A.2 Calculations, Spanning Trees, and First Percolation

1. This is a standard analysis proof. Fix \((x_n)\) a real sequence satisfying the subadditivity inequality, and let \(\ell = \inf_n \{x_n/n : n \in \mathbb{N}\}\). Fix \(\epsilon > 0\), then there exists a \(k \in \mathbb{N}\) such that

\[ k(\ell + \epsilon) > x_k \geq k\ell. \]

Let \(b = \max_{1 \leq i \leq k} x_i\). For \(m \geq k\) we may write \(m = qk + r\) for some \(q \in \mathbb{N}\) and \(r \in \{0, \ldots, k-1\}\). Then, by subadditivity, we have

\[ x_m = x_{qk+r} \leq qx_k + x_r \leq qx_k + b. \]

Thus,

\[ \ell \leq \frac{x_m}{m} \leq \frac{qx_k}{m} + \frac{b}{m} \leq \frac{(m-r)(\ell + \epsilon)}{m} + \frac{b}{m} \to \ell + \epsilon \text{ as } m \to \infty. \]

2. A direct application of Fekete’s Lemma. Consider the sequence \(x_n = \log \sigma_n\), observe that if \(\alpha = \lim_{n \to \infty} x_n/n\) exists then by continuity so does \(\lim_{n \to \infty} \sigma_n^{1/n}\), and \(\kappa(d) = e^\alpha\). Since each SAW of length \(m + n\) is a SAW of length \(m\) concatenated with a SAW if length \(n\) (with the extra constraint that the two SAWs also can not intersect), it follows that \(\sigma_{n+m} \leq \sigma_n \sigma_m\), i.e. \(x_n\) satisfies the subadditivity property. Therefore \(\lim_{n \to \infty} x_n/n\) exists, as required. We saw an upper bound in notes of \(\sigma_n(d) \leq 2d(2d-1)^{n-1}\), which gives the required upper bound on \(\kappa\), the lower bound follows by taking paths that always increase in one of the \(d\)-directions at each step, there are \(d^n\) such increasing paths.

3. (One dimensional percolation) We consider Bernoulli percolation on the subset of the \(\mathbb{Z}^2\) lattice whose sites contain \(\mathbb{Z} \times \{0, 1, \ldots, n\}\). Let \(q = 1 - p\) then

\[ \mathbb{P}(|C_0| = \infty) \leq \mathbb{P}(0 \leftrightarrow \{m, m\} \times \{0, \ldots, n\}) \]

for the event on the right hand side to occur each of the \(2m + 1\) columns must contain at least one edge, i.e.

\[ \mathbb{P}(0 \leftrightarrow \{m, m\} \times \{0, \ldots, n\}) \leq (1 - q^n)^{2m+1} \to 0 \quad \text{as } m \to \infty, \]

and so \(\mathbb{P}(|C_0| = \infty)\) as required.

4. (Monotonicity in \(p\)) Consider the coupling described in lectures for independent Bernoulli percolation, using uniform edge weights, call the measure \(\mathbb{P}\) and expectation \(\mathbb{E}\). Notice that if \(p_1 \leq p_2\) then by construction \(\mathbb{P}(\omega_{p_1} \leq \omega_{p_2}) = 1\). Since \(f\) is increasing \(f(\omega) \leq f(\omega')\) if \(\omega \leq \omega'\) and hence

\[ \mathbb{E}[f(\omega_{p_1}) - f(\omega_{p_2})] \leq 0, \]

Also \(\mathbb{E}[f(\omega_{p_1})] = \mathbb{E}_{p_1}[f]\) (and similarly for \(p_2\)), the result follows.

