A.4 Vacation Sheet

1. (Russo-Seymour-Welsh)


On the event $V(S)$ there clearly exists (a.s.) a right most path connecting the top and bottom of $S$ (we do not worry about proving this statement precisely), call this path $\Gamma$. We first argue that

$$\mathbb{P}_{1/2}[X(S') | V(S) \cap \{\Gamma = \gamma\}] \geq \mathbb{P}_{1/2}[H(S')]/2.$$ 

Let $P_\gamma$ be the (not necessarily open) path formed by $\gamma \cup \rho(\gamma)$, this path crosses from the top to the bottom of $S'$. With probability $\mathbb{P}_{1/2}[H(S')]$ there is an open path that crosses from the left to the right of $S'$, which therefore must cross $P_\gamma$, and in particular has a unique ‘first point’ that it hits $P_\gamma$ (following the path ‘left to right’). By symmetry, the probability there is such a path, and this path first hits $P_\gamma$ on $\gamma$ is $\mathbb{P}_{1/2}[H(S')]/2$. Hence the probability of the event $Y(S')$ that there is a path $P_2$ in $S'$ to the left of $P_\gamma$ joining some point of $\gamma$ to the left-hand side of $S'$ is at least $\mathbb{P}_{1/2}[H(S')]/2$. Also it is clear that the event $Y(S')$ is independent of $V(S) \cap \{\Gamma = \gamma\}$ since they depend on disjoint edges ($V(S) \cap \{\Gamma = \gamma\}$ only depends on the edges on and to the right of $\gamma$). Furthermore $X(S') \cap V(S) \cap \{\Gamma = \gamma\} = Y(S') \cap V(S) \cap \{\Gamma = \gamma\}$ and hence

$$\mathbb{P}_{1/2}[X(S') | V(S) \cap \{\Gamma = \gamma\}] = \mathbb{P}_{1/2}[Y(S') | V(S) \cap \{\Gamma = \gamma\}] = \mathbb{P}_{1/2}[Y(S')] \geq \mathbb{P}_{1/2}[H(S')]/2.$$
Now, by the law of total expectation, we have
\[
\mathbb{P}_p[X(S')] \geq \sum_{\gamma} \mathbb{P}_{1/2}[X(S') \mid V(S) \cap \{\Gamma = \gamma\}] \mathbb{P}_{1/2}[V(S) \cap \{\Gamma = \gamma\}]
\]
\[
\geq \frac{\mathbb{P}_{1/2}[H(S')]}{2} \sum_{\gamma} \mathbb{P}_{1/2}[V(S) \cap \{\Gamma = \gamma\}] = \mathbb{P}_{1/2}[V(S)] \mathbb{P}_{1/2}[H(S')] / 2,
\]
as required.

Let \(X(S'')\) be the event that there exists an open path between the top and bottom of the rectangle \(S\) (inside of \(S\)) and a path inside of \(S'' = [0,2n] \times [-n,n]\) connecting a point on this path to the right hand side of \(S''\) (this should really be added to the picture). By the obvious symmetry we have \(\mathbb{P}_{1/2}[X(S'')] = \mathbb{P}_{1/2}[X(S')]\). To complete the proof we observe that the event \(X(S') \cap X(S'') \cap V(S)\) is contained in \(H(R)\) (the implication is clear from a picture), and then since each of the events \(X(S'), X(S'')\) and \(V(S)\) are increasing the result follows by applying the Harris inequality.

2. (Markov and Gibbs random fields)

(a) Let \(\pi\) be a probability measure on \(\Omega\). First we note that the local Markov property follows trivially from the global Markov property. We now prove that the local Markov property is equivalent to the following property: For all \(A \subseteq \Lambda\) and any pair \(u,v \in \Lambda\) with \(u \notin A\), \(v \in A\) and \(u \sim v\)

\[
\frac{\pi(A \cup u)}{\pi(A)} = \frac{\pi(A \cup u \setminus v)}{\pi(A \setminus v)}. \tag{6}
\]

Let \(u,v\) be as above and denote by \(\Delta u\) the external boundary of \(u\). Assuming we have the local Markov property, then we have the following.

\[
\frac{\pi(A \cup u)}{\pi(A) + \pi(A \cup u)} = \pi(\sigma_u = 1 \mid \sigma_{\Lambda \setminus u} = A)
\]
\[
= \pi(\sigma_u = 1 \mid \sigma_{\Delta u} = A \cap \Delta u)
\]
\[
= \pi(\sigma_u = 1 \mid \sigma_{\Lambda \setminus u} = A \setminus v)
\]
\[
= \pi(A \cup u \setminus v) / (\pi(A \setminus v) + \pi(A \cup u \setminus v)).
\]

The second equality was the local Markov property, and the third equality was using that \(v \notin \Delta u\). Since we are given that (6) is equivalent to the above this gives one direction of the equivalence, furthermore the above calculation also gives the reverse direction.

