \)
Chapter 7

Further percolation

We start this section with some important results on ergodicity (with respect to shifts) and some important results about the infinite cluster. Recall the discussion in Section 4.3 where we discussed the details of the measurable structure of \( \Omega \), and we defined shift-ergodic

**Lemma 7.1.** The measure \( P_p \) is invariant under translations and is shift-ergodic.

**Proof.** We use repeatedly that any measurable event \( A \in F \) can be approximated by an event that depends on finitely many edges, in the sense that for all \( \varepsilon > 0 \) there exists an event \( B \) depending on finitely many edges with \( P_p(A \Delta B) < \varepsilon \). It is clear that for any event \( B \) depending on finitely many edges \( P_p(B) = P_p(\tau_x B) \), since \( P_p \) is a homogeneous product measure, hence \( P_p \) is invariant under shifts. Fix \( A \in F \) translation invariant, we which to show that \( P_p(A) \in \{0, 1\} \).

Corollary 7.2. If \( p > p_c \) then there is an infinite cluster \( P_p \)-a.s.

Note we already saw one proof using Kolmogorov’s zero-one law earlier, but it is also an immediate consequence of the previous lemma. It turns out that the infinite cluster (if it exists) is also unique.

**Theorem 7.3.** If \( p > p_c \) then there is a unique infinite cluster \( P_p \)-a.s.

This result was first proved by Aizenman, Kesten and Newman in 1987, and there have been various alternative proof since. See for example Burton am Keane 1989. I will add references.

7.1 Exponential decay in the subcritical regime

The arguments in this section and the next section on Kestern’s theorem are taken from Duminil-Copin and Tassion 2016 [9]. It turns out that (strictly) inside the subcritical region, not only is there almost surely no infinite cluster, but the probability that the cluster containing the origin is large decays exponentially quickly in the diameter. More precisely:

**Theorem 7.4** (Exponential decay in the diameter). Fix \( d \geq 2 \), for every \( p \in (0, p_c) \) there exists \( c_p > 0 \) such that for all \( n \geq 1 \) we have \( P_p(0 \leftrightarrow \partial^c \Lambda_n) \leq e^{-c_dn} \). Furthermore, there exists a \( c > 0 \) such that for \( p \in (0, p_c) \), \( \theta(p) \geq c(p - p_c) \).

This theorem was first proved by Aizenman and Barksy in 1987, but we now present are more recent proof by Duminil-Copin and Tassion from 2016. Clearly \( P_p(0 \leftrightarrow \partial^c \Lambda_n) \) is at least \( p^n = e^{-(log 1/p)n} \) since this is the probability that the path from the origin straight up to the boundary is open.
Proof. Let $\theta_n(p) = \mathbb{P}_p(\emptyset \leftrightarrow \partial^c \Lambda_n)$, and denote the edge boundary of $S \subset \mathbb{Z}^d$ by $\Delta S = \{ e \in E : e^- \in S, e^+ \notin S \}$, and finally define

$$\varphi_p(S) = p \sum_{e \in \Delta S} \mathbb{P}_p(0 \leftrightarrow e^-),$$

which is the expected number of open edges in the boundary of the connected component of $S$ containing 0. Clearly if $0 \notin S$ then $\varphi_p(S) = 0$ The proof now relies on the following claims (we postpone their proof to the end):

**Claim 7.5.** If there exists a finite set $S$ containing 0 such that $\varphi_p(S) < 1$, then there exists $c_p > 0$ such that for all $n \geq 1$ we have $\theta_n(p) \leq e^{-c_p n}$.

For $n \geq 1$ define the random subset of $\Lambda_n$,

$$\mathcal{S} = \{ x \in \Lambda_n : x \not\leftrightarrow \partial^c \Lambda_n \}.$$

The boundary of $\mathcal{S}$ is the ‘out-most blocking surface’ from the boundary of $\Lambda_n$, you can ‘discover’ $\mathcal{S}$ by exploring all the sites you can reach following open paths started from the boundary, this will leave ‘unexplored’ islands of sites which can’t be reached from the boundary (belonging to $\mathcal{S}$), in a ‘sea’ of the connected component of the boundary. Thus the event $\{ \mathcal{S} = S \}$ is measurable with respect to all the edges which have at one end outside of $S$.

**Claim 7.6.** For every $p \in (0, 1)$ we have

$$\theta'_n(p) = \frac{1}{p(1-p)} \mathbb{E}_p[\varphi_p(\mathcal{S})]. \quad (7.1.1)$$

**Remark 7.7.** Since the edge boundary of $\mathcal{S}$ must be closed (or else the vertex attached by an open edge would also be in $\mathcal{S}$), we can interpret $\varphi_p(S)$ (up to constant factors) as the expected number of closed pivotal points of $\{ 0 \leftrightarrow \partial^c \Lambda_n \}$ conditioned on the connected component of the boundary being $S^c$.

We now complete the proof in light of these claims. This is achieved through a new definition of the critical point that turns out to be equivalent; let

$$\tilde{p}_c := \sup \{ p \in [0, 1] : \exists \text{ a finite set } S \text{ containing } 0 \text{ such that } \varphi(S) < 1 \}.$$

By Claim 7.5, for $p < \tilde{p}_c$ there exists a $c_p > 0$ such that $\theta_n(p) \leq e^{-c_p n}$ for every $n \geq 1$, in particular $\theta(p) = 0$. Therefore $p \leq p_c$, so $\tilde{p}_c \leq p_c$. We now show $\tilde{p}_c \geq p_c$, i.e. $\theta(p) > 0$ for all $p > \tilde{p}_c$. Fix $p > \tilde{p}_c$, so by definition $\varphi(S) \geq 1$ for all $S$ containing the origin. Claim 7.6 then gives

$$\theta'_n(p) = \frac{1}{p(1-p)} \mathbb{E}_p[\varphi_p(\mathcal{S})] \geq \frac{1}{p(1-p)} \mathbb{P}_p(0 \in \mathcal{S}) = \frac{1}{p(1-p)}(1 - \theta_n).$$

Some basic re-arranging then gives

$$\left( \log \left( \frac{1}{1 - \theta_n} \right) \right)' \geq \left( \log \left( \frac{p}{1 - p} \right) \right)'$$

which by integrating between $\tilde{p}_c$ and $p_c$ implies that for all $n \geq 1$, $\theta_n(p) \geq \frac{p - \tilde{p}_c}{p(1 - p)}$. Letting $n$ tend to infinity we have a positive lower bound on $\theta(p)$ since $p > \tilde{p}_c$. This completes the proof that $\tilde{p}_c = p_c$, together with Claim 7.5 this completes the proof of the first part of the theorem. Also, substituting $\tilde{p}_c = p_c$ into the previous bound on $\theta(p)$ gives $\theta(p) \geq c(p - p_c)$ as required (the so-called mean field bound). \qed
Proof of Claim 7.5. A sketch could be helpful here. Fix $S$ such that $varphi_p(S) < 1$, since $S$ is finite there exists an $L > 0$ such that $S \subseteq \Lambda_{L-1}$. Now fix $k \in \mathbb{N}$ and consider the 'big box' $\Lambda_k$, then by the same argument as in the proof of Corollary 6.13, using the BK inequality,

$$\theta_k(p) \leq p \sum_{e \in \Delta S} \mathbb{P}_p(0 \xleftarrow{\mathcal{S}} e^-) \mathbb{P}_p(e^+ \leftrightarrow \partial^o \Lambda_k) \leq \varphi_p(S) \theta_{k-1}(p),$$

where in the second inequality we used translation invariance and the fact that $e^+$ must be a distance at least $(k-1)L$ of $\partial^o \Lambda_k$. [The proof of the above inequality is left as an exercise on Sheet 3]. The claim follows by iteration on $k$, which give $\theta_k(p) \leq \varphi_p(S)^k$, i.e. exponential decay of $\theta_n(p)$ in $n$.

Proof of Claim 7.6. Here we use Russo’s formula. Let $A = \{0 \leftrightarrow \partial^o \Lambda_n\}$, then note that

$$(1-p)\mathbb{P}_p(e \text{ is pivotal for } A) = \mathbb{P}_p(\{e \text{ is pivotal for } A, \omega(e) = 0\})$$

$$= \mathbb{P}_p(\{e \text{ is pivotal for } A, 0 \not\leftrightarrow \partial^o \Lambda_n\}),$$

since (as discussed previously) $\{e \text{ is pivotal for } A\}$ does not depend on the configuration of edge $e$. The by Russo’s formula,

$$\theta'_n(p) = \frac{1}{1-p} \sum_{e \in E_n} \mathbb{P}_p(\{e \text{ is pivotal for } A, 0 \not\leftrightarrow \partial^o \Lambda_n\})$$

$$= \frac{1}{1-p} \sum_{e \in E_n} \mathbb{P}_p(0 \leftrightarrow e^-, e^+ \leftrightarrow \partial^o \Lambda_n, 0 \not\leftrightarrow \partial^o \Lambda_n) \tag{7.1.2}$$

where $E_n$ are the edges between vertices both in $\Lambda_n$. Now we partition on $\{\mathcal{S} = S\}$, and observe that on the event $\{\mathcal{S} = S\}$ the event on the right hand side above can be interpreted nicely,

$$\{0 \leftrightarrow e^-, e^+ \leftrightarrow \partial^o \Lambda_n, 0 \not\leftrightarrow \partial^o \Lambda_n, \mathcal{S} = S\} = \{0 \xleftarrow{\mathcal{S}} e^-, e^+ \not\in S, 0 \in S, \mathcal{S} = S\}.$$ 

it follows immediately that we have

$$\theta'_n(p) = \frac{1}{p(1-p)} \sum_{S \subseteq \Lambda_n : 0 \in S} \sum_{e \in \Delta S} p \mathbb{P}_p(0 \xleftarrow{\mathcal{S}} e^-, \mathcal{S} = S)$$

$$= \frac{1}{p(1-p)} \sum_{S \subseteq \Lambda_n : 0 \in S} \left( \sum_{e \in \Delta S} p \mathbb{P}_p(0 \xleftarrow{\mathcal{S}} e^-) \right) \mathbb{P}_p(\mathcal{S} = S)$$

$$= \frac{1}{p(1-p)} \mathbb{E}_p[\mathcal{S}\omega(S)],$$

where in the second equality we used that $\{0 \xleftarrow{\mathcal{S}} e^-\}$ depends only on edges strictly inside $S$, and $\{\mathcal{S} = S\}$ can be determined from all the other edges (exploring from the boundary in using open paths).

\section{7.2 Kesten’s theorem}

We will rely heavily on the self duality of the $\mathbb{Z}^2$ lattice previously discussed. In light of this self duality there are many heuristic arguments for why $p_c(2) = 1/2$, arguably the most informative is in terms of “box crossing’s”. Let $R_n = [0, n] \times [0, n - 1]$ and

$$\mathcal{H}_n = \{\omega \in \Omega : \{0\} \times [0, n - 1] \leftrightarrow \{n\} \times [0, n - 1]\},$$

$$\theta_k(p) \leq p \sum_{e \in \Delta S} \mathbb{P}_p(0 \xleftarrow{\mathcal{S}} e^-) \mathbb{P}_p(e^+ \leftrightarrow \partial^o \Lambda_k) \leq \varphi_p(S) \theta_{k-1}(p),$$

where in the second inequality we used translation invariance and the fact that $e^+$ must be a distance at least $(k-1)L$ of $\partial^o \Lambda_k$. [The proof of the above inequality is left as an exercise on Sheet 3]. The claim follows by iteration on $k$, which give $\theta_k(p) \leq \varphi_p(S)^k$, i.e. exponential decay of $\theta_n(p)$ in $n$. \qed
be the event that there is a right to left crossing of $R_n$. It turns out that studying crossing probabilities, of boxes and more general shapes, is instrumental in understanding the random geometry of percolation and related models, in particular at criticality. This technology used to study this is generally called Russo-Seymour-Welsh theory.

Notice that the compliment of $\mathcal{H}_n$ is the event that there exists a closed path in the dual percolation ‘from top to bottom’ of the rectangle $R_n$ under a rotation of $\pi/2$. Checking this remark carefully it follows immediately that

$$\mathbb{P}_\frac{1}{2}(\mathcal{H}_n) = \frac{1}{2}, \quad \forall \ n \geq 1.$$  \hspace{1cm} (7.2.1)

If $p < p_c$ we know from the previous section that clusters are typically small, so that the probability of crossing $R_n$ goes to zero as $n \to \infty$, therefore $p_c \leq 1/2$. On the other hand if $p > p_c$ then its is believable that the infinite cluster is omnipresent, in the sense that if you look at any large-enough box you will see part of the infinite cluster, and therefore if $n$ is sufficiently big then the probability of crossing $R_n$ from left to right is close to one (i.e. $\mathbb{P}_p(\mathcal{H}_n) \to 1$ as $n \to \infty$ for $p > p_c$), and hence $p_c \geq 1/2$. In order to prove that $p_c = 1/2$ we will make both of these argument rigorous, the former really follows almost immediately from the exponential decay of cluster size at sub-criticality, and we prove the latter now using uniqueness of the infinite cluster.

**Proposition 7.8.** If $\theta(p) > 0$ then $\lim_{n \to \infty} \mathbb{P}_p(\mathcal{H}_n) = 1$.

**Proof.** Fix $n > k > 1$, if $\Lambda_k = [-k,k]^2$ is a box of ‘size’ $k$ inside $\Lambda_n = [-n,n]^2$, then a path from $\Lambda_k$ to $\Lambda_n$ ends up on either the top, bottom, left or right of $\Lambda_n$, so by the Harris inequality it follows that

$$\mathbb{P}_p(\Lambda_k \leftrightarrow \{\text{the left side of } \Lambda_n\}) \geq 1 - \mathbb{P}_p(\Lambda_k \not\leftrightarrow \infty)^{1/4}, \tag{7.2.2}$$

this is a special case of the ‘square-root trick’ that will be part of the third example sheet.

Now set $n' = \lfloor \frac{n-1}{2} \rfloor$, and let $A_n$ be the event that the box of size $k$ shifted to the (left) centre of $R_n$, given by $\Lambda_k = \Lambda_k + (n',n')$, is connected to the left of $R_n$, and the box of size $k$ shifted to the (right) centre of $R_n$, given by $\Lambda_k = \Lambda_k + (n'+2,n')$ is connected to the right of $R_n$ (notice that we are a little careful about what we mean by ‘centred’ so that the bounds are obvious). Then from the bound above (7.2.2) we have

$$\mathbb{P}_p(A_n) \geq 1 - 2\mathbb{P}_p(\Lambda_k \not\leftrightarrow \infty)^{1/4}.$$  

If $A_n$ occurs but not $\mathcal{H}_n$, then there are two distinct clusters from the box in the centre of $R_n$ to the boundary. The intersection over $n \geq 1$ of this event is included in the event that there are two distinct infinite clusters. It follows that $\mathbb{P}_p(A_n \setminus \mathcal{H}_n) \to 0$ as $n \to \infty$. Therefore,

$$\lim_{n \to \infty} \inf \mathbb{P}_p(\mathcal{H}_n) = \lim_{n \to \infty} \inf \mathbb{P}_p(A_n) \geq 1 - 2\mathbb{P}_p(\Lambda_k \not\leftrightarrow \infty)^{1/4}.$$  

Taking the limit as $k$ tends to infinity we deduce that $\mathbb{P}_p(\mathcal{H}_n) \to 1$ as required. \hfill \blackqed

Combining this proposition with Eq. (7.2.1) we have the following corollary.

**Corollary 7.9.** There is no infinite cluster at $p = 1/2$, i.e. $\theta(1/2) = 0$, in particular $p_c \geq 1/2$.

To complete Kesten’s theorem we apply the subcritical exponential bounds.