5. (Approximation) This is a fairly standard measure theory question, see for example P. Halmos Measure Theory (Chapter III). The \(\sigma\)-algebra \(\mathcal{F}\) is generated by cylinder sets, which them selves form an algebra (not a \(\sigma\)-algebra), an element of this algebra only depends on finitely many edges. In general we may prove that if \(\mathcal{F}\) is a \(\sigma\)-algebra generated by an algebra \(\mathcal{A}\) (note \(\mathcal{A}\) was used in the question in this form without explanation, sorry) then any element \(A \in \mathcal{F}\) can be approximated by some element \(B \in \mathcal{A}\). One way is to start from the ‘outer-measure’ characterisation of \(\mathbb{P}\), i.e.

\[ \mathbb{P}(A) = \inf \left\{ \sum_{i=1}^{\infty} \mathbb{P}(A_i) : A \subset \bigcup_{i=1}^{\infty} A_i, A_i \in \mathcal{A}, i \in \mathbb{N} \right\}, \]
and proceed by approximation.

Alternatively, let
\[ G = \{ E \in F : \forall \epsilon > 0, \exists B \in A \text{ such that } \mathbb{P}(E \Delta B) < \epsilon \}. \]

We will show that \( G \) is a \( \sigma \)-algebra, and then since \( A \subseteq G \), we have \( F \subseteq G \) (since \( F \) is the smallest \( \sigma \)-algebra containing \( A \)) which completes the proof.

It is straightforward to show \( G \) is an algebra, as follows. Clearly \( \emptyset, \Omega \in G \), now fix \( E \in G \) and \( \epsilon > 0 \). Then there exists an \( B \in A \) such that \( \mathbb{P}(E \Delta B) < \epsilon \). Since \( E \Delta B = E^c \Delta B^c \) and \( B^c \in A \) it follows that \( E^c \in G \).

Finally, suppose that \( (E_i)_{i \in \mathbb{N}} \) is a countable collection of elements of \( G \), let \( E = \bigcup_{i \in \mathbb{N}} E_i \) and \( B_n = \bigcup_{i=1}^n E_n \). Since \( \lim_{n \to \infty} \mathbb{P}(B_n) = \mathbb{P}(E) \) there exists and \( n \in \mathbb{N} \) such that \( |\mathbb{P}(B_n) - \mathbb{P}(E)| < \epsilon \), which implies that \( \mathbb{P}(E \Delta B_n) < \epsilon \). For \( i = 1, \ldots, n \), there exists an \( x_i \in A \) such that \( \mathbb{P}(E_i \Delta E_i) < \epsilon/n \), which implies that \( \mathbb{P}(\bigcup_{i=1}^n E_i \Delta B_n) < \epsilon \). It follows that \( \mathbb{P}(\bigcup_{i=1}^n E_i \Delta E) < 2\epsilon \), so in particular we have \( E \in G \) as required.

6. (n-th root trick) As mentioned in lectures, this is an application of the FKG inequality. The first important observation here is that the same inequality holds also in the case that both events are decreasing. Suppose \( A_1 \) and \( A_2 \) are both increasing events, then by the FKG (Harris) inequality we have
\[ \mathbb{P}(A_1 \cap A_2) \geq \mathbb{P}(A_1) \mathbb{P}(A_2), \]
and so (by inclusion-exclusion)
\[ \mathbb{P}(A^c_1 \cap A^c_2) = 1 - \mathbb{P}(A_1 \cup A_2) = 1 - \mathbb{P}(A_1) - \mathbb{P}(A_2) + \mathbb{P}(A_1 \cap A_2) \]
\[ \geq 1 - \mathbb{P}(A_1) - \mathbb{P}(A_2) + \mathbb{P}(A_1) \mathbb{P}(A_2) = \mathbb{P}(A_1^c) \mathbb{P}(A_2^c), \]
i.e. \( A_1^c \) and \( A_2^c \) satisfy the FKG inequality. Now if \( A_1, A_2, \ldots, A_n \) are all increasing it follows that
\[ \mathbb{P}(\bigcap_{i=1}^n A_i^c) \geq \prod_{i=1}^n \mathbb{P}(A_i^c) \geq \left( \min_{i \leq n} \mathbb{P}(A_i) \right)^n = \left( 1 - \max_{i \leq n} \mathbb{P}(A_i) \right)^n, \]
which gives the desired inequality.