All that remains is to show that the global Markov property follows from the local one. We use Theorem 9.1: Let \(K\) denote the maximal set of cliques, then there exists functions \(f_K : \Omega \mapsto [0, \infty)\) with \(K \in K\) such that

\[
\pi(A) = \prod_{K \in K} f_K(A \cap K).
\]
Now letting $W \subseteq \Lambda$, $A \subseteq W$ and $C \subseteq \Lambda \setminus W$, Theorem 9.1 gives

$$\pi(\sigma_W = A \mid \sigma_{\Lambda \setminus W} = C) = \frac{\prod_{K \in \mathcal{K}} f_K((A \cup C) \cap K)}{\sum_{B \subseteq W} \prod_{K \in \mathcal{K}} f_K((B \cup C) \cap K)}.$$

Any clique $K$ such that $K \cap W = \emptyset$ makes the same contribution of $f_K(C \cap K)$ to both the numerator and the denominator in the above, so can be canceled. Thus we can consider only those cliques that are subsets of $W_1 = W \cup \Delta W$,

$$\pi(\sigma_W = A \mid \sigma_{\Lambda \setminus W} = C) = \frac{\prod_{K \subseteq \mathcal{K}, K \subseteq W_1} f_K((A \cup C) \cap K)}{\sum_{B \subseteq W} \prod_{K \subseteq \mathcal{K}, K \subseteq W_1} f_K((B \cup C) \cap K)}.$$

Now note that right hand side does not depend on $\sigma_{\Lambda \setminus W_1}$, thus

$$\pi(\sigma_W = A \mid \sigma_{\Lambda \setminus W} = C) = \pi(\sigma_W = A \mid \sigma_{\Delta W} = C \cap \Delta W),$$

which is what we needed for the global Markov property.

(b) A straightforward application of the inclusion-exclusion principle.

(c) Since $C \notin \mathcal{K}$ we may choose $u, v \in C$ such that $u \not\sim v$. Then by the assumption given in part (b) we have

$$\phi_C = \sum_{L \subseteq C \setminus \{u, v\}} (-1)^{|C \setminus L|} \log \left( \frac{\pi(L \cup u \cup v)}{\pi(L \cup u)} \frac{\pi(L)}{\pi(L \cup v)} \right).$$

The local Markov property and an application of Proposition 9.5 gives that $\phi_C = 0$.

(d) Let $A \subseteq \Lambda$ with $u \notin A$, $v \in A$ with $u \not\sim v$. Then using part (b) and (c) we get

$$\log \left( \frac{\pi(A \cup u)}{\pi(A)} \right) = \sum_{K \subseteq A \cup u, u \in K} \phi_K = \sum_{K \subseteq A \cup u \setminus v, u \in K} \phi_K = \log \left( \frac{\pi(A \cup u \setminus v)}{\pi(A \setminus v)} \right),$$

where the second equality came from $u \not\sim v$ and $K \in \mathcal{K}$. We are done by Proposition 9.5.
3. **(The spectral gap)** Firstly we recall the backward-equation, $\frac{d}{dt}P_t f(\sigma) = \mathcal{L}P_t f(\sigma)$ for $\sigma \in \Omega$. Setting $u(t) = \|P_t f\|_{2,\pi}^2$, we have that

$$u'(t) = -2 \sum_{\sigma \in \Omega} P_t f(\sigma) \cdot -\mathcal{L}P_t f(\sigma) \cdot \pi(\sigma)$$

$$= -2 \langle P_t f, -\mathcal{L}P_t f \rangle_{\pi} = -2D_\mathcal{L}(P_t f),$$

which handles part (a).

Noting that $P_t f - \pi(f) = P_t (f - \pi(f))$ and applying part (a) to $u(t) = \|P_t (f - \pi(f))\|_{2,\pi}^2$, we get that

$$u'(t) \overset{(a)}{=} -2D_\mathcal{L}(P_t f - \pi(f)) = -2D_\mathcal{L}(P_t f) \leq -2\text{gap} \text{Var}_\pi(P_t f) =: -2\text{gap} \cdot u(t),$$

where the second equality came from the definition of $D_\mathcal{L}$, and we applied the Poincaré inequality. Now integrating $u'(t)/u(t)$ (applying Grönwall’s inequality) gives

$$u(t) \leq e^{-2\text{gap} t}u(0),$$

and since $u(0) = \text{Var}_\pi(f)$ this gives part (b).