**Theorem 7.10** (Kesten’s theorem). For Bernoulli percolation on the $\mathbb{Z}^2$ lattice, $p_c(2) = 1/2$, and furthermore there is no infinite cluster at $p_c$. 

7.3. THE RUSSO-SEYMOUR-WELSH THEORY

Proof. It remains to show that if $p < p_c$ then $P_p(H_n) \to 0$ as $n \to \infty$. Fix $p < p_c$ then by Theorem 7.4 there exists $c_p > 0$ such that

$$P_p(0 \leftrightarrow \partial \Lambda_n) \leq e^{-c_p n}.$$ 

We use a simple union bound over the left hand edge of $R_n$ to get

$$P_p(H_n) \leq \sum_{k=0}^{n-1} P_p((0, k) \leftrightarrow \{n\} \times [0, n - 1]) \leq nP_p(0 \leftrightarrow \partial \Lambda_n) \leq ne^{-c_p n},$$

where in the second inequality we used the obvious inclusion (i.e. $R_n \subset \tau(0,k) \Lambda_n$) and translation invariance. Again by (7.2.1) this implies that $p < 1/2$. \qed

7.3 The Russo-Seymour-Welsh theory

In the previous section we saw estimates of the probability of crossing a box at the critical point (in particular $P_{1/2}(H_n) = 1/2$). It turns out that understanding the probabilities of crossing more complicated shapes is extremely informative about the structure of the connected components. Estimates of rectangle crossing probabilities can be used to estimate the probability of crossing more complicated structure, or for example the probability of an open cycle in an annulus, it turn these estimates have been pivotal for the proof of Cardy’s formula, conformal invariance, and scaling limits.

We did not have time to cover this in lectures, but we might explore some of these concepts on the longer final example sheet (Sheet 4 Question 2).
Chapter 8

Exclusion, contact and voter models

We will look at some classical interacting particle systems, in the sense of Markov processes on $\Omega = \{0, 1\}$ $^A$. Along the way we will develop some very general tools which turn out to be surprisingly powerful, such as graphical constructions, coupling of processes (along with stochastic monotonicity), and duality. We start with some general setup.

8.1 Semigroups and generators

Throughout this section we will consider compact state space $\Omega$, i.e. finite local state space, with the topological and measurable structure discussed previously.

Let $C(\Omega) = \{f : \Omega \to \mathbb{R} \text{ continuous}\}$ regarded as a Banach space with respect to the sup-norm

$$\|f\|_\infty = \sup_{\eta \in \Omega} |f(\eta)|.$$  

Since $\Omega$ is compact $\|f\|_\infty < \infty$ for all $f \in C(\Omega)$.

**Definition 8.1.** A (homogeneous) Markov process on $\Omega$ is a collection $\{P_\zeta : \zeta \in \Omega\}$ of probability measures on $D[0, \infty)$ with the following properties:

(a) $P_\zeta[\eta_0 = \zeta ] = 1$ for all $\zeta \in \Omega$, i.e. $P_\zeta$ is normalized on all paths with initial condition $\eta_0 = \zeta$.

(b) The mapping $\zeta \mapsto P_\zeta[A]$ is measurable for every $A \in \mathcal{F}$.

(c) $P_\zeta[\eta_t \in A | \mathcal{F}_t] = P_{\eta_t}[A]$, $P_\zeta$-a.s. for all $\zeta \in \Omega$, $A \in \mathcal{F}$ and $t > 0$. (Markov property)

Suppose $\{P_\eta : \eta \in \Omega\}$ is a Markov process on $\Omega$. For $f \in C(\Omega)$ we define

$$S(t)f(\eta) := E_\eta[f(\eta_t)] = \int_{D[0, \infty)} f(\eta_t)dP_\eta,$$  

(8.1.1)

Recall $P^n f(\eta) = E_\eta[f(\eta_n)]$ in discrete-time, and $P(t)f(\eta) = E_\eta[f(\eta_t)]$ for countable state continuous time Markov processes, in this sense $S(t)$ is the continuous time analogue of the transition matrix.

**Definition 8.2.** A Markov process is called a Feller process if $S(t)f \in C(\Omega)$ for every $f \in C(\Omega)$ and $t \geq 0$.

**Proposition 8.3.** (and Definition of Markov semigroup) Suppose $\{P_\eta : \eta \in \Omega\}$ is a Feller process on $\Omega$. Then the collection of linear operators $(S(t) : t \geq 0)$ on $C(\Omega)$ is a Markov semigroup, i.e.

(a) $S(0) = I$, (identity operator at $t = 0$).

(b) $t \mapsto S(t)f$ from $[0, \infty)$ to $C(\Omega)$ is right-continuous for all $f \in C(\Omega)$, (right-continuity)

(c) $S(t+s)f = S(t)S(s)f$ for all $f \in C(\Omega)$, $s, t \geq 0$, (semigroup/Markov property)

(d) $S(t)1 = 1$ for all $t \geq 0$, (conservation of probability)
(e) \( S(t)f \geq 0 \) for all non-negative \( f \in C(\Omega) \).

**Proof.** Part (a) is equivalent to (a) of Definition 8.1, \( S(0)f(\zeta) = \mathbb{E}_\zeta[f(\eta_0)] = f(\zeta) \) since \( \eta_0 = \zeta \).

For fixed \( \eta \in \Omega \) the right-continuity of \( t \mapsto S(t)f(\eta) \) in \( t \) follows directly from right-continuity of \( \eta_t \) and continuity of \( f \). The proof of uniformity in \( \eta \in \Omega \), required to show (b), is more involved and can be found in Section 1 of Chapter IX of Yosida (1980).

Part (c) follows from the Markov property of \( \eta_t \), Definition 8.1 (c),

\[
S(t+s)f(\zeta) = \mathbb{E}_\zeta f(\eta_{t+s}) = \mathbb{E}_\zeta \left[ \mathbb{E}_\zeta \left( f(\eta_{t+s}|\mathcal{F}_t) \right) \right] = \mathbb{E}_\zeta \left[ \mathbb{E}_\eta f(\eta_s) \right] = \\
= \mathbb{E}_\zeta \left[ (S(s)f)(\eta_t) \right] = S(t)S(s)f(\zeta).
\]

Part (d): \( S(t)1 = \mathbb{E}_\eta(1) = \mathbb{E}^\eta(1\{\Omega\}(\eta_t)) = 1 \) since \( \eta_t \in \Omega \) for all \( t \geq 0 \) (conservation of probability). Similarly for all constant functions \( c \) we must have \( S(t)c = c \).

Part (e): is immediate by definition. \( \square \)

**Remark 8.4.** Note that (b) implies in particular \( S(t)f \to f \) as \( t \to 0 \) for all \( f \in C(\Omega) \), which is usually called strong continuity of the semigroup (see e.g. [? ], Section 19). Furthermore, \( S(t) \) is also contractive, i.e. for all \( f \in C(X) \)

\[
\|S(t)f\|_\infty \leq \|S(t)f\|_\infty \leq \|f\|_\infty \|S(t)1\|_\infty = \|f\|_\infty,
\]

which follows directly from conservation of probability (d). Strong continuity and contractivity imply that \( t \mapsto S(t)f \) is actually uniformly continuous for all \( t > 0 \). Using also the semigroup property (c) we have for all \( t > \epsilon > 0 \) and \( f \in C(X) \)

\[
\|S(t)f - S(t-\epsilon)f\|_\infty = \|S(t-\epsilon)(S(\epsilon)f - f)\|_\infty \leq \|S(\epsilon)f - f\|_\infty,
\]

which vanishes for \( \epsilon \to 0 \) and implies left-continuity in addition to right-continuity (b).

The importance of Markov semigroups lies in the fact that each one corresponds to a Markov process, as can be seen from the following inverse of Proposition 8.3.

**Theorem 8.5.** Suppose \( (S(t))_{t \geq 0} \) is a Markov semigroup on \( C(\Omega) \). Then there exists a unique Markov process \( \{P_\eta : \eta \in \Omega \} \) such that

\[
S(t)f(\eta) = \mathbb{E}_\eta[f(\eta_t)]
\]

for all \( f \in C(\Omega) \), \( \eta \in \Omega \), and \( t \geq 0 \).

**Proof.** Proofs can be found in Chapter I of Blumenthal and Getoor (1968) and Chapter I of Gihman and Skorohod (1975). \( \square \)

Let \( \mathcal{P}(\Omega) \) denote the set of all probability measures on \( \Omega \), with the topology of weak convergence: \( \mu_n \to \mu \) in \( \mathcal{P}(\Omega) \) iff \( \int f d\mu_n \to \int f d\mu \) for all \( f \in C(\Omega) \). Note that \( \mathcal{P}(\Omega) \) is compact since \( \Omega \) is. Recall if \( \mu \in \mathcal{P}(\Omega) \) then the Markov process started from initial distribution \( \mu \) is given by

\[
P_\mu = \int_\Omega P_\eta \mu(d\eta).
\]

Thus,

\[
\mathbb{E}_\mu[f(\eta_t)] = \int_\Omega (S(t)f)(\zeta) \mu(d\zeta) = \int_\Omega S(t)f d\mu \text{ for all } f \in C(\Omega),
\]

This motivates the following definition.
**Definition 8.6.** Suppose \( (S(t))_{t \geq 0} \) is a Markov semigroup on \( C(\Omega) \). Given an initial distribution \( \mu \in P(\Omega) \) we denote the distribution at time \( t \) by \( \mu S(t) \in P(\Omega) \), defined by

\[
\int_{\Omega} f d[\mu S(t)] = \int_{\Omega} S(t) f d\mu \quad \forall f \in C(\Omega).
\] (8.1.5)

Note that we will sometimes write \( \mu_t = \mu S(t) \). This is reminiscent of \( \pi_n = \pi P^n \) in discrete-time countable state space case. The fact that \( \mu S(t) \) is uniquely determined by (8.1.5) is a consequence of the Riesz representation theorem (See for example [? , Theorem 2.14]).

Since \( (S(t))_{t \geq 0} \) has the semigroup property we expect it has the form of an exponential generated by the linearization \( S'(0) \);

\[
S(t) = e^{tS'(0)} = \text{Id} + S'(0)t + o(t) \quad \text{as} \quad t \to 0 \quad \text{where} \quad S'(0) = \text{Id}'.
\]

In countable state space this makes sense in terms of matrix exponentials of the \( Q \)-matrix. Otherwise we make it precise as follows.

**Definition 8.7.** The generator \( L : \mathcal{D}_L \to C(\Omega) \) for the process \( (S(t))_{t \geq 0} \) is given by

\[
L f := \lim_{t \searrow 0} \frac{S(t)f - f}{t} \quad \text{for all} \quad f \in \mathcal{D}_L,
\] (8.1.6)

where the domain \( \mathcal{D}_L \subseteq C(\Omega) \) is the set of all functions for which the limit above exists. Note that the limit is with respect to the sup-norm \( \| \cdot \|_{\infty} \).

**Proposition 8.8.** (and Definition of Markov generator) \( L \) as defined above in Definition 8.7 is a Markov generator, i.e.

(a) \( 1 \in \mathcal{D}_L \) and \( L 1 = 0 \),

(b) for \( f \in \mathcal{D}_L \) and \( \lambda \geq 0 \): \( \min_{\xi \in \Omega} f(\xi) \geq \min_{\xi \in \Omega} (f - \lambda L f)(\xi) \),

(c) \( \mathcal{D}_L \) is dense in \( C(\Omega) \) and the range \( \mathcal{R}(\text{Id} - \lambda L) = C(\Omega) \) for all \( \lambda > 0 \) sufficiently small.

**Theorem 8.9.** (Hille-Yosida) There is a one-to-one correspondence between Markov generators and Markov semigroups on \( C(\Omega) \) given by Definition 8.7 and

\[
S(t)f := \lim_{n \to \infty} \left( \text{Id} - \frac{t}{n} L \right)^{-n} f \quad \forall f \in C(\Omega), \ t \geq 0.
\]

Furthermore for \( f \in \mathcal{D}_L \) we have \( S(t)f \in \mathcal{D}_L \) and

\[
\frac{d}{dt} S(t)f = S(t)Lf = LS(t)f
\]
called the forward and backward equations respectively.

**Remark 8.10.** We see that the forward equation corresponds, in words, to differentiating with respect to a small change in time at the end of the path, whereas the backward equation corresponds to a derivative in the initial time. Formally,

\[
S(t)Lf(\eta) = S(t)g(\eta) \quad \text{where} \quad g(\eta) = \frac{d}{ds} \mathbb{E}_\eta [f(\eta_s)]
\]

\[
= \mathbb{E}_\eta \left[ \frac{d}{ds} \mathbb{E}_{\eta_s} [f(\eta_s)] \right]
\]

and

\[
LS(t)f(\eta) = Lh(\eta) \quad \text{where} \quad h(\eta) = \mathbb{E}_\eta [f(\eta_t)]
\]

\[
= \frac{d}{ds} \mathbb{E}_{\eta_s} [f(\eta_s)]
\]
Remark 8.11. It may appear that the forward and backward equations are one and the same. In general this is not the case. While $\mathcal{L}$ and $S(t)$ formally commute, the domains of definition of the operators are not necessarily identical. The difference between the forward and backward equations becomes significant, for instance, when dealing with certain boundary conditions where there is instantaneous return from boundary points (or points at infinity) to another state. However, if the generator $\mathcal{L}$ on a countable state space has the property that the absolute values of the diagonal entries satisfy a uniform bound, then the forward and backward equations have the same solution. In general, the backward equation has more solutions than the forward equation and its minimal solution is also the solution of the forward equation. A full discussion of such matters demands more technical analysis which is not relevant for the rest of this course.

Remark 8.12. The Hille-Yosida Theorem usually is used as follows. The process is first described in terms of the jump rates as we did for the examples. These are used to define a generator, formally written
\[ \mathcal{L}f(\eta) = \sum_{\eta' \neq \eta} c(\eta, \eta')(f(\eta') - f(\eta)). \]  
(8.1.7)

Note that the sum on the right-hand side doesn’t technically makes sense for uncountable state space. For the process we have introduced this makes sense for sufficiently smooth functions, specifically or $f \in D(\Omega) = \{ f \in C(\Omega) : \sum_{x \in \Lambda} \sup_{\eta \in \Omega} |f(\eta^x) - f(\eta)| < \infty \}$. Given reasonable conditions on the jump rates, the closure of the operator $\mathcal{L}$ on $D(\Omega)$ is a generator on $C(\Omega)$ (in this case $D(\Omega)$ is called a core of the generator). For more details see [7], Chapter 1 or [7], Chapter 3 and 4.

An interacting particle system on $\Omega = \{0, 1\}^\Lambda$ is called a reaction diffusion process if the only possible transitions are those of the form $\eta \rightarrow \eta^x$ or $\eta \rightarrow \eta^{x,y}$ for some $x, y \in \Lambda$.

Definition 8.13. The jump rates of a reaction-diffusion process on $\{0, 1\}^\Lambda$ are said to be of finite range $R > 0$ if for all $x \in \Lambda$ there exists a finite $\Delta_x \subseteq \Lambda$ with $|\Delta_x| \leq R$ such that
\[ c(\eta^x, (\eta^x)^z) = c(\eta, \eta^x) \quad \text{and} \quad c(\eta^x, (\eta^x)^{x,y}) = c(\eta, \eta^{x,y}) \quad \forall \eta \in \Omega \quad \text{and} \quad z \notin \Delta_x, \]

furthermore
\[ |\{y \in \Lambda : c(\eta, \eta^{x,y})\}| \leq R \quad \forall \eta \in \Omega, \ x \in \Lambda. \]
That is the rate at which the state changes at a site $x$ only depends on the configuration in at most $R$ lattice sites, and a particle can move to at most $R$ sites.