7. (FKG/positive correlations for the uniform measure) This is an application of the same ‘trick’ we saw in lectures in a very simple setting. If \( f \) and \( g \) have the same monotonicity then \( (f(x) - f(y))(g(x) - g(y)) \geq 0 \) for all \( x, y \in [0, 1] \), now integrate and the result follows.

8. (Monotonicity and attractive are the same for spin-flip dynamics) We sweep some of the technical issues under the carpet, for details see for example Liggett Interacting particle systems.

Firstly suppose that the process (with semigroup \( \{S(t) : t \geq 0\} \) is monotone, i.e. if \( f \) is increasing then \( S(t)f \) is increasing for all \( t \geq 0 \). Let \( f(\eta) = \eta(x) \), observe that since \( f \) only depends on the configuration at \( x \in \Lambda \) then \( f \in \mathcal{D}_\mathcal{L} \), in particular
\[ \mathcal{L}f(\eta) = \lim_{t \to 0} \frac{S(t)f(\eta) - f(\eta)}{t}. \]
If \( \eta \leq \sigma \) and \( \eta(x) = \sigma(x) \), i.e. \( f(\eta) = f(\sigma) \), then by monotonicity
\[ \mathcal{L}f(\eta) = \lim_{t \to 0} \frac{S(t)f(\eta) - f(\eta)}{t} \leq \lim_{t \to 0} \frac{S(t)f(\sigma) - f(\sigma)}{t} = \mathcal{L}f(\sigma). \]
The result follows by applying the generator

\[ \mathcal{L}f(\eta) = c(\eta, \eta^x)(1 - 2\eta(x)). \]

To show that if the process is attractive then it must be monotone we will construct a coupling, we will do this by moving the chains together as much as possible when the local state agrees, and flip independently where the local state is not the same. Consider the Markov process on \( \Omega \times \Omega \) with jump rates

\[
\begin{align*}
&c((\eta, \sigma), (\eta^x, \sigma^x)) = g(\eta, \sigma, x), \\
&c((\eta, \sigma), (\sigma^x, \sigma^x)) = c(\eta, \eta^x) - g(\eta, \sigma, x), \\
&c((\eta, \sigma), (\eta^x, \sigma)) = c(\sigma, \sigma^x) - g(\eta, \sigma, x), \\
&c((\eta, \sigma), (\eta^x, \sigma)) = c(\eta^x), \\
&c((\eta, \sigma), (\eta^x, \sigma^x)) = c(\sigma, \sigma^x),
\end{align*}
\]

where \( g(\eta, \sigma, x) := \min\{c(\eta, \eta^x), c(\sigma, \sigma^x)\} \) i.e. generator

\[ \tilde{\mathcal{L}}f(\eta, \sigma) = \sum_{x : \eta(x) = \sigma(x)} g(\eta, \sigma, x)[f(\eta^x, \sigma^x) - f(\eta, \sigma)] \\
+ \sum_{x : \eta(x) = \sigma(x)} (c(\eta, \eta^x) - g(\eta, \sigma, x))[f(\eta^x, \sigma) - f(\eta, \sigma)] \\
+ \sum_{x : \eta(x) = \sigma(x)} (c(\sigma, \sigma^x) - g(\eta, \sigma, x))[f(\eta, \sigma^x) - f(\eta, \sigma)] \\
+ \sum_{x : \eta(x) \neq \sigma(x)} c(\eta, \eta^x)[f(\eta^x, \sigma) - f(\eta, \sigma)] \\
+ \sum_{x : \eta(x) \neq \sigma(x)} c(\sigma, \sigma^x)[f(\eta, \sigma^x) - f(\eta, \sigma)],
\]