Let $d(t) = \max_{\eta \in \Omega} \|P_t(\eta, \cdot) - \pi(\cdot)\|_{TV}$ and recall that $\|\mu - \nu\|_{TV} = \frac{1}{2} \sum_{\sigma \in \Omega} |\mu(\sigma) - \nu(\sigma)|$ for two measure $\mu$ and $\nu$ on $\Omega$ finite. For the lower bound take orthonormal eigenfunctions of $P_t$ (a finite stochastic matrix - it may be worth discussing some basic linear algebra results here, such as Perron-Frobenius theorem), with $f_1 \equiv 1$ for the eigenfunction corresponding to eigenvalue 1. Let $f$ be the eigenfunction corresponding to the eigenvalue $e^{-\text{gap} t}$ of $P_t$.

By orthogonality $\langle f, f_1 \rangle_\pi = \pi(f) = 0$. Then we have that

$$e^{-\text{gap} t} |f(\eta)| = |P_t f(\eta)| = \sum_{\sigma \in \Omega} P_t(\eta, \sigma)f(\sigma) - \pi(\sigma)f(\sigma) \leq 2d(t)\|f\|_{\infty}$$

Taking $\eta \in \Omega$ such that $|f(\eta)| = \|f\|_{\infty}$ gives that $e^{-\text{gap} t} \leq 2d(t)$. Setting $t = T_{\text{mix}}(\epsilon)$ implies $e^{-\text{gap} T_{\text{mix}}(\epsilon)} \leq 2\epsilon$ and taking logs and rearranging finishes the lower bound.

For the upper bound if $f_\sigma(\eta) = 1_{\eta = \sigma}/\pi(\sigma)$, then $P_t f_\sigma(\eta) = P_t(\eta, \sigma)/\pi(\sigma)$. Noting that $\pi(f_\sigma) = 1$ and $\text{Var}_\pi(f_\sigma) = (1 - \pi(\sigma))/\pi(\sigma)$ via an application of part (b) we obtain the following

$$\|P_t f_\sigma - 1\|_{2,\pi}^2 \leq \frac{e^{-2\text{gap} t}(1 - \pi(\sigma))}{\pi(\sigma)}.$$

So in particular we have that

$$\|P_t f_\sigma - 1\|_{2,\pi} \leq e^{-\text{gap} t}/\sqrt{\pi(\sigma)}. \quad (\heartsuit)$$

We now proceed as in the hint:

$$|P_t f_\sigma(\eta) - 1| = \left| \sum_{\zeta \in \Omega} (P_{t/2} f_\sigma(\zeta) - 1)(P_{t/2} f_\sigma(\zeta) - 1)\pi(\zeta) \right|$$

$$\leq \|P_{t/2} f_\sigma - 1\|_{2,\pi} \cdot \|P_{t/2} f_\sigma - 1\|_{2,\pi}.$$

were we used the Cauchy-Schwarz inequality on the second line. In particular by (2)
\[
\left| \frac{P_t(\eta, \sigma)}{\pi(\sigma)} - 1 \right| \leq \frac{e^{-\text{gap}_t}}{\sqrt{\pi(\sigma)\pi(\eta)}}.
\]

Multiplying by $\pi(\sigma)$ and summing over $\sigma \in \Omega$ gives
\[
2\|P_t(\eta, \cdot) - \pi(\cdot)\|_{TV} \leq e^{-\text{gap}_t} \sum_{\sigma \in \Omega} \frac{\pi(\sigma)}{\sqrt{\pi(\eta)\pi(\sigma)}} \leq \frac{e^{-\text{gap}_t}}{\pi^*}.
\]

Choosing $t$ such that the term on the right is equal to $\epsilon$ we have $t \geq T_{\text{mix}}(\epsilon)$ and since $t = \frac{1}{\text{gap}} \log \left( \frac{1}{\epsilon \pi^*} \right)$ we are done.

4. **(The bottleneck inequality)** Choose $f$ to be $\mathbf{1}_{\{\eta: \eta \in A\}}$ in the Poincaré inequality. In particular we have that
\[
D_L(\mathbf{1}_{\{\eta: \eta \in A\}}) = \sum_{\eta \in A \atop \sigma \in A^c} \pi(\eta) L(\eta, \sigma) =: Q(\partial A), \quad \text{and}
\]
\[
\text{Var}_\pi(\mathbf{1}_{\{\eta: \eta \in A\}}) = \pi(A) - \pi(A)^2.
\]

Applying the Poincaré inequality it follows that
\[
\lambda(1 - \pi(A)) \leq \frac{Q(\partial A)}{\pi(A)},
\]
where $A \subset \Omega$ was arbitrary. Minimising over $\pi(A) \leq 1/2$ we obtain
\[
\frac{\lambda}{2} \leq \min_{A \subset \Omega \atop \pi(A) \leq 1/2} \frac{Q(\partial A)}{\pi(A)} =: \Phi = \frac{\Phi}{2},
\]
which gives the desired result by choosing $\lambda$ to be optimal.