Proposition 8.14. For a finite range reaction-diffusion process for each $f \in D(\Omega)$ we have $\|\mathcal{L}f\|_\infty < \infty$ and the Markov generator $\mathcal{L}$ is uniquely determined by its action on $D(\Omega)$, furthermore
\[ \mathcal{L}f(\eta) = \sum_{x \in \Lambda} c(\eta, \eta^x)(f(\eta^x) - f(\eta)) + \sum_{x \in \Lambda} c(\eta, \eta^{x,y})(f(\eta^{x,y}) - f(\eta)) \quad \forall f \in D(\Omega). \]

Proof. The proof that $\|\mathcal{L}f\|_\infty < \infty$ for all $f \in D(\Omega)$ is left as an exercise. The second part is more technical, see for example [7], Theorem 3.9. \qed

Definition 8.15. A measure $\pi$ on $\Omega$ is called stationary (or invariant) if $\pi S(t) = \pi$ for all $t > 0$. We denote the set of all stationary probability measures by $\mathcal{I}$.

Proposition 8.16. Suppose $D$ is a core for the generator $\mathcal{L}$ of a Markov semigroup $S(t)$ then
\[ \pi \in \mathcal{I} \iff \pi(\mathcal{L}f) = \int_{\Omega} \mathcal{L}f(\eta)\pi(d\eta) = 0 \quad \forall f \in D(\Omega). \]

Theorem 8.17. For every Feller process on compact $\Omega$: 

8.2. CONTACT PROCESS AND GRAPHICAL CONSTRUCTION

1. \( I \) is non-empty, compact and convex,

2. If the weak limit \( \pi = \lim_{t \to \infty} \mu S(t) \) exists from some probability measure \( \mu \) then \( \pi \in I \).

Proof. Convexity of \( I \) follows from the following two basic facts:

a) A convex combination of probability measures is again a probability measure, if \( \mu_1, \mu_2 \in P(\Omega) \) then
\[
\nu = \lambda \mu_1 + (1 - \lambda) \mu_2 \in P(\Omega) \quad \forall \lambda \in [0, 1].
\]

b) The stationarity condition is linear, i.e. if \( \mu_1, \mu_2 \in I \) then \( \nu \in I \) since
\[
\nu(\mathcal{L}f) = \lambda \mu_1(\mathcal{L}f) + (1 - \lambda) \mu_2(\mathcal{L}f) = 0 \quad \forall f \in \mathcal{D}.\]

Suppose \( \mu_1, \mu_2 \in I \) then by the two properties above
\[
\nu := \lambda \mu_1 + (1 - \lambda) \mu_2 \in I.
\]

If \( \mu_n \in I \) is a sequence satisfying \( \mu_n \to \mu \) weakly then \( \mu_n(\mathcal{L}f) = 0 \) for all \( n \in \mathbb{N}, \) and \( \mathcal{L}f \in C(\Omega), \) hence \( \mu(\mathcal{L}f) = \lim_{n \to \infty} \mu_n(\{f\}) = 0. \) Therefore \( I \) is a closed subset of a compact space, and hence compact.

The proof of part 2. is omitted.

Definition 8.18. A Feller process is called ergodic if;

1. \( I = \{ \pi \} \) is a singleton and,

2. \( \lim_{t \to \infty} \mu S(t) = \pi \) for all \( \mu \) a probability measure on \( \Omega. \)

8.2 Contact process and graphical construction

We may think of the contact process as a simple model for the spread of infection on a countable graph \( G = (\Lambda, E) \). The state space is again \( \Omega = \{0, 1\}^\Lambda \) with the usual topological and measurable structure. We think of a site \( x \in \Lambda \) being in state 0 as ‘healthy’ and 1 as infected. We define a Markov process in which each infected site recovers independently with rate 1 and infects its neighbour (independently) with rate \( \lambda \) while it is infected, i.e. for \( \eta \in \Omega \)
\[
\eta \to \eta^x \quad \text{with rate} \quad \begin{cases} 1 & \text{if } \eta(x) = 1 \text{ (recovery)}, \\ \lambda \sum_{y \sim x} \eta(y) & \text{if } \eta(x) = 0 \text{ (infection)}. \end{cases}
\]

Note that in this model the number of particles (i.e. infections) is not conserved, and \( \eta \equiv 0 \) is absorbing, so if \( G \) is finite the process is a.s. absorbed at 0. We can follow the construction described very briefly in the previous section with
\[
\mathcal{L}f(\eta) = \sum_{z \in \Lambda} \left( \eta(z) + \lambda(1 - \eta(z)) \sum_{y \sim z} \eta(y) \right) (f(\eta^z) - f(\eta)) \quad \forall f \in D(\Omega).
\]

However it turns out that there is an equivalent definition on a much larger probability space which will have a lot of useful features, namely the graphical construction. The rigorous proof that the graphical construction gives rise to the same process goes back to Harris in the late 70’s.

For each site \( x \in \Lambda \) we draw a vertical ‘time line’ \([0, \infty)\), and on each line \( \{x\} \times [0, \infty) \) we place a Poison point process \( R^x \) (equivalently the arrival times of a Poison process \((R^x_t)_{t \geq 0}\)) with intensity 1, independently in \( x \), which we associate with possible recovery events. To each pair \( x, y \in \Lambda \) such that \( x \sim y \) we associate independent Poison point process \( B^{x,y} \) with intensity
8.2. CONTACT PROCESS AND GRAPHICAL CONSTRUCTION

\[ Z \]

\[ \text{time} \]

\[ 0 \quad 2 \quad 1 \quad -1 \quad -2 \quad -3 \quad -4 \quad 3 \quad 4 \]

Figure 8.1: Contact process on \( \{0, 1\}^Z \) graphical construction started from two (ordered) initial conditions

\( \lambda \), which we associate with possible pass of infection from \( x \) to \( y \) (see Figure 8.1). This defines the probability space and measure \( P^\lambda \).

The contact process is constructed on the space above in terms of (directed) paths. We say there exists a direct path from \((x, s)\) to \((y, t)\) if there is a path which is increasing in time, does not cross any recovery events, and can follow the direct arrows (see Fig. 8.1). More precisely, there exists a direct path from \((x, s)\) to \((y, t)\) if there exists

\[ (x, s) = (x_0, t_0), (x_0, t_1), (x_1, t_1), (x_1, t_2), \ldots, (x_n, t_{n+1}) = (y, t) \]

with \( t_0 \leq t_1 \leq \ldots \leq t_{n+1} \) with;

1. each interval \( \{x_i\} \times [t_i, t_{i+1}] \) contains no points of \( \mathbb{R}^{x_i} \),
2. \( t_i \in B^{x_i, x_{i+1}} \) for \( i = 1, 2, \ldots, n \).

If such a path exists we say that an infection at \( x \) at time \( s \) can give rise to an infection at \( y \) at time \( t \).

For an initial infection \( \eta_0 \in \Omega \) we define \( \eta_t \in \Omega \), for \( t \in [0, \infty) \), by \( \eta_t(y) = 1 \) if and only if there exists an \( x \in \Lambda \) such that \( \eta_0(x) = 1 \) and there is a directed path from \((x, 0)\) to \((y, t)\). Then \((\eta_t)_{t \geq 0}\) is a contact process.

8.2.1 Monotonicity

The contact process belongs to an important class of IPS called spin systems. We call a Feller process a spin system if \( \Omega = \{0, 1\}^\Lambda \) or \( \{-1, 1\}^\Lambda \), i.e. the local state space has cardinality 2, and the dynamics are such that at most one site can change state at any time (single 'spin-flips').

**Definition 8.19** (Stochastic monotonicity). A Feller process \((\eta_t)_{t \geq 0}\) with semigroup \( S(t) \) is called monotone if it preserves stochastic order; i.e. if \( f \in C(\Omega) \) is an increasing function then \( S(t)f \) is an increasing function for all \( t \geq 0 \). Equivalently \( \mu \leq \nu \) implies \( \mu S(t) \leq \nu S(t) \) for all \( t \geq 0 \).

It turns out, and we will prove on example sheet 3, that for spin systems this property can be identified in a rather simple way in terms of the spin flip rates. In particular we call the spin system attractive if the presence of more 1s in a configuration cannot decrease the rate of flipping to 1 - i.e. the local configuration is 'attracted' by the state around it. More precisely:
Definition 8.20 (Attractive). A spin system is called attractive if and only if

\[ \eta \preceq \sigma \implies \begin{cases} 
  c(\eta, \eta^x) \leq c(\sigma, \sigma^x) & \text{if } \eta(x) = \sigma(x) = 0, \\
  c(\eta, \eta^x) \geq c(\sigma, \sigma^x) & \text{if } \eta(x) = \sigma(x) = 1.
\end{cases} \]

It turns out that for spin systems these two concepts are equivalent.

Theorem 8.21. A spin system is attractive if and only if it is monotone.

Proof. See Sheet 3.

Lemma 8.22. The contact process is monotone.

Proof. The fact that the contact process is monotone follows immediately from the graphical construction.

In fact the graphical construction immediately gives rise to an even stronger property.

Lemma 8.23. The graphical construction is additive: Given the usual bijection between \( \Omega \) and the power set \( 2^\Lambda \) given by identifying a configuration with the subset of \( \Lambda \) of occupied sites, i.e. we identify \( \eta \in \Omega \) with \( \{ x \in \Lambda : \eta(x) = 1 \} \subset \Lambda \), if \( \eta \) we write \( \eta = \eta^A \), then \( \eta^A \cup \eta^B = \eta^A \cup \eta^B \).

There is a further monotonicity present in the contact process which is evident in the graphical construction, in-particular monotonicity in \( \lambda \).

Lemma 8.24. Let \( (\eta_t)_{t \geq 0}, (\sigma_t)_{t \geq 0} \) be two contact processes with semigroups \( S_{\lambda_1} \) and \( S_{\lambda_2} \), and infection rates \( \lambda_1 \leq \lambda_2 \) respectively. Then

\[ \mu^{\lambda_1} \leq \mu^{\lambda_2} \implies \mu^{\lambda_1} S_{\lambda_1}(t) \leq \mu^{\lambda_2} S_{\lambda_2}(t) \quad \forall t \geq 0. \]

Proof. We provide a Markov coupling of the two processes such that if \( h_0 \leq \sigma_0 \) then \( \eta_t \leq \sigma_t \) for all \( t \geq 0 \) with probability one. The result then follows from Strassen’s theorem (Theorem ?). The coupling is given by the graphical construction with infection rate \( \lambda_2 \), to construct the contact process with smaller infection rate \( \lambda_2 \) we thin the infection processes \( (B^x,y) \), by removing each arrow with probability \( \lambda_1 / \lambda_2 \). It is clear that this gives a coupling such that if \( \eta_0 = \sigma_0 \) then \( \eta_t \leq \sigma_t \) for all \( t \geq 0 \). The result follows by stochastic monotonicity of the process.

8.2.2 Duality

Another extremely useful tool for studying interacting particle systems is duality. This property allows us to examine the probability of certain key events under one processes by relating them to events under a hopefully simpler Markov process.

Definition 8.25. Two Markov processes \( (\eta_t)_{t \geq 0} \) on \( \Omega \) and \( (\zeta_t)_{t \geq 0} \) on \( \tilde{\Omega} \) with path measures \( P_\eta \) and \( \tilde{P}_\zeta \) respectively are said to be dual with respect to duality function \( H : \Omega \times \tilde{\Omega} \to \mathbb{R} \) if \( H \) is non-negative and continuous, and

\[ \mathbb{E}_\eta [H(\eta, \zeta)] = \mathbb{E}_\zeta [H(\eta, \zeta)], \]

equivalently

\[ S(t)H(\cdot, \zeta)(\eta) = \tilde{S}(t)H(\eta, \cdot)(\zeta) \quad \forall \eta \in \Omega, \ \zeta \in \tilde{\Omega}. \]

If \( \Omega = \tilde{\Omega} \) and \( P_\eta = \tilde{P}_\eta \) then \( (\eta_t)_{t \geq 0} \) is called self-dual.

The following result is extremely useful for checking, and recognising, duality conditions.
Proposition 8.26. In the context of the previous definition, if \((\eta_t)_{t \geq 0}\) (respectively \((\zeta_t)_{t \geq 0}\)) has Markov generator \(L\) (respectively \(\tilde{L}\)) they the processes are dual if and only if
\[
\mathcal{L}H(\cdot, \zeta)(\eta) = \tilde{\mathcal{L}}H(\eta, \cdot)(\zeta) \quad \forall \eta \in \Omega, \zeta \in \tilde{\Omega},
\]
provided that \(\mathcal{L}H(\cdot, \zeta)\) and \(\tilde{\mathcal{L}}H(\eta, \cdot)\) are well defined.

**Proof.** Omitted, see Liggett *Interacting Particle Systems*.

Example 8.27. Birth-death process on \(\mathbb{N} = \{0, 1, \ldots\}\) with a trap at 0.

\[
\begin{align*}
n &\to n + 1 \quad \text{at rate} \quad \beta(n) > 0 \quad \forall n \geq 1 \quad \beta(0) = 0, \\
n &\to n - 1 \quad \text{at rate} \quad \delta(n) > 0 \quad \forall n \geq 1 \quad \delta(0) = 0.
\end{align*}
\]

Let \(H(\eta, \zeta) = \mathbb{1}\{\eta \leq \zeta\}\), we shall see that this defines a duality function and that the dual process is 'simpler' in the sense that it is irreducible (no absorbing state).

\[
\mathcal{L}H(\cdot, \zeta)(\eta) = \beta(\eta) [H(\eta + 1, \zeta) - H(\eta, \zeta)] + \delta(\eta) [H(\eta - 1, \zeta) - H(\eta, \zeta)]
\]

\[
= \beta(\eta) \begin{cases} 0 & \text{if } \eta \neq \zeta \\
-1 & \text{if } \eta = \zeta \end{cases} + \delta(\eta) \begin{cases} 0 & \text{if } \eta \neq \zeta + 1 \\
1 & \text{if } \eta = \zeta + 1 \end{cases}
\]

\[
= \beta(\zeta) [H(\eta, \zeta - 1) - H(\eta, \zeta)] + \delta(\zeta + 1) [H(\eta, \zeta + 1) - H(\eta, \zeta)] := \tilde{\mathcal{L}}H(\eta, \cdot)(\zeta),
\]

where

\[
\tilde{L}f(\eta) = \beta(\eta) (f(\zeta - 1) - f(\zeta)) + \delta(\zeta + 1) (f(\zeta + 1) - f(\zeta))
\]

is the Markov generator of a process for which \(n \to n + 1\) at rate \(\delta(n + 1)\) and \(n \to n - 1\) at rate \(\beta(n)\). Notice that the chain described by \(\tilde{L}\) is irreducible, and the roles of \(\beta\) and \(\delta\) are reversed. Now in terms of the Markov semigroup

\[
S(t)H(\cdot, \zeta)(\eta) = S(t)\mathbb{1}\{\cdot \leq \zeta\}(\eta) = \mathbb{E}_\eta [\mathbb{1}\{\eta_t \leq \zeta\}] = \mathbb{P}_\eta(\eta_t \leq \zeta)
\]

\[
= \tilde{S}(t)\mathbb{1}\{\eta \leq \cdot\}(\zeta) = \tilde{P}_\zeta(\zeta_t \geq \eta).
\]

So taking the limit \(t \to \infty\) we see that \(\eta_t\) has a positive probability of escaping to \(\infty\) if and only if \(\zeta_t\) is positive recurrent, and in this case \(\mathbb{P}_\eta(\eta_t\text{ is absorbed at } 0) = \sum_{\zeta \geq \eta} \pi(\zeta)\) where \(\pi\) is the stationary distribution of \((\zeta_t)_{t \geq 0}\).

