A.4 Vacation sheet

Hand in work by Wednesday 6pm in Statistics in week 1(?) HT.

1. (Russo-Seymour-Welsh) Consider Bernoulli percolation on the $\mathbb{Z}^2$ lattice at criticality, i.e. $p = 1/2$. For this question a picture could be very helpful, you may use Equation (7.2.1) from lecture notes without proof (i.e. at $p = 1/2$ the probability that there is an open path from the left to the right of any integer square on the lattice is 1/2).

Consider two rectangles

\[ S := [0, n]^2, \quad S' := [-n, n]^2, \quad \text{and} \quad R = [-n, 2n] \times [-n, n]. \]

Let $V(S)$ be the event that there is an open path (inside of $S$) between the bottom and top of the square $S$ and $H(S')$ be the event that there is an open path between the right and left of the rectangle $S'$ inside of $S'$. Let $X(S')$ be the event that there is an open path $P_1$ between the top and bottom of the rectangle $S$ (inside $S$) and a path $P_2$ inside of $S'$ connecting a point on $P_1$ to the left hand side of $S'$. For a path $P$ from between the top and bottom of $S$ let $\rho(P)$ be its reflection with respect to $\mathbb{Z} \times \{0\}$. By conditioning on the event $V(S)$ and that the right most open path in $S$ between the top and bottom is $P$ show that

\[ \mathbb{P}_{1/2}[X(S')] \geq \mathbb{P}_{1/2}[V(S)] \mathbb{P}_{1/2}[H(S')] / 2 = 1/8. \]

Hence prove that if $H(R)$ is the event that there is a horizontal crossing of $R$ inside of $R$ then

\[ \mathbb{P}_{1/2}[H(R)] \geq 1/128. \]

2. (Markov and Gibbs random fields) Let $\Omega = \{0, 1\}^\Lambda$, and recall the usual bijection between subsets of $\Lambda$ and configurations in $\Omega$.

(a) Prove proposition 9.5 in the lecture notes. Hint: Equation (9.1.2) is equivalent to

\[ \pi(A \cup u) \pi(A \cup u) = \pi(A \cup u \setminus v) \pi(A \setminus v) + \pi(A \cup u \setminus v), \]

where, with a slight abuse of notation, we write $u$ instead of $\{u\}$ for singleton sets.

You may also use Theorem 9.1 from lecture notes without proof.

(b) Use the inclusion-exclusion principle to show that if $\pi$ is a positive Markov field on $\Omega$ and

\[ \phi(C) = \sum_{L \subseteq C} (-1)^{|C \setminus L|} \log \pi(L), \quad C \subseteq \Lambda. \]

then

\[ \log \pi(A) = \sum_{C \subseteq A} \phi(C), \quad A \subseteq \Lambda. \]

(c) Use the local Markov property and Proposition 9.5 to show that $\phi_C = 0$ for $C \notin \mathcal{K}$.

(d) To complete the proof of Theorem 9.4, use Proposition 9.5 to show that if $\pi$ is a Gibbs field with potential function $\phi$ then $\pi$ satisfies the global Markov property.
3. **(The spectral gap)** Suppose that $L$ is the generator of an irreducible continuous time Markov process on finite statespace $\Omega$, which is reversible with respect to $\pi$. Recall that, if we think of $L$ as a Matrix (the Q-matrix), then the Dirichlet form is given by

$$D_L(f) = \langle f, -Lf \rangle_\pi = \frac{1}{2} \sum_{\eta, \zeta \in \Omega} \pi(\eta)L(\eta, \zeta)|f(\eta) - f(\zeta)|^2.$$ 

Clearly 0 is an eigenvalue of $-L$, the Perron-Frobenius Theorem implies that the smallest non zero eigenvalue of $-L$ is simple, and this controls the rate of convergence of the process to the stationary measure (which we will explore in more detail bellow). We define the spectral gap (i.e. the smallest non-zero eigenvalue of $-L$) by

$$\text{gap} = \inf_{f \text{ non const.}} \frac{D_L(f)}{\text{Var}_\pi(f)},$$

i.e. the spectral gap is the optimal constant $\lambda > 0$ in the Poincaré inequality

$$\lambda \text{Var}_\pi(f) \leq D_L(f) \text{ for all } f.$$ 

(a) Prove that, if $P_t = e^{tL}$ is the associated semigroup then

$$\frac{d}{dt} \|P_t f\|_{2,\pi}^2 = -2\mathcal{E}_L(P_t f),$$

where $\|f\|_{2,\pi}^2 = \langle f, f \rangle_\pi$. 

(b) Show that

$$\text{Var}_\pi(P_t f) = \|P_t f - \pi(f)\|^2_{2,\pi} \leq e^{-2\text{gap} t} \text{Var}_\pi(f).$$

(c) Show that

$$\frac{1}{\text{gap}} \log \left( \frac{1}{2\varepsilon} \right) \leq T_{\text{mix}}(\varepsilon) \leq \frac{1}{\text{gap}} \log \left( \frac{1}{\varepsilon \pi_*} \right),$$

where $\pi_* = \min_{\eta \in \Omega} \pi(\eta)$. *Hint: for the upper bound use $P_t(\eta, \sigma) = \sum_{\zeta \in \Omega} P_{t/2}(\eta, \zeta)P_{t/2}(\zeta, \sigma)$, and the Cauchy-Schwarz inequality.*

4. **(The bottleneck inequality)** Under the same setting as the previous question define the bottleneck constant (alternatively called the Cheeger or isoperimetric constant) by

$$\Phi = \min_{\pi(A) \leq 1/2} \frac{2Q(\partial A)}{\pi(A)},$$

where $Q(\partial A) = \sum_{\eta \in A, \sigma \in A^c} \pi(\eta)L(\eta, \sigma)$. By an appropriate choice of test function in the Poincaré inequality, show that

$$\text{gap} \leq \Phi.$$