note that for any pair \((\eta, \sigma)\) either the second or third term is zero. It is clear (plugging in) that if \( f(\eta, \sigma) = h(\eta) \) then \( \tilde{\mathcal{L}}f(\eta, \sigma) = h(\eta) \), i.e. the marginal on the first coordinate is the original process (and the same for the marginal on the second coordinate). Also if we start with \( \eta \leq \sigma \) then under this coupling \( \eta_t \leq \sigma_t \), with probability 1, since the rate of jumping out of this is zero by the conditions on the jump rates (attractive). The result follows by Strassen’s theorem. Equivalently, you may identify \( \Omega^2 \) with \( \{(0, 0), (0, 1), (1, 0), (1, 1)\}^\Lambda \) and write down all the ‘local’ transition rates as we did in class, notice if \( \eta \leq \sigma \) then the dynamics stay in the set \( \{(0, 0), (0, 1), (1, 1)\}^\Lambda \) almost surely.

9. (Linear voter model and duality) The generator of the linear voter model is given by

\[ \mathcal{L}f(\eta) = \sum_{x,y \in \Lambda} p(x, y) (f(\eta^x) - f(\eta)), \quad \text{for } f \in \mathcal{D}. \]

The following duality relation is very useful for characterising the stationary measures. Let

\[ H(\eta, A) = \prod_{x \in A} \eta(x) = 1, \quad \text{for } \eta \in \Omega, \text{ and } A \subset \Lambda \text{ finite,} \]

where as usual we identify a set \( A \subset \Lambda \) with a configuration in \( \sigma_A \in \Omega \) by \( \sigma_A(x) = 1 \) if \( x \in A \) and \( \sigma_A(x) = 0 \) otherwise. We will aim to write everything in \( \mathcal{L}H(\cdot, A)(\eta) \) in terms of products over \( \eta(x) \). Observe

\[
H(\eta^x, A) = \begin{cases} 
H(\eta, A) & \text{if } x \notin A, \\
(1 - \eta(x))H(\eta, A \setminus \{x\}) & \text{if } x \in A,
\end{cases}
\]
therefore \( H(\eta^x, A) - H(\eta, A) = (1 - 2\eta(x))H(\eta, A \setminus \{x\}) \) if \( x \in A \), and also \( 1(\eta(y) \neq \eta(x)) = \eta(x)(1 - \eta(y)) + \eta(y)(1 - \eta(x)) \). It follows that

\[
\mathcal{L}H(\cdot, A)(\eta) = \sum_{x \in A} \sum_{y \in \Lambda} p(x, y) H(\eta, A \setminus \{x\})(1 - 2\eta(x)) [\eta(x)(1 - \eta(y)) + \eta(y)(1 - \eta(x))]
\]

\[
= \sum_{x \in A} \sum_{y \in \Lambda} p(x, y) H(\eta, A \setminus \{x\}) [\eta(y) - \eta(x)]
\]

\[
= \sum_{x \in A} \sum_{y \in \Lambda} p(x, y) [H(\eta, (A \setminus \{x\}) \cup \{y\}) - H(\eta, A)] = \tilde{\mathcal{L}}H(\eta, \cdot)(A),
\]

which we interpret as follows. Each particle in \( A \), say at site \( x \), moves to a neighbour site \( y \) at rate \( p(x, y) \), if \( y \) is empty then site \( x \) becomes empty and site \( y \) becomes occupied, if \( y \) is already occupied the site \( x \) becomes empty and the state at \( y \) remains the same. This is exactly coalescing random walks that move according to the matrix \( p(\cdot, \cdot) \).

For the final part of the question observe

\[
P_\eta(\eta_t \equiv 1 \text{ on } A) = E_\eta[H(\eta_t, A)] = S(t)H(\cdot, A)(\eta) = \tilde{S}(t)H(\eta, \cdot)(A) = \tilde{P}_A(\eta \equiv 1 \text{ on } A_t),
\]

where \( \tilde{S}(t) \) and \( \tilde{P}_A \) are, respectively, the semigroup and path measure associate with \( \tilde{\mathcal{L}} \).