**Theorem 8.28.** The contact process is self dual (with a contact process with finitely many infected sites) with respect to the duality function

\[
H(\eta, A) = \prod_{x \in A} (1 - \eta(x)) = 1(\eta \equiv 0 \text{ on } A), \quad \text{for } \eta \in \Omega, \text{ and } A \subset \Lambda \text{ finite},
\]

where (as usual) we identify subsets of \(\Lambda\) with configurations in \(\Omega\). That is, for \(A, B \subset \Lambda\)

\[
\mathbb{P}_A(\eta_t^A \cap B \neq 0) = \mathbb{P}_B(\eta_t^B \cap A \neq 0),
\]

where \((\eta_t^A)_{t \geq 0}\) denotes the process started from infection in \(A \subset \Lambda\) (there is some obvious redundancy in the notation \(\mathbb{P}_A\) at this point).

**Proof.** Using the graphical construction we see that the event \(\{\eta_t^A \cap B \neq 0\}\) is the union over all \(a \in A\) and \(b \in B\) of the event that there is a directed path from \((a, 0)\) to \((b, t)\). If we reverse the direction of time and all the arrows in the graphical construction the probability measure induced on path space started from time \(t\) and run back to 0 is the same as we have for the process started from time zero and run forwards. Given these two observation the event \(\{\eta_t^A \cap B \neq 0\}\) occurs with the same probability as \(\{\eta_t^B \cap A \neq 0\}\).

As an alternative proof, you could mirror 'style' of the calculation in the previous example. \(\square\)
8.2.3 Stationary measures and percolation

To simplify the presentation we will now focus on the contact process on the $\mathbb{Z}^d$ lattice. We would like to characterise the stationary measures of the process, by Theorem 8.17 it is sufficient to find all the extremal measures. Let $\delta_0$ be the probability measure that puts unit mass on the configuration “everyone is healthy”, i.e. $\delta_0(\{\eta = 0\}) = 1$, and $\delta_1$ is defined by $\delta_1(\{\eta = 1\}) = 1$.

Clearly $\delta_0$ is invariant (since no infection can “appear out of nowhere”), and $\delta_0 \leq \mu$ for all $\mu$ a probability measure on $\Omega$. $\delta_0$ is called the minimal (invariant) measure.

The maximal (invariant) measure is constructed by the weak limit of the contact process started from “everyone infected”.

**Proposition 8.29.** We have $\delta_1 S(t) \leq \delta_1 S(s)$ for all $0 \leq s \leq t$ and hence the weak limit $\bar{\nu}_\lambda = \lim_{t \to \infty} \delta_1 S(t)$ exists, furthermore $\bar{\nu}_\lambda \in \mathcal{I}_\kappa$. We call $\bar{\nu}_\lambda$ the upper invariant measure.

**Proof Sketch.** $\mu_r := \sigma_1 S(r) \leq \delta_1$ by construction. Let $t = s + r$, then

$$\mu_t = \sigma_1 S(r) S(s) = \mu_r S(s) \leq \delta_1 S(s) = \mu_s,$$

where the final inequality follows from stochastic monotonicity. We now use the fact that the set of probability measures over $\Omega$ is relatively compact (see Section 4.3.1). The fact that $\bar{\nu} \in \mathcal{I}_\kappa$ follows from the next proposition.

**Proposition 8.30.** We have $\delta_0 \leq \nu \leq \bar{\nu}$ for all $\nu \in \mathcal{I}$.

**Proof.** Apply stochastic monotonicity and take week limits.

To complete the proof of the previous proposition suppose $\bar{\nu} = \alpha \mu_1 + (1 - \alpha) \mu_2$ for $\mu_1, \mu_2 \in \mathcal{I}$ and $\alpha \in [0, 1]$. Then for any increasing function $f$ we have, by the Prop 8.30, we have $\mu_1(f) \leq \bar{\nu}(f)$, and $\mu_2(f) \leq \bar{\nu}(f)$. Together with $\bar{\nu}(f) = \alpha \mu_1(f) + (1 - \alpha) \mu_2(f)$ these imply $\nu = \mu_1 = \mu_2$, as required. Since $\delta_1$ is clearly translation invariant, and $\bar{\nu}$ is defined as the week limit of $\delta_1 S(t)$ as $t \to \infty$, we also have that $\bar{\nu}$ is translation invariant.

As a consequence of the results above, the contact process is ergodic if and only if $\bar{\nu} = \delta_0$. Trying to identify when this is the case reduces to a percolation-type of problem. In analogy with the percolation probability we define

$$\theta(\lambda) = \mathbb{P}_{\{0\}}(\eta^0_t \neq \emptyset \text{ for all } t \geq 0) \quad (8.2.2)$$

this is exactly the probability that the origin $(0, 0)$ is ‘connected to $\infty$’ by a directed path in the graphical construction.

**Proposition 8.31.** The density of infected sites under $\bar{\nu}$ is equal to $\theta(\lambda)$, i.e.

$$\theta(\lambda) = \bar{\nu}(\sigma(x) = 1), \quad x \in \mathbb{Z}^d,$$

which is clearly independent of $x$.

**Proof.** We use the duality result, Theorem 8.28 and translation invariance. Clearly $\{\eta^0_t \neq \emptyset\} \subseteq \{\eta^0_s \neq \emptyset\}$ for $t > s$, hence

$$\theta(\lambda) = \lim_{t \to \infty} \mathbb{P}_{\{0\}}(\eta^0_t \neq \emptyset).$$

Now applying the self duality, Theorem 8.28, with with $B = \mathbb{Z}^d$ we have

$$\mathbb{P}_{\{0\}}(\eta^0_t \neq \emptyset) = \mathbb{P}_{\delta_1}(\eta_t(0) = 1),$$

and by construction

$$\mathbb{P}_{\delta_1}(\eta_t(0) = 1) = \delta_1 S(t)(\{\eta(0) = 1\}) \to \bar{\nu}(\eta(0) = 1) \quad \text{as } t \to \infty,$$

the claim follows by translation invariance of $\bar{\nu}$. \qed
By the monotonicity of the contact process in the infection rate, Lemma 8.24, $\theta(\lambda)$ must be non-decreasing in $\lambda$. We define the critical value of $\lambda$ by

$$
\lambda_c = \lambda_c(d) = \sup\{\lambda \geq 0 : \theta(\lambda) = 0\}.
$$

Then (c.f. percolation)

$$
\theta(\lambda) = \begin{cases} 
0 & \text{if } \lambda < \lambda_c, \\
> 0 & \text{if } \lambda > \lambda_c,
\end{cases}
$$

and by Proposition 8.31,

$$
\varrho = \begin{cases} 
\delta_0 \text{ (ergodic)} & \text{if } \lambda < \lambda_c, \\
\neq \delta_0 \text{ (not ergodic)} & \text{if } \lambda > \lambda_c.
\end{cases}
$$

In fact the connections between the contact process and percolation go much deeper than we have time to go into any further here, but see for example Grimmett *Probability on Graphs* and Liggett *Interacting Particle Systems* (Chapter on the contact process) for an introduction to this area and further references.

### 8.3 Exclusion process

The exclusion process is a prototypical IPS in which the particle number is locally conserved. Roughly speaking particles make independent random walks until they meet each other, and then they interact simply by exclusion (as the name suggests) meaning that at most one particle can occupy any site. The most simple forms of this process have been studied intensely from a theoretical point of view and have given rise to a whole host of rich and beautiful results, also with deep connections to many other areas of probability theory. Many variants have also been studied and applied to modelling systems in biology and physics.

Again, let $\Omega = \{0, 1\}^\Lambda$ and for $\eta \in \Omega$ we say there is a particle at site $x \in \Lambda$ if $\eta(x) = 1$, and otherwise the site is empty. Fix $p(x, y)$ a transition matrix of a (discrete time) irreducible random walk on $\Lambda$ such that

$$
\sum_x \sum_y p(x, y) < \infty,
$$

this technical condition is just a simple way of insuring we construct a Feller process. Then $\eta \to \eta^{x,y}$ at rate $p(x, y)$ if $\eta(x) = 1$ and $\eta(y) = 0$, where

$$
\eta^{x,y}(z) = \begin{cases} 
\eta(z) & \text{if } z \notin \{x, y\}, \\
1 - \eta(z) & \text{if } z \in \{x, y\}.
\end{cases}
$$

Correspondingly, the generator of the process is given by

$$
\mathcal{L}f(\eta) = \sum_{x, y \in \Lambda} p(x, y)\eta(x)(1 - \eta(y)) \left(f(\eta^{x,y}) - f(\eta)\right), \quad \text{for } f \in \mathcal{D}_\mathcal{L}.
$$

Here we may take the core of the generator to $\mathcal{D}_\mathcal{L}$ to be the set of functions which depend on only finitely many sites (cylinder functions).

**Remark 8.32.** Unlike ‘spin-flip’ dynamics we saw in the contact process, in the exclusion process the particle number is conserved, i.e. particles move but are never created or destroyed.

Our main aim in this section will again be to characterise the stationary measures of the process. With this in mind we will find the following definition rather useful.
**Definition 8.33.** For a given density profile $\rho : \Lambda \to [0, 1]$ the probability measure $\nu_\rho$ is called product measure on $\Omega$ if for all $k \in \mathbb{N}$ and $x_1, \ldots, x_k \in \Lambda$, mutually different, and $n_1, \ldots, n_k \in \{0, 1\}$,

$$\nu_\rho [\eta(x_1) = n_1, \eta(x_2) = n_2, \ldots, \eta(x_k) = n_k] = \prod_{i=1}^k \nu_\rho [\eta(x_i) = n_i]$$

$$= \prod_{i=1}^k \rho(x_i)^{n_i} (1 - \rho(x_i))^{1-n_i}.$$

With a slight abuse of notation, if $\rho \in [0, 1]$ then by $\nu_\rho$ we mean $\nu_\varphi$ with $f(x) = \rho$ for all $x \in \Lambda$.

Observe, the above notation simply says that under $\nu_\rho$ we have $\eta(x) \sim \text{Ber}(\rho(x))$ independently in $x \in \Lambda$.

**Theorem 8.34.** (a) Suppose $p(x, y)$ is doubly stochastic, i.e.

$$\sum_{y' \in \Lambda} p(x, y') = \sum_{x' \in \Lambda} p(x', y) = 1 \quad \forall x, y \in \Lambda$$

then $\nu_\rho \in \mathcal{I}$ for all $\rho \in [0, 1]$ (uniform density stationary measures).

(b) If $\lambda(x) \geq 0$ and $\lambda(x)p(x, y) = \lambda(y)p(y, x)$ for all $x, y \in \Lambda$ (i.e. $p(x, y)$ is reversible), then

$$\nu_\rho \in \mathcal{I} \quad \text{where} \quad \rho(x) = \frac{\lambda(x)}{1 + \lambda(x)} \quad \forall x \in \Lambda.$$

**Proof.** By Proposition 8.16, we need to check

$$\nu_\rho (\mathcal{L}f) = 0, \quad \text{for all } f \in \mathcal{D}_\mathcal{L}.$$

By linearity of this condition it is sufficient to check that $\nu_\rho (\mathcal{L}f_\Delta) = 0$ for simple functions

$$f_\Delta(\eta) = \begin{cases} 1 & \text{if } \eta(x) = 1 \quad \forall x \in \Lambda, \\ 0 & \text{otherwise,} \end{cases}$$

where $\Delta \subset \Lambda$ finite. The rest of the proof is a calculation

$$\nu_\rho (\mathcal{L}f_\Delta) = \sum_{x, y \in \Lambda} p(x, y) \int_\Omega \eta(x)(1 - \eta(y)) (f_\Delta(\eta^{x,y}) - f_\Delta(\eta)) \, d\nu_\rho.$$  \hspace{1cm} (8.3.2)

For $x \not= y$ (we take $p(x, x) = 0$ for all $x \in \Lambda$) the integral terms in the sum look like

$$\int_\Omega f_\Delta(\eta) \eta(x)(1 - \eta(y)) \, d\nu_\rho = \begin{cases} 0 & \text{if } y \in \Delta, \\ (1 - \rho(y)) \prod_{u \in \Delta \setminus \{x\}} \rho(u) & \text{if } y \not\in \Delta \end{cases}$$

$$\int_\Omega f_\Delta(\eta^{x,y}) \eta(x)(1 - \eta(y)) \, d\nu_\rho = \begin{cases} 0 & \text{if } x \in \Delta, \\ (1 - \rho(y)) \prod_{u \in \Delta \setminus \{x\} \setminus \{y\}} \rho(u) & \text{if } x \not\in \Delta \end{cases}.$$  \hspace{1cm} (8.3.3)

This follows from the fact that the integrands take values only in $\{0, 1\}$, so the right-hand side is the probability of the integrand being 1. Re-arranging the sum we get

$$\nu_\rho (\mathcal{L}f_\Delta) = \sum_{x \in \Delta \setminus \{y\} \not\in \Delta} \left[ \rho(y)(1 - \rho(x)) p(y, x) - \rho(x)(1 - \rho(y)) p(x, y) \right] \prod_{u \in \Delta \setminus \{x\}} \rho(u).$$  \hspace{1cm} (8.3.4)
Assumption of (b) is equivalent to
\[
\frac{\rho(x)}{1 - \rho(x)} p(x, y) = \frac{\rho(y)}{1 - \rho(y)} p(y, x),
\]
so the square bracket vanishes for all \(x, y\) in the sum (8.3.4). For \(\rho(x) \equiv \rho\) in (a) we get
\[
\nu_\rho(\mathcal{L} f_\Delta) = \rho|\Delta^c| (1 - \rho) \sum_{x \in \Delta, y \not\in \Delta} [p(y, x) - p(x, y)] = 0
\]
due to \(p(\cdot, \cdot)\) being proportional to a doubly-stochastic matrix.

We have found sufficient conditions for stationary measures, in fact they turn out to be close to necessary. In the next two sections we focus on two different special cases.

### 8.3.1 ASEP and blocking measures

Now we consider an asymmetric simple exclusion process on \(\Lambda = \mathbb{Z}\) (ASEP) with \(p(x, x + 1) = p\) and \(p(x, x - 1) = q\) with \(p > q\). The underlying RW is transient but \(\lambda(x) = c \left( \frac{x}{q} \right)^2\) satisfies the detailed balance condition (it is certainly not a stationary probability measure for the RW since it can not be normalized). For \(c \geq 0\) it blows up exponentially as \(x \to \infty\). By Theorem 8.34 we know \(\nu_\rho\) is stationary for this ASEP, where
\[
\rho(x) = \frac{c(p/q)^x}{1 + c(p/q)^x} = \frac{(p/q)^{x-x_0}}{1 + (p/q)^{x-x_0}},
\]
and \(x_0 = \log_{p/q}(1/c)\). See picture from lectures. Note, by the Borel-Cantelli Lemma, \(\nu_\rho\) concentrates on configuration with only finitely many particles on the left and only finitely many vacancies on the right, i.e.
\[
\nu_\rho \left[ \left\{ \eta \in \Omega : \sum_{x<0} \eta(x) < \infty, \sum_{x \geq 0} (1 - \eta(x)) < \infty \right\} \right] = 1.
\]
These are called blocking measures (there is a blockage in this case on the right), and it turns out that for \(p \neq q\) they are reversible. Let
\[
\Theta_{x_0} = \left\{ \eta \in \Theta : \sum_{x < x_0} \eta(x) = \sum_{x \geq x_0} (1 - \eta(x)) \right\}
\]
where \(\Theta = \{ \eta \in \Omega : \sum_{x<0} \eta(x) < \infty, \sum_{x \geq 0} (1 - \eta(x)) < \infty \}\). Then the process starting from a point \(\eta_0 \in \Theta_{x_0}\) remains in \(\Theta_{x_0}\) for all times. This holds since - when a particle crosses the bond \((x_0 - 1, x_0)\) then simultaneously a hole goes the other way. So \((\eta_t)_{t \geq 0}\) is an irreducible, countable state, Markov chain on \(\Theta_{x_0}\) which is positive recurrent since \(\pi_{x_0}[\cdot] = \nu_\rho[\cdot | \Theta_{x_0}]\) is stationary (note \(\pi_{x_0}\) is independent of \(\rho\)). T. M. Ligget proved that if \(p \neq q\) then \(\mathcal{I}_e = \{\nu_\rho : \rho \in [0, 1]\} \cup \{\pi_n : n \in \mathbb{Z}\}\).

### 8.3.2 SSEP stationary measure, coupling and duality

We again focus on \(\Lambda = \mathbb{Z}\) but now instead we assume the underlying RW is symmetric, i.e. \(p(x, y) = p(0, y - x) = p(y, x)\), called the symmetric simple exclusion process (SSEP).

**Theorem 8.35.** For the SSEP on \(\Lambda = \mathbb{Z}\) and with \(p(x, y) = p(0, y - x) = p(y, x)\) the transition probabilities of a symmetric irreducible discrete time random walk on \(\mathbb{Z}\), then
Now since \( p \) there are only two cases in the sum above for which the argument is non-zero, either \( \eta \in \text{birth-death chain example)\), exclusion process acts on \( H \) please check this calculation yourself. We now examine how the generator of the symmetric generator of the symmetric exclusion process acts on \( H \) as a function of its first variable (and mirror the calculation we did in the birth-death chain example),

\[
\mathcal{L} H(\cdot, A) = \sum_{x,y \in \Lambda} p(x,y) (1 - \eta(y)) (H(\eta, A) - H(\eta, y)) + p(x,y) (1 - \eta(y)) (H(\eta, A) - H(\eta, A)),
\]

\[
= \sum_{x,y \in \Lambda} p(x,y) (1 - \eta(y)) (H(\eta, A) - H(\eta, y)) + p(x,y) (1 - \eta(y)) (H(\eta, A) - H(\eta, A)).
\]

There are only two cases in the sum above for which the argument is non-zero, either \( x \in A \) and \( y \) is not or vice versa. So, also applying the following obvious identities \( \eta(y)(1 - \eta(y)) = 0 \) and \( \eta(x)\eta(y) = \eta(x)\),

\[
\mathcal{L} H(\cdot, A) = \sum_{x \in A, y \in A} p(x,y) (1 - \eta(y)) (H(\eta, A) - H(\eta, y)) + \sum_{x \in A, y \notin A} p(x,y) (1 - \eta(y)) (H(\eta, A) - H(\eta, y)) + \sum_{x \notin A, y \in A} p(x,y) (1 - \eta(y)) (H(\eta, A) - H(\eta, y)) + \sum_{x \notin A, y \notin A} p(x,y) (1 - \eta(y)) (H(\eta, A) - H(\eta, y))
\]

Now since \( p(x,y) = p(y,x) \) we have

\[
\mathcal{L} H(\cdot, A) = \sum_{x \in A, y \notin A} p(x,y) (1 - \eta(y)) (H(\eta, A) - H(\eta, y)) + \sum_{x,y \in A} p(x,y) (1 - \eta(y)) (H(\eta, A) - H(\eta, y)) + \sum_{x \notin A, y \in A} p(x,y) (1 - \eta(y)) (H(\eta, A) - H(\eta, y)) + \sum_{x \notin A, y \notin A} p(x,y) (1 - \eta(y)) (H(\eta, A) - H(\eta, y)) := \tilde{\mathcal{L}} H(\cdot, A)
\]

\( \text{Proposition 8.36. The SSEP on } \Omega \text{ is self-dual with a finite exclusion process. More precisely, for } \eta \in \Omega \text{ and } A \subset \Lambda \text{ finite define}

\[
H(\eta, A) = \prod_{x \in A} \eta(x) = \mathbb{1}_{\eta|A=1} = \begin{cases} 1 & \text{if } \eta \equiv 1 \text{ on } A, \\ 0 & \text{otherwise.} \end{cases}
\]

then

\[
S(t)H(\cdot, A)(\eta) = S(t)H(\eta, \cdot)(A) \quad \text{for all } \eta \in \Omega \text{ and } A \subset \Lambda \text{ finite.}
\]

Here again we identify subsets of \( \Lambda \) with elements of \( \Omega \).

**Proof.** We extend our notation for configuration’s after a jump from \( x \) to \( y \) to finite sets as follows, for a finite \( A \subset \Lambda \) let

\[
A^{x,y} = \begin{cases} A & \text{if } x, y \in A, \text{ or } x, y \notin A, \\ (A \setminus \{x\}) \cup \{y\} & \text{if } x \in A \text{ and } y \notin A, \\ (A \cup \{x\}) \setminus \{y\} & \text{if } x \notin A \text{ and } y \in A. 
\end{cases}
\]

Now observe that

\[
H(\eta^{x,y}, A) = H(\eta, A^{x,y}),
\]

please check this calculation yourself. We now examine how the generator of the symmetric exclusion process acts on \( H \) as a function of its first variable (and mirror the calculation we did in the birth-death chain example),

\[
\mathcal{L} H(\cdot, A) = \sum_{x,y \in \Lambda} p(x,y) (1 - \eta(y)) (H(\eta, A) - H(\eta, y)) + p(x,y) (1 - \eta(y)) (H(\eta, A) - H(\eta, y))
\]

\[
= \sum_{x,y \in \Lambda} p(x,y) (1 - \eta(y)) (H(\eta, A) - H(\eta, y)) + p(x,y) (1 - \eta(y)) (H(\eta, A) - H(\eta, y)).
\]

\[\text{Now since } p(x,y) = p(y,x) \text{ we have}\]

\[
\mathcal{L} H(\cdot, A) = \sum_{x \in A, y \notin A} p(x,y) (1 - \eta(y)) (H(\eta, A) - H(\eta, y)) + \sum_{x,y \in A} p(x,y) (1 - \eta(y)) (H(\eta, A) - H(\eta, y)) + \sum_{x \notin A, y \in A} p(x,y) (1 - \eta(y)) (H(\eta, A) - H(\eta, y)) + \sum_{x \notin A, y \notin A} p(x,y) (1 - \eta(y)) (H(\eta, A) - H(\eta, y)) := \tilde{\mathcal{L}} H(\cdot, A)
\]
where $\tilde{L}$ is the generator of a process for which
\[ A \rightarrow A^{x,y} \quad \text{at rate} \quad p(x,y)\mathbb{I}_A(x)\left(1 - \mathbb{I}_A(y)\right). \]
This is just an exclusion process with a finite number of particles with different notation. That is we may associate a finite configuration with the set $A$ via $\zeta$ we may associate a finite configuration with the set $A$ via $\zeta(x) = \mathbb{I}_A(x)$, or equivalently identify $\eta \in \Omega$ such that $\sum_x \eta(x) < \infty$ with a finite subset of $\Lambda$ by $A = \{x \in \Lambda : \eta(x) = 1\}$ (See pictures from the lectures).

The self duality is useful since it allows us to write that probability that a finite exclusion process, started from $\eta$, occupies the initial configuration of the process we were originally interested in. Much more succinctly
\[ S(t)H(\cdot,A)(\eta) = \mathbb{E}_\eta[\mathbb{I}_{\{\eta_t | A = 1\}]} = \mathbb{P}_\eta[\eta_t \equiv 1 \text{ on } A] = S(t)H(\eta,\cdot)(A) = \mathbb{P}_A[\eta \equiv 1 \text{ on } A_t]. \]

The basic coupling of two copies of the exclusion process is the process $(\eta_t, \zeta_t)_{t \geq 0}$ with the following transitions at rate $p(x,y)$:
\[
\begin{align*}
(\eta, \zeta) \rightarrow (\eta^{x,y}, \zeta^{x,y}) & \quad \text{if } \eta(x) = \zeta(x) = 1 \text{ and } \eta(y) = \zeta(y) = 0, \\
(\eta, \zeta) \rightarrow (\eta^{x,y}, \zeta) & \quad \text{if } \eta(x) = 1, \eta(y) = 0 \text{ and } (\zeta(x) = 0 \text{ or } \zeta(y) = 1), \\
(\eta, \zeta) \rightarrow (\eta, \zeta^{x,y}) & \quad \text{if } \zeta(x) = 1, \zeta(y) = 0 \text{ and } (\eta(x) = 0 \text{ or } \eta(y) = 1).
\end{align*}
\]
In other words, particles move together whenever possible. We say there is a discrepancy at site $x \in \Lambda$ if $\eta(x) \neq \zeta(x)$. An important property of the coupling is that, although discrepancies can move and be destroyed, they can not be created. Isolated discrepancies move according to a continuous time random walk on $\Lambda$ with rates $p(x,y)$. Also if $\eta_0 \preceq \zeta_0$ then $\eta_t \preceq \zeta_t$ for all $t \geq 0$ under the coupling, so the process is attractive. In lectures we drew some pictures, these are very helpful here, they may appear in time in these printed notes, but in the meantime it is worth drawing your own. For some important consequences see for example the Liggett notes on Interacting Particle Systems - An introduction [?].

For the proof of Theorem 8.35 we will use a coupling of two symmetric exclusion processes with the same finite number of particles, which initially have two discrepancy (the minimal number). We will use the finite subset representation of configurations described in the previously. Consider two initial conditions $A_0$ and $B_0$ such that they both contain $n$ particles, $|A_0| = |B_0| = n$, and which agree everywhere except for two particles (again draw pictures). Write $A_t = C_t \cup \{X_t\}$ and $B_t = C_t \cup \{Y_t\}$ where $C_t = A_t \cap B_t$. The two discrepancies at $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ perform independent random walks until they meet, at which point they annihilate each other, from this random time onwards the two processes $A_t$ and $B_t$ move together, they are said to be coupled. We call this random time the coupling time, $\tau_{\text{couple}} := \inf\{t > 0 : A_t = B_t\} = \inf\{t > 0 : X_t = Y_t\}$. For $t < \tau_{\text{couple}}$ the basic coupling can be re-written as follows (in terms of the overlap $(C_t)$ and the discrepancies $(X_t)$ and $(Y_t)$), for $u \in C$ and $v \notin C \cup \{x,y\}$
\[
(C,x,y) \rightarrow \begin{cases} 
(C^{u,v},x,y) & \text{at rate } p(u,v), \\
(C,v,y) & \text{at rate } p(x,v), \\
(C,x,v) & \text{at rate } p(y,v), \\
(C,x,x) & \text{at rate } p(x,y), \\
(C,y,y) & \text{at rate } p(x,y), \\
(C^{u,x},u,y) & \text{at rate } p(u,x), \\
(C^{u,y},x,u) & \text{at rate } p(u,y).
\end{cases}
\]
For \( t > \tau_{\text{couple}} \) the two processes move together, i.e. \( A_t = B_t \). Since we construct this coupling only in the symmetric case \( p(x,y) = p(y,x) \) we can check that the marginal process truly move with the rates we described initially, that is: The pair \((X_t, Y_t)\) has the transition rates of two independent random walks with transition rates \( p(\cdot, \cdot) \) until the first time they meet, at which point they move together, and \( C_t \) evolves as a single exclusion process. This is just the basic coupling but we are careful to keep track of the locations of the two discrepancies, and when they meet they annihilate each other.

We will use this coupling to give a neat proof of a Maximum Principle (c.f. Theorem 2.1) in this particular setting (which is countable so Theorem 2.1 does not apply directly). We will prove that the finite symmetric exclusion process, restricted to an irreducible component, has no non-constant bounded harmonic functions. Recall a bounded function \( f \) is called harmonic for the process if \( \mathbb{E}_\eta [f(\eta_t)] = f(\eta) \).

**Lemma 8.37.** A bounded function \( f \) is harmonic function for the finite exclusion process, i.e.

\[
\mathcal{L} f (A) = 0 \quad \forall A \subset \Lambda \text{ finite},
\]

if and only if \( f \) is constant on \( \{ A \subset \Lambda : |A| = n \} \) for all \( n \geq 1 \).

**Proof.** Since the exclusion process conserves particle number we immediately observe that any function which is constant on \( \{ A \subset \Lambda : |A| = n \} \) for each \( n \geq 1 \) is harmonic.

We will present the converse only for the case that the random walk \( p(\cdot, \cdot) \) is recurrent. Suppose \( f \) is a bounded harmonic function and fix \( A_0 \) and \( B_0 \) such that \( |A_0| = |B_0| = n \). Let \( m = n - |A_0 \cap B_0| \) since \( A_0 \) and \( B_0 \) both have cardinality \( n \) there exist a sequence of sets \( A_0 = A^{(1)}, A^{(2)}, \ldots, A^{(m)} = B_0 \) such that \( |A^{(i)}| = n \) and \( A^{(i)} \cap A^{(i+1)} = n - 1 \) (i.e. we can reduce to exactly the case we constructed the coupling for above). So

\[
|f(A_0) - f(B_0)| = |\mathbb{E}_{A_0}[f(A_t)] - \mathbb{E}_{B_0}[f(B_t)]| \\
\leq m \max_{1 \leq i \leq m-1} \left| \mathbb{E}_{A^{(i)}}[f(A_t^{(i)})] - \mathbb{E}_{A^{(i+1)}}[f(A_t^{(i+1)})] \right|.
\]

We denote path measure and expectation with respect to the coupling of two finite exclusion process started from just two discrepancies, constructed above, by \( \tilde{\mathbb{P}} \) and \( \tilde{\mathbb{E}} \). Now the right hand side above can be written in terms of the coupled process, so

\[
|f(A_0) - f(B_0)| \leq m \max_{1 \leq i \leq m-1} \tilde{\mathbb{E}}_{(A^{(i)}, A^{(i+1)})} \left| f(A_t^{(i)}) - f(A_t^{(i+1)}) \right| \\
\leq m \| f \|_{\infty} \max_{1 \leq i \leq m-1} \tilde{\mathbb{P}}_{(A^{(i)}, A^{(i+1)})} \left[ A_t^{(i)} \neq A_t^{(i+1)} \right].
\]

Since the two discrepancies move as independent random walks with transition rates \( p(\cdot, \cdot) \), until the first time they meet \( \tau_{\text{couple}} \), by the recurrence assumption on \( p(\cdot, \cdot) \) (in particular \((X_t - Y_t)_{t \geq 0}\) is a random walk with rates \( 2p(\cdot, \cdot) \)), we have

\[
\tilde{\mathbb{P}}_{(A^{(i)}, A^{(i+1)})} [\tau_{\text{couple}} < \infty] = 1 \quad \text{for } i \in \{1, \ldots, m - 1\},
\]

and so the right hand side above tends to zero as \( t \to \infty \). Since the left hand side is independent of \( t \) this implies \( f(A_0) = f(B_0) \), and so \( f \) is constant on \( \{ A \subset \Lambda : |A| = n \} \).

If the random walk is not recurrent then one can proceed by comparison with independent random walks (for which the result still holds), using a different coupling \footnote{[? , Section VII.4]}. \( \square \)

To complete the proof of Theorem 8.35 we need one more standard result, in particular we will use de Finetti's Theorem. To this end we have to introduce the concept of exchangeable random variables, which you may have come across before in different settings. For more details see for example \footnote{[? , Section VII.4]}.
Definition 8.38. The random variables \((\eta(x))_{x \in \Lambda}\) with probability distribution \(\mu\) are exchangeable if all finite dimensional marginals are the same under finite permutations of the coordinates, i.e. for each \(n \geq 1, x_1, \ldots, x_n \in \Lambda\) mutually different and \(\varepsilon_1, \ldots, \varepsilon_n \in S\)

\[
\mu[\eta(x_1) = \varepsilon_1, \ldots, \eta(x_n) = \varepsilon_n] = \mu[\eta(x_{\pi_1}) = \varepsilon_1, \ldots, \eta(x_{\pi_n}) = \varepsilon_n]
\]

for every permutation \(\pi \in S_n\).

Remark 8.39. On \(\Omega = \{0, 1\}^\Lambda\) the measure \(\mu\) is exchangeable if and only if \(\mu[\{\eta \in \Omega : \eta \equiv 1 \text{ on } A\}]\) depends on \(A\) only through its cardinality \(|A|\).

Theorem 8.40 (de Finetti). Every exchangeable measure \(\mu\) on \(\{0, 1\}^\Lambda\) is a mixture of homogeneous (flat) product Bernoulli measures, i.e. there exists a measure \(\gamma\) on \([0, 1]\) such that

\[
\mu = \int_0^1 \nu \gamma(d\alpha).
\]

Equivalently

\[
\mu[\eta(x_1) = 1, \ldots, \eta(x_k) = 1, \eta(x_{k+1}) = 0, \ldots, \eta(x_n) = 0] = \int_0^1 \alpha^k (1 - \alpha)^{n-k} \gamma(d\alpha).
\]

We are now in a position to complete the proof of Theorem 8.35.

Proof of Theorem 8.35 (a). By de Finetti’s theorem it is sufficient to show a measure is stationary if and only if it is exchangeable. Since linear combinations of function of the form \(H(\cdot, A)\) for \(A \subset \Lambda\) finite are dense in \(C(\Omega)\), any measure probability measure on \(\Omega\) is uniquely determined by its values on sets of the form \(\{\eta \mid \eta \equiv 1 \text{ on } A\}\) for \(A \subset \Lambda\) finite. Fix an initial probability measure \(\mu\) on \(\Omega\), and let \(\mu_t = \mu S(t)\), applying the duality (Proposition 8.36) we have

\[
\mu_t(\{\eta \mid \eta \equiv 1 \text{ on } A\}) = \int_\Omega \mathbb{P}_\eta (\eta_t \equiv 1 \text{ on } A) \mu(d\eta) = \int_\Omega \mathbb{P}_A (\eta \equiv 1 \text{ on } A_t) \mu(d\eta) = \sum_{B \subset A \mid |B| = |A|} \mathbb{P}_A (A_t = B) \mu(\{\eta \mid \eta \equiv 1 \text{ on } B\}) .
\]

(8.3.7)

If we assume that \(\mu\) is exchangeable then the last term on the right hand side is just \(\mu(\{\eta \mid \eta \equiv 1 \text{ on } A\})\), and so \(\mu\) is stationary. Now suppose \(\mu\) is stationary so \(\mu_t = \mu\), and let \(f(A) := \mu(\{\eta \mid \eta \equiv 1 \text{ on } A\})\), (8.3.7) implies

\[
f(A) = \mu_t(\{\eta \mid \eta \equiv 1 \text{ on } A\}) = \sum_{B \subset A \mid |B| = |A|} \mathbb{P}_A (A_t = B) f(B) = S(t) f(A),
\]

where the first inequality was due to stationarity. So \(f\) is harmonic, and so by Lemma 8.37 it can only depend on \(A\) through its cardinality, therefore \(\mu\) is exchangeable. This completes the proof. \(\square\)
Chapter 9

Ising, Potts and Random Cluster models

9.1 Gibbs States

The large scale time stationary, or equilibrium, behaviour of spatial random systems on lattices (or more generally graphs) is described in terms of the corresponding 'infinite volume' probability distributions. In classical equilibrium statistical physics this is made precise in terms of Gibbs states (which are in fact probability measures). These are defined starting from certain potentials, or energy functions, in finite volume, which broadly speaking describe how the total energy can be calculated by summing over interactions between components of the system. The Gibbs measures are then given by maximum entropy measures with fixed average energy.

It turns out that the infinite volume Gibbs measures, at certain parameter values, may not be unique - this gives rise to a rigorous way of describing phase transitions in such systems. Also, this non-uniqueness is intimately linked to ergodicity of natural dynamics, for which the Gibbs measures are reversible, which often turn out to also be physically relevant. This very general approach turns out to be extremely powerful, although the details were worked out for models in condensed matter physics the applications are now numerous across physics, biology, economics and more. For more details then we will have time to cover in this Chapter see Liggett IPS (Chapter IV) and Georgii Gibbs Measures and Phase Transitions.

We take \( \Omega = S^\Lambda \) where \( S \) is finite and for now we also assume \( \Lambda \) is finite. To begin we need some definitions: Given \( (\eta(x))_{x \in \Lambda} \) with distribution \( \mu \), we write \( x \perp y \) if \( \eta(x) \) and \( \eta(y) \) are conditionally independent given \( (\eta(z))_{z \in \Lambda \setminus \{x,y\}} \) (clearly \( \perp \) is symmetric). This property gives rise to the dependency graph \( G = (\Lambda, E) \) where \( E = \{\{x,y\} : x \not\sim y\} \). A fully connected subgraph of \( G \) is called a clique, and we denote the set of all cliques by \( K \ni \emptyset \). A clique \( K \) is called maximal if no superset of \( K \) is contained in \( K \). The set of all maximal cliques is denoted by \( M \).

**Theorem 9.1.** Let \( \pi \) be a positive probability measure on \( \Omega \) (i.e. \( \pi(\eta) > 0 \) for all \( \eta \in \Omega \)), then there exists a family of functions \( f_K : \Omega^K \to [0, \infty) \), for \( K \in M \), such that

\[
\pi(\eta) = \prod_{K \in M} f_K(\eta|_K).
\]

**Proof.** The result follows from Brook’s Lemma, see for example Grimmett Probability on Graphs for further details. \( \square \)

We now introduce a type of spatial analogue of the Markov property. This very natural property arrises often in random geometry, in particular in statistical mechanics and statistics.

**Definition 9.2.** A positive probability measure \( \pi \) on \( \Omega \) is called a Markov random field if it satisfies

\[
\pi(\eta|_W) = s|_W \quad \eta|_{\Lambda \setminus W} = s|_{\Lambda \setminus W} = \pi(\eta|_W = s|_W \mid \eta|_{\partial W} = s|_{\partial W}),
\]

for all \( s \in \Omega, W \subset \Lambda, \) where \( \partial W = \{x \in W^c : x \sim y \text{ for some } y \in W\} \) is the external boundary of \( W \).
It turns out that Markov random fields can be written in terms of potential functions, i.e. as Gibbs states. For simplicity in the presentation we restrict to \( \Omega = \{0, 1\}^\Lambda \) with \( \Lambda \) finite, so that we have the usual bijection between configurations and subsets of \( \Lambda \).

**Definition 9.3.** A probability measure \( \pi \) on \( \Omega \) is called a Gibbs random field if there exists a potential function \( \phi : 2^\mathcal{L} \to \mathbb{R} \) satisfying \( \phi(C) = 0 \) if \( c \notin \mathcal{K} \) and

\[
\pi(B) = \exp \left( \sum_{K \subseteq B} \phi(K) \right), \quad \text{for } B \subseteq \Lambda,
\]

i.e. there exists a function a family of functions \( f_K : \{0, 1\}^K \to \mathbb{R} \), such that

\[
\pi(\sigma) = \exp \left( \beta \sum_{K \in \mathcal{K}} f_K(\sigma |_K) \right), \quad \text{for } \sigma \in \Lambda,
\]

for some \( \beta > 0 \).

The motivation behind Gibbs measures comes from statistical mechanics, in particular the total energy of the system is a random variable \( E(\sigma) \) which is written in terms of the sum over the contribution of each of the components \( E(\sigma) = -\sum_{K \in \mathcal{K}} f_K(\sigma |_K) \). Then the Gibbs measures are exactly the maximum (Shannon) entropy measures subject to the constraint that the mean of \( E \) is fixed. The parameter \( \beta \) is the Lagrange multiplier (or conjugate parameter) which fixes the mean of \( E \). In the physics context \( \beta \) is called the inverse temperature.

**Theorem 9.4.** A positive probability measure on \( \Omega \) is a Markov random field if and only if it is a Gibbs random field. The potential function corresponding to a Markov random field \( \pi \) is given by

\[
\phi(K) = \sum_{L \subseteq K} (-1)^{|K \setminus L|} \log \pi(L), \quad K \in \mathcal{K}.
\]

A measure which satisfies Eq. 9.1.1, for all \( s \in \Omega \) and \( W \subseteq \Lambda \), is said to satisfy the global Markov property. It turns out that it is sufficient to check the condition on singleton sets \( W = \{w\} \). A positive probability measure is said to satisfy the local Markov property if Eq. 9.1.1 holds for all \( s \in \Omega \), and all singleton sets \( W = \{w\} \), for \( w \in \Lambda \). It is clear that the global property implies the local one, however the converse also holds.

**Proposition 9.5.** Let \( \pi \) be a positive probability measure on \( \Omega \), then the following three statements are equivalent:

(a) \( \pi \) satisfies the global Markov property.

(b) \( \pi \) satisfies the local Markov property.

(c) For all \( A \subseteq \Lambda \) and any pair \( x, y \in \Lambda \) with \( x \notin A \), \( y \in A \) and \( x \not\sim y \),

\[
\frac{\pi(A \cup \{x\})}{\pi(A)} = \frac{\pi((A \cup \{x\}) \setminus \{y\})}{\pi(A \setminus \{y\})}
\]

(9.1.2)

**Proof.** The proof will be a guided exercise on the final problem sheet.

**Proof of Theorem 9.4.** The Theorem follows from an application of the previous proposition.
9.2 Ising and Potts Models

The Ising model is a surprisingly effective ‘toy’ model of magnetic materials, which is extremely well studied. As way of motivation, imagine applying a strong external magnetic field to a piece of iron and then turning the field off. At sufficiently low temperature, if \( T < T_c \) where \( T_c \) is called the Curie temperature, the sample will remain magnetic. However, if the temperature is increased past the critical point, it will lose its magnetisation. This is an example of a phase transition (in analogy with changes of phase for example as a liquid is heated and becomes a gas), i.e. over a very small region of the parameter \( T \) the physical properties of the material changed abruptly.

In the statistical physics such phase transitions are often described in terms of uniqueness/non-uniqueness transitions of infinite volume Gibbs measures (or states). This is also why phase transitions of the IPS we have seen previously are sometimes described in terms of the set \( I_e \) of extremal stationary measures.

Let \( G = (\Lambda, E) \) be a finite graph and \( \Omega = \{-1, +1\}^\Lambda \), where, for \( \sigma \in \Omega \) we say that the state at site \( x \in \Lambda \) is a down-spin if \( \sigma(x) = -1 \) and an up-spin if \( \sigma(x) = 1 \).

We define the finite volume Gibbs measures (which are just a family of probability measures on \( \Omega \)) in terms of a Hamiltonian (energy function). Let a coupling constant \( J \in \mathbb{R}_+ \) an external field \( h \in \mathbb{R} \), then define the Hamiltonian \( H : \Omega \to \mathbb{R} \) by

\[
H(\sigma) = -J \sum_{e \in E} \sigma(e^-)\sigma(e^+) - h \sum_{x \in \Lambda} \sigma(x).
\]  

(9.2.1)

Then, for a fixed inverse temperature \( \beta \) we define the finite volume Gibbs measures on \( \Lambda \) by

\[
\mu^{\Lambda, \beta}(\sigma) = \frac{1}{Z(\beta)} e^{-\beta H(\sigma)}, \quad \text{for all } \sigma \in \Omega,
\]

(9.2.2)

where \( Z(\beta) \) is the normalization to ensure we have a probability measure, which in the physics jargon is called the partition function, \( Z(\beta) = \sum_{\sigma \in \Omega} e^{-\beta H(\sigma)} \). Since \( J > 0 \), these measures favour configurations where the spins are aligned (ferromagnetic). How strongly these are energetically favorable is controlled by the inverse temperature \( \beta \), and there is clearly a competition between energy and entropy in the sense that there are few configurations with extremely low energy.

One obvious extension of the Ising model is to make the local state space larger (still finite), this is called the Potts model. Let \( G \) be as above, and \( \Omega_q = \{1, 2, \ldots, q\}^\Lambda \), that is the local state can take any of \( q \) colours (the model will reduce to the Ising model - with slightly different parameters which can be calculate - when \( q = 2 \)). For \( e \in E \) let \( \delta_e(\sigma) = 1(\sigma(e^-) = \sigma(e^+)) \), then the Hamiltonian of the Potts model is given by

\[
H_q(\sigma) = -\sum_{e \in E} \delta_e(\sigma),
\]

(9.2.3)

note that in this case there is no external field (we only consider the case \( h = 0 \) here). the associated measure is then given by

\[
\pi_q^{\Lambda, \beta}(\sigma) = \frac{1}{Z_q(\beta)} e^{-\beta H_q(\sigma)}, \quad \text{for all } \sigma \in \Omega.
\]

(9.2.4)

These are two of the most well studied models in statistical mechanics. Notice that the models are just described in terms of measures on \( \Omega \), at this point there are no dynamics. The study of such models is often called equilibrium statistical mechanics, although there is now some ambiguity in the literature over non-equilibrium being reserved for non-reversible dynamics. We will come to dynamics later.
9.3 Random-cluster model

The random-cluster model developed by Fortuin and Kasteleyn, sometimes called the FK-percolation, brings together percolation, the Ising model (more generally Potts model), electrical networks and USTs.

As we did when we studied percolation let \( G = (\Lambda, E) \) be a finite connected graph, and \( \Omega = \{0,1\}^E \) (with the usual measurable structure). Let \( k(\omega) \) be the number of connected components (including isolated vertices) in \( \omega \in \Omega \). For \( p \in [0,1] \) and \( q \in (0,\infty) \) the random cluster model is a probability measure on \( \Omega \) defined by

\[
\phi_{p,q}(\omega) = \frac{1}{Z_{p,q}} \left( \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right) q^k(\omega), \quad \omega \in \Omega. \tag{9.3.1}
\]

Note that \( q = 1 \) is exactly Bernoulli bond percolation on \( G \). In the limit \( p, q \to 0 \) while \( p/q \to 0 \) we recover the UST measure. It also turns out that connections in the FK-percolation correspond in some way to correlation structure in the Ising and Potts models.

9.3.1 Coupling with the Potts model

In this section \( \Omega_q = \{1,\ldots,q\}^\Lambda \) and \( \Omega = \{0,1\}^E \). Given and FK-realisation, \( \omega \in \Omega \), independently colour each component a colour chosen uniformly from \( \{1,\ldots,q\} \). It turns out that this gives a realisation of the Potts model with the correct distribution. We now make this more precise.

**Definition 9.6** (FK and Potts coupling). Let \( F \subset \Omega_q \times \Omega \) be the set given by

\[
F = \{(\sigma, \omega) : \sigma(e^-) = \sigma(e^+) \; \forall \; e \in E \text{ such that } \omega(e) = 1\}, \tag{9.3.2}
\]

i.e. all pairs of compatible Potts and FK configurations (the configurations are called compatible if every connected component is monochromatic). Then we define the coupling by

\[
\mu(\sigma, \omega) \propto \phi_{p,1}(\omega) \mathbb{1}_F(\sigma, \omega), \quad (\sigma, \omega) \in \Omega_q \times \Omega. \tag{9.3.3}
\]

We now observe that the two marginals of \( \mu \) are as desired.

**Marginal on \( \Omega_q \):** Fix \( \sigma \in \Omega_q \) and sum over \( \omega \in \Omega \), we find:

\[
\mu(\sigma) \propto \sum_{\omega \in \Omega} \phi_{p,1}(\omega) \mathbb{1}(\omega(e) = 0 \text{ if } \sigma(e^-) \neq \sigma(e^+))
\]

\[
\propto \sum_{\omega \in \Omega} \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \mathbb{1}(\omega(e) = 0 \text{ if } \sigma(e^-) \neq \sigma(e^+))
\]

\[
\propto \prod_{e \in E} (1-p)^{1-\delta_e(\sigma)} \propto e^{\beta \sum_{e \in E} \delta_e(\sigma)} \propto \pi^{L,\beta}_q(\sigma), \quad \text{where } p = 1 - e^{-\beta},
\]

where in the first line in the third line we used that for each \( e \) sating \( \omega(e^-) = \omega(e^+) \) there is no constraint on the value of \( \omega(e) \). In particular if we fix the value of \( \omega \) on all edges except one monochromatic edge, then we see that by summing over \( \omega(e) \in \{0,1\} \) the contribution to the product from this edge becomes \( p+(1-p) = 1 \). After we have done this on all monochromatic edges, we know that each non-monochromatic edge \( w(e) = 0 \) which gives the expression.

**Marginal on \( \Omega \):** For a given \( \omega \in \Omega \) we sum over all the \( \sigma \in \Omega_q \) which are constant on open
9.3. RANDOM-CLUSTER MODEL

There are \( q^k(\omega) \) such configurations (simply by picking colour 1, 2, \ldots, \( q \) for each cluster), and \( \mu(\sigma, \omega) = \mu(\sigma', \omega) \) for any two such colourings \( \sigma \) and \( \sigma' \). It follows that

\[
\mu(\omega) \propto \left( \prod_{e} p^{\omega(e)} (1 - p)^{1 - \omega(e)} \right) q^k(\omega) \propto \phi_{p,q}(\omega).
\]

Conditional measures:

(i) \( \mu(\sigma \mid \omega) \) is obtained by putting uniform random colours on each component in \( \omega \).

(ii) \( \mu(\omega \mid \sigma) \) is obtained by flipping a \( \text{Ber}(p) \) coin for each monochrome edge in \( \sigma \) to decide if they are connected, all non-monochrome edges are necessarily not connected.

9.3.2 Potts model two point correlations

If we think of the local configurations as vectors \( \sigma(x) \in \{1, 2, \ldots, q\} \) as a vector \((0, \ldots, 0, 1, 0, \ldots, 0)\) with a single 1 in the \( \sigma(x) \)th position. This is useful for concepts such as average colour, if the order of the colours does not matter. Since we will not use this representation much beyond motivating the following definition of the two-point correlation we will not discuss this in any more detail here, you may simply take the right handside of the following display as the definition,

\[
\tau_{\beta,q}(x,y) = \text{Cov}(\sigma(x), \sigma(y)) = E[\sigma(x) \cdot \sigma(y)] - E[\sigma(x)] \cdot E[\sigma(y)] = \pi_{q,\beta}^{\left[\sigma(x) = \sigma(y)\right]} - \frac{1}{q},
\]

(9.3.4)

for \( x, y \in \Lambda \) (where all expectations are with respect to the Potts measure \( \pi_{q,\beta} \)).

**Theorem 9.7.** For \( q \in \{2, 3, \ldots\} \), \( \beta \geq 0 \) and \( p = 1 - e^{-\beta} \),

\[
\tau_{\beta,q}(x,y) = \left(1 - \frac{1}{q}\right) \phi_{p,q}(x \leftrightarrow y).
\]

(9.3.5)

**Proof.** We firstly write \( \tau_{\beta,q}(x,y) = E[\mathbb{1}\{\sigma(x) = \sigma(y)\} - \frac{1}{q}] \) and then use \( \mu(\sigma \mid \omega) \),

\[
\tau_{\beta,q}(x,y) = \sum_{\sigma,\omega} \left(\mathbb{1}\{\sigma(x) = \sigma(y)\} - \frac{1}{q}\right) \mu(\sigma, \omega)
= \sum_{\omega} \phi_{p,q}(\omega) \sum_{\sigma} \mu(\sigma \mid \omega) \left(\mathbb{1}\{\sigma(x) = \sigma(y)\} - \frac{1}{q}\right)
= \sum_{\omega} \phi_{p,q}(\omega) \left(1 - \frac{1}{q}\right) \mathbb{1}_{\{x \leftrightarrow y\}}(\omega) + 0 \mathbb{1}_{\{x \not\leftrightarrow y\}}(\omega)
= \left(1 - \frac{1}{q}\right) \phi_{p,q}(x \leftrightarrow y).
\]

\( \square \)

9.3.3 Basic properties of the FK model

**Theorem 9.8.** The measure \( \phi_{p,q} \) has positive correlations, i.e. it satisfies the FKG lattice condition (or Holley Criterion), see Theorem 6.8.

**Proof.** The result is clear for \( p = 0, 1 \). Fix \( p \in (0, 1) \), the Holley inequality (condition of Theorem 6.8) is equivalent to

\[
k(\omega \vee \omega') + k(\omega \wedge \omega') \geq k(\omega) + k(\omega') \quad \omega, \omega' \in \Omega.
\]

(The equivalence and the proof of the inequality above are left as an exercise.)
Theorem 9.9 (Comparison - monotonicity in p and q). The following comparisons hold:

- \( \phi_{p',q'} \leq \phi_{p,q} \) if \( p' \leq p, \quad q' \geq q, \quad q' \geq 1. \)
- \( \phi_{p',q'} \geq \phi_{p,q} \) if \( \frac{p'}{q'(1-p')} \geq \frac{p}{q(1-p)} \) and \( q \geq q', q \geq 1. \)

Proof. Apply the FKG inequality. \qed

The BK inequality does not hold for \( \phi_{p,q} \) due to long range dependencies which are present. These are in a sense through the “boundaries” of simple regions e.f. the spatial Markov property.

9.3.4 \( \infty \)-volume and the phase transition

See Section 8.3 in Grimmett Probability on Graphs. We construct the \( \infty \)-volume measure by taking weak limits. There are alternative constructions using the Dobrushin-Lanford-Ruelle (DLR) formalism. The two approaches are not quite equivalent in ways we will not get into here (Add ref). For clarity we consider subsets of the usual integer lattice \( \mathbb{Z}^d \) with associated edge set \( E^d \) and finite subsets \( \Lambda \subset \mathbb{Z}^d \) with associated edge set \( E_\Lambda \) given by the set of all nearest neighbors pairs (in \( \ell_1 \)).

Let \( \Omega^\tau_\Lambda = \{ \omega \in \{ 0, 1 \}^{E^d} : \omega(e) = \tau \ \forall e \notin E^c_\Lambda \} \) where \( \tau \in \{ 0, 1 \} \) is considered to be the boundary condition imposed on the finite region \( \Lambda \). If \( \tau = 1 \) we call the boundary wired (every edge outside the finite region is open or wired together), and if \( \tau = 0 \) we call the boundary condition free. Notice that the set \( \Omega^\tau_\Lambda \) is finite since the configuration is fixed on all but finitely many edges. It turns out that these boundary conditions can have an interesting effect on the large volume limit.

On \( \Omega^\tau_\Lambda \) we define the random cluster measure with boundary condition \( \tau \), and \( p \in [0, 1], \quad q \in (0, \infty) \) as

\[
\phi^{\Lambda,\tau}_{p,q}(\omega) = \frac{1}{Z^\tau_{p,q}} \left( \prod_{e \in E_\Lambda} p^{\omega(e)}(1-q)^{1-\omega(e)} \right) q^{k(\omega|\Lambda)}, \quad \omega \in \Omega^\tau_\Lambda.
\]

The next theorem states that the infinite volume measure exists, see Section 4.3.

Theorem 9.10. Let \( p \geq 1 \). The weak limits

\[
\phi^\tau_{p,q} = \lim_{\Lambda \nearrow \mathbb{Z}^d} \phi^{\Lambda,\tau}_{p,q}, \quad \tau \in \{ 0, 1 \},
\]

exist and are translation invariant and (shift) ergodic.

The infinite volume limit is often called the ‘thermodynamic limit,’ in analogy with physical models where relevant limits are often to take volume diverging as other parameter such as pressure or temperature are kept constant.

It turns out that the measure \( \phi^{0/1}_{p,q} \) are extremal in the following sense

\[
\phi^{0/1}_{p,q} \leq \phi^\tau_{p,q} \leq \phi^1_{p,q} \quad \text{for more general } \tau,
\]

where \( \phi^\tau_{p,q} \) is the obvious generalization of \( \phi^{0/1}_{p,q} \) by taking other ‘wirings’ outside of the finite region \( \Lambda \). In particular, if \( \phi^0_{p,q} = \phi^1_{p,q} \) then the infinite volume measure is unique. By weak and very general arguments (using convexity of the free energy function) it can be shown that \( \phi^1_{p,q} = \phi^0_{p,q} \) for all but countably many values of \( p \) for fixed \( q \). This means that there is no phase transition in \( p \). However, there is a percolation transition, which translates into a uniqueness/non-uniqueness phase transition for the associated Potts (Ising) model.

For \( \tau \in \{ 0, 1 \} \) let \( \theta^\tau(p,q) = \phi^\tau_{p,q}(0 \leftrightarrow \infty) \) and \( p^\tau_c(p,q) = \sup\{ p : \tau^\tau(p,q) = 0 \} \). By the discussion in the previous paragraph (no phase transition) \( p^0_c = p^1_c = p_c(q) \) where we dropped the \( \tau \) dependence in the notation on the right hand side since it is not required.
Theorem 9.11. Fix $q \geq 1$, $d \geq 1$, then

- For $p < p_c$ the $\infty$-volume FK measure has no $\infty$-cluster (a.s.).
- For $p > p_c$ the $\infty$-volume FK measure has an $\infty$-cluster (a.s.).

In terms of the associated Potts model (when $q \in \mathbb{N}$) implies the following. Let $\beta_c$ be given by $1 - e^{-\beta_c} = p_c$, then (by Theorem 9.7) we have

$$\pi_{\beta,q}^{\Lambda,1}(\sigma_0 = 1) - \frac{1}{q} = (1 - q^{-1})\phi_{p,q}^{\Lambda,1}(0 \leftrightarrow \partial \Lambda),$$

so

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \pi_{\beta,q}^{\Lambda,1}(\sigma_0 = 1) = \begin{cases} 0 & \text{if } \beta < \beta_c, \\ m^* > 0 & \text{if } \beta > \beta_c, \end{cases}$$

where $m^*$ is $(1 - q^{-1})$ times the probability that $0$ belongs to the $\infty$-cluster in the associated FK model. The value $m^*$ is often called magnetization in analogy with the $q = 2$ cases which can be mapped to the Ising model. Following the calculation above $m^* = \lim_{\Lambda \nearrow \mathbb{Z}^d} \left( \pi_{\beta,q}^{\Lambda,1}(\sigma_0 = 1) - \frac{1}{q} \right)$ which measures the effect of the boundary condition $1$ on the spin at the origin. If $\beta < \beta_c$ (small inverse temperature i.e. high temperature) the origin does not ‘feel’ the boundary condition in the limit. However, if $\beta > \beta_c$ (low temperature) then the origin prefers to be aligned to the boundary even as the distance to the boundary diverges. The limiting infinite volume Potts measure must then depends on which boundary conditions were chosen, i.e. the infinite volume measure is not unique.
Chapter 10

Stochastic Ising model and Mixing times

Mixing times and relaxation times of Markov processes were explored further on the vacation assignment sheet (Sheet 4).

It turns out that phase transitions in the sense of the previous section are often associated with “dynamic” transitions for very natural (local) associated dynamics. That is dynamics that have the measures of interest as invariant measures. The dynamics are then used to model how the system reaches its equilibrium which is characterized by the stationary measure. When there is non-uniqueness of the infinite-volume measure as described previously then the system ‘feels’ the effect of boundaries even when arbitrarily large. These long range effects typically make it difficult (slow) to mix (reach equilibrium).

Here we will consider local ‘spin-flip’ type dynamics (meaning the configuration can change at most at one site at a time). The dynamics can be defined immediately in infinite volume (see for example the Liggett book ref), but we will be interested in how characteristic times associated with the dynamics grow as the system size increases.

Recall the Ising measure on finite volume \( \Lambda \subset \mathbb{Z}^d \) with edge set \( E_\Lambda \) taken to be all the edges in the integer lattice that have both ends in \( \Lambda \) (note we could include boundary conditions by including in \( E_\Lambda \) all edges that have at least one end in \( \Lambda \) and following the formalism of the previous section, what we do here is sometimes called free boundary conditions for the Ising model),

\[
\pi_\beta^\Lambda (\sigma) = \frac{1}{Z_\beta^\Lambda} e^{-\beta H_\Lambda(\sigma)}, \quad \text{for } \sigma \in \Omega_\Lambda
\]

where \( H_\Lambda(\sigma) = \sum_{e \in E_\Lambda} \sigma(e^-)\sigma(e^+) \) (note we take zero external field i.e. \( h = 0 \)). To simplify notation we will drop the \( \Lambda \) and \( \beta \) dependence, i.e.

\[
\pi(\sigma) = \frac{1}{Z} e^{\frac{\beta}{2} \sum_{x \sim y} \sigma(x)\sigma(y)}.
\]

Note the re-parametrisation of the sum above means that we include each edge twice, hence the extra factor of \( 1/2 \) in the exponent.

For simplicity of the presentation we will consider discrete time dynamics, though similar analysis hold also for continuous time dynamics (where each site updates at the arrival times of i.i.d. Poisson process). The discrete time Markov dynamics are define as follows: At each step pick a vertex \( x \in \Lambda \) uniformly at random and then sample a new local state from \( \pi \) conditioned on the configuration outside of \( \{x\} \), i.e. from \( \pi(\cdot \mid \sigma \rceil_{\Lambda \setminus \{x\}}) \) (independent of the current state, so this may be a ‘lazy’ jump). This description gives rise to the following transition matrix elements (check!)

\[
P(\sigma, \eta) = \frac{1}{|\Lambda|} \sum_{z \in \Lambda} \mathbb{I}(\sigma(y) = \eta(y) \text{ for } y \neq x) \frac{e^{\frac{\beta}{2} (\eta(x)S(\sigma,x)) \frac{e^{\frac{\beta}{2} S(\sigma,x)} + e^{-\frac{\beta}{2} S(\sigma,x)}}{e^{\frac{\beta}{2} S(\sigma,x)} + e^{-\frac{\beta}{2} S(\sigma,x)}}}{1 + \tanh \left( \frac{\beta}{2} S(\sigma,x) \right)}, \quad (10.0.1)
\]

where \( S(\sigma,x) = \sum_{z \sim x} \sigma(z) \). Using standard identities we can write the term under the sum as

\[
\frac{e^{\frac{\beta}{2} \eta(x)S(\sigma,x)}}{e^{\frac{\beta}{2} S(\sigma,x)} + e^{-\frac{\beta}{2} S(\sigma,x)}} = \frac{1}{2} \left( 1 + \tanh \left( \frac{\beta}{2} S(\sigma,x) \right) \right).
\]
10.1 Phase transition

Clearly $P$ is reversible with respect to $\pi$, so for fixed boundary conditions (or free) we have a finite state irreducible and reversible process. It turns out that if we had defined things appropriately on the infinite lattice then the set of Gibbs states would have coincided with the set of reversible infinite volume invariant measures (which are not necessarily unique if there is a phase transition).

For simplicity of the analysis from this point on we will focus on the complete graph rather than the integer lattice. In physics language this is called the mean field approximation. That is from now on $G = (\Lambda, E)$ where $E = \Lambda^2$. In the mean field case each term in the sum in the Hamiltonian is order of $n$, that is each site has $(n-1)$ neighbor’s. It is therefore helpful to parameterize by setting $\alpha = \frac{\beta}{2n}$.

10.1 Phase transition

Using results developed on random graphs, such as $G(n, p)$ i.e. Bernoulli percolation on the complete graph (this will be covered in more detail in the Probabilistic Combinatorics course), Bollobás, Grimmett and Janson (1995) showed that the FK model on the complete graph undergoes a percolation transition at

$$\lambda_c(q) = \begin{cases} q, & \text{if } 0 < q \leq 2, \\ 2 \left(\frac{q-1}{q-2}\right) \log(q-1), & \text{if } q > 2, \end{cases}$$

where $p = \lambda/n$. (10.1.1)

There is therefore a corresponding phase transition the the associated Potts model at $\beta_c = -\log(1 - p_c)$.

It turns out that when $q = 2$ (i.e. the Ising case) the mean field Potts (or Ising) model is exactly solvable, meaning that it is possible to calculate many things explicitly. These exact methods were used to fully characterize the phase transition by Wu (1982), and Kesten and Schonmann (1990). It can be shown that the critical value of $\beta_c = 1/n$ i.e. $\alpha_c = \frac{\beta_c}{n} = 1$. Note this is consistent with the FK result on the complete graph.

If $\alpha < 1$ (i.e. $\beta < 1/n$) then there is a unique $\infty$-volume Gibbs measure. Whereas if $\alpha > 1$ then there are exactly two extremal measures, one corresponds to taking $+$ boundary conditions, i.e. $\pi^\Lambda_\beta^+$, and one corresponding to $-$ boundary conditions, i.e. the weak limit of $\pi^\Lambda_\beta^-$. Clearly the average magnetization of any single fixed vertex under $\pi^\Lambda_\beta^+$ will be positive and negative under $\pi^\Lambda_\beta^-$. We will denote these average magnetizations by

$$\pi^+_\beta(\sigma(0)) = \lim_{\Lambda \nearrow \mathbb{Z}^d} \pi^\Lambda_\beta^+(\sigma(0)) = m^+ > 0 \quad \text{and}$$

$$\pi^-_\beta(\sigma(0)) = \lim_{\Lambda \nearrow \mathbb{Z}^d} \pi^\Lambda_\beta^-(\sigma(0)) = m^- > 0.$$

On a finite lattice with free boundary conditions there there is an obvious symmetry with respect to flipping all the spins. It turns out as we will see in the next section, if there is no boundary condition or external field to break the symmetry then at low temperature, $\beta > \beta_c$, the stationary measure concentrates on configurations that have magnetization close to $m^+$ or $m^-$. 

10.2 Large deviations of the empirical magnetization

We now focus exclusively on the Ising model on the complete graph of $n$ vertices with free boundary conditions, i.e.

$$\pi(\sigma) \propto e^{\frac{1}{T} \sum_{x \in \Lambda} \sum_{y \neq x} \sigma(x)\sigma(y)}.$$
Recall, from (10.1.1) it follows that if $\beta = \alpha/n$ then the critical point is specified by $\alpha_c = 1$. The weak limit (thermodynamic limit) as $|\Lambda| \to \infty$ of $\pi$ when $\alpha > 1$ is a mixture of two measure $\pi^+$ and $\pi^-$ which concentrate, respectively, on positive and negative (empirical) mean magnetisation

$$m_n(\sigma) = \frac{1}{n} \sum_{x \in \Lambda} \sigma(x).$$

Figure 10.1: The energy function (large deviation rate function) associated with the empirical magnetisation $m_n$, described by (10.2.1).

We can define an effective energy landscape under the stationary measure as follows (as called a rate function in the language of large deviation theory),

$$\psi^{(n)}(m) = -\frac{1}{n} \log \pi_\alpha(m_n(\sigma) = m), \quad m \in \{-1, -1 + \frac{2}{n}, \ldots, 1 - \frac{2}{n}, 1\}. \quad (10.2.1)$$

Notice that the last term on the right hand side is the normalisation and is independent of $m$, it turns out that it converges as $n \to \infty$ and we call the limit $f(\alpha)$. This is often called the free energy in the physics literature. By applying Stirling’s approximation we find

$$\psi^{(n)}(m) \xrightarrow{n \to \infty} \psi(\alpha)(m) := m \log m + (1 - m) \log(1 - m) - \frac{\alpha}{2} (1 - 2m)^2 - f(\alpha).$$

The first two terms are related to the Binary entropy function. Taking derivatives shows that

$$\psi'(\alpha)(0) = 0, \quad \text{and} \quad \psi''(\alpha)(0) = 4(1 - \alpha),$$

hence $m = 0$ is a critical point of $\psi_\alpha$, and is a local maximum or minimum depending on the value of $\alpha$ (see Fig. 10.1). When $m = 0$ is a local maximum it can be interpreted as an ‘energy barrier’ in the sense that we will discuss in more detail now. Intuitively this is because under the dynamics the value of $m_n$ can only make ‘local moves’ (in fact in this case it makes a RW with step size $2/n$), and to go from one region with large probability, $m_n < 0$, to another, $m_n > 0$, the process must cross a state with very low stationary probability. This is often referred to as a bottleneck.

Let

$$S = \{ \sigma \in \Omega_\Lambda : m_n(\sigma) < 0 \},$$

then by symmetry $\pi_\alpha(S) \leq 1/2$ and to go from $S$ to $S^c$ the process must have to go through the ‘unlikely’ set $\partial S = \{ \sigma : m_n = 1 - \frac{2}{n} \lfloor \frac{n}{2} \rfloor \}$. 

10.3 Mixing time

The ideas of mixing and relaxation time were explored more on the final problem sheet, see also Levin, Peres, Wilmer Markov chains and mixing times (2008) - available online.

**Definition 10.1** (Mixing time). The total variation mixing time is defined by the first time that the distribution of the chain is close to the stationary distribution, uniformly in the initial conditions. Fix $\varepsilon > 0$,

$$T_{mix}(\varepsilon) = \inf \{ t > 0 : \max_{\eta \in \Omega} \| P^t(\eta, \cdot) - \pi \|_{TV} < \varepsilon \}, \quad (10.3.1)$$

where

$$\| \nu - \mu \|_{TV} = \frac{1}{2} \sum_{\eta \in \Omega} |\nu(\eta) - \mu(\eta)| = \sup_{f : \| f \|_\infty \leq 1} |\mu(f) - \nu(f)| = \inf \{ \mathbb{P}(\sigma \neq \eta) : (\sigma, \eta) \sim \mathbb{P} \text{ is a coupling of } \mu, \nu \}.$$

To simplify notation we let $T_{mix} := T_{mix}(1/4)$. 

10.3.1 Bottle neck

We use the existence of the bottleneck discussed above to show that the system is slow mixing (i.e. exponentially large mixing time in the system size), when $\alpha > 1$ (low temperature).

**Theorem 10.2** (Slow mixing). If $\alpha > 1$ then there exists a positive function $r(\alpha) > 0$ and constant $C > 0$ such that

$$T_{mix} \geq Ce^{r(\alpha)n}.$$ 

**Proof.** Recall the definition of the conductance $c(\eta, \sigma) = \pi(\eta)P(\eta, \sigma)$, let $Q(A, B) = \sum_{\eta \in A, \sigma \in B} c(\eta, \sigma)$ denote the probability to go from $A$ to $B$ in one step starting from the stationary measure $\pi$. We measure the perimeter of a set $S \subset \Omega$ by

$$\Phi(S) = 2Q(S, S^c) / \pi(S).$$

Then define the bottleneck constant (also known as the Cheeger or Isoperimetric constant) by

$$\Phi_* = \min_{S : \pi(S) \leq \frac{1}{2}} \Phi(S).$$

We proved the following lemma on the final exercise sheet (Sheet 4)

**Lemma 10.3.**

$$T_{mix} \geq \frac{1}{\Phi_*}.$$ 

to complete the proof we simply show that $S = \{ \sigma \in \Omega : m_n(\sigma) < 0 \}$ gives rise to an exponentially narrow bottleneck. Roughly the argument proceeds as follows (and can be made precise with only a little effort). To cross from $S$ to $S^c$ in a single jump of the process the starting configuration must have a magnetization close to 0 plus or minus something of order $1/n$. It follows that

$$Q(S, S^c) \leq \left[ \frac{n/2}{n} \right] \pi(m_n \approx 0) \lesssim e^{-n\Psi_\alpha(0)},$$

and $\pi(S) \geq \pi(m_n = m^-)$, so

$$\Psi(S) \leq \frac{e^{-n\Psi_\alpha(0)}}{e^{-n\Psi(m^-)}} \lesssim e^{-n(\Psi_\alpha(0) - \Psi_\alpha(m^-))}.$$ 

To conclude we observe that $(\Psi_\alpha(0) - \Psi_\alpha(m^-)) > 0.$
10.3.2 Path coupling

To derive upper bounds on the mixing time of the stochastic Ising model we will use a method called path coupling.

Since the magnetisation of the stochastic Ising model on the complete graph is Markov (can be checked) and completely characterizes the full state (by symmetry), we can reduce the problem to the mixing time of a simple random walk. However we will take a more general approach that can be applied with only slightly more technicalities to more general models of this type.

**Theorem 10.4.** The Glauber dynamics for the Ising model on the complete graph is fast mixing if \( \alpha < 1 \) (at high temperature),

\[
T_{\text{min}} \leq \frac{n(\log n + \log \alpha)}{1 - \alpha}.
\]

The proof relies on the following more general path coupling result. Let

\[
d(t) = \max_{\eta \in \Omega} \| P^t(\eta, \cdot) - \pi \|_{\text{TV}}.
\]

Since \( \pi \) is stationary it follows that \( d(t) \leq \max_{\eta, \sigma \in \Omega} \| P^t(\eta, \cdot) - P^t(\sigma, \cdot) \|_{\text{TV}} \). In particular if \( P_{\eta, \sigma} \) is a Markov coupling of \( (\eta_t)_t \geq 0 \) and \( (\sigma_t)_t \geq 0 \), such that they both have transition matrix \( P(\cdot, \cdot) \) and initial conditions \( \eta_0 = \eta, \sigma_0 = \sigma \) (a.s.) respectively, then \( d(t) \leq P_{\eta, \sigma}(\eta_t \neq \sigma_t) \) (this follows from the coupling characterization the total variation norm). The aim, as we have seen previously, is to construct a Markov coupling of the process started from different points such that the two trajectories move towards each other as much as possible.

Let \( \ell : E \to \mathbb{R}_+ \) be a positive function define on the edge sets, which we consider to be edge lengths, such that \( \ell(\eta, \sigma) \geq 1 \) for each \( (\eta, \sigma) \in E \). Then we define an associated path metric by

\[
\rho(\sigma, \eta) = \min\{|\gamma| : \gamma \text{ a path from } \sigma \text{ to } \eta\},
\]

where we define the length of path by \( |\gamma| = \sum_{e \in \gamma} \ell(e) \). Observe that \( \rho(\sigma, \eta) \geq 1(\eta \neq \sigma) \), so

\[
P_{\eta, \sigma}(\eta_t \neq \sigma_t) \leq E_{\eta, \sigma}(\rho(\eta_t, \sigma_t)),
\]

for any coupling. We define the transportation metric on probability measures, with respect to a path metric \( \rho \), by

\[
d_\rho(\mu, \nu) = \inf\{E[\rho(X, Y)] : (X, Y) \text{ is a coupling of } \mu \text{ and } \nu\}.
\]

**Theorem 10.5** (Budley and Dyer). Let \( \rho \) be a path metric and for each \( (\eta, \sigma) \in \Omega^2 \) with \( P(\eta, \sigma) > 0 \) there exists a coupling of \( P(\eta, \cdot) \) and \( P(\sigma, \cdot) \) such that

\[
E_{\eta, \sigma}(\rho(\eta_1, \sigma_1)) \leq \rho(\eta, \sigma)e^{-c} = \ell(\eta, \sigma)e^{-c}.
\]

Then for each \( \mu, \nu \) probability measure on \( \Omega \),

\[
d_\rho(\mu P, \nu P) \leq e^{-c}d_\rho(\mu, \nu).
\]

**Corollary 10.6.**

\[
d(t) \leq e^{-ct} \text{diam}(\Omega),
\]

and hence

\[
T_{\text{mix}} \leq \left\lceil \frac{\log(1/\varepsilon) + \log \text{diam}(\Omega)}{c} \right\rceil.
\]
Sketch proof of Theorem 10.5. Choose \( \eta, \sigma \in \Omega \) then pick the shortest path and apply the triangle inequality to get

\[
d_{\rho}(P(\eta, \cdot), P(\sigma, \cdot)) \leq e^{-c \rho(\eta, \sigma)}.
\]

Finally pick the optimal coupling.

We now apply these results to the stochastic Ising model to complete the proof of Theorem 10.4.

Let \( \rho(\eta, \sigma) = \frac{1}{2} \sum_{x \in \Lambda} |\eta(x) - \sigma(x)| \) (which counts the number of differences between the two configurations). Fix \( \eta, \sigma \in \Omega \) with \( \rho(\eta, \sigma) = 1 \), and without loss of generality let \( x \in \Lambda \) be the vertex at which they differ and \( \eta(x) = 1, \sigma(x) = -1 \). We now describe a one-step coupling that on average reduces the number of differences between the two configurations. Pick the same vertex \( z \) to update in both configurations. If we pick \( z = x \) then the neighbours of \( z \) have the same spin in both configurations and so we can update to the same thing. Otherwise, if \( z \neq x \) we use ‘the same randomness’ to update both configurations. Let \( U \sim \text{Unif}[0, 1] \) independent of everything else and let

\[
\eta_1(z) = \begin{cases} 
+1 & \text{if } U \leq P(\eta, \eta^z,+), \\
-1 & \text{if } U > P(\eta, \eta^z,+),
\end{cases}
\]

\[
\sigma_1(z) = \begin{cases} 
+1 & \text{if } U \leq P(\sigma, \sigma^z,+), \\
-1 & \text{if } U > P(\sigma, \sigma^z,+).
\end{cases}
\]

Thus on the event \( z = x \) then \( \rho(\eta_1, \sigma_1) = 0 \) (which occurs with probability \( 1/n \)). If \( z \neq x \) and \( P(\sigma, \sigma^z+) < U < P(\eta, \eta^z+) \) then \( \rho(\eta_1, \sigma_1) = 2 \) otherwise the two configurations move together and \( \rho(\eta_1, \sigma_1) = 1 \). It follows that

\[
\mathbb{E}_{\eta, \sigma}[\rho(\eta_1, \sigma_1)] \leq 1 + \frac{n-1}{n} \left( P(\eta, \eta^z+) - P(\sigma, \sigma^z+) \right) - \frac{1}{n} \leq \tanh(\beta).
\]

So

\[
\mathbb{E}_{\eta, \sigma}[\rho(\eta_1, \sigma_1)] \leq e^{-c(\beta)/n}, \text{ if } \beta < \frac{1}{n}.
\]