A.2 Calculations, Spanning Trees, and First Percolation

Hand in work by Wednesday 6pm in Statistics in week 5.

1. Prove Lemmas 3.1 and 3.2, i.e. the Series and Parallel law, by checking that the discussed potential is still harmonic in the appropriate sense, and the flow given is still the associated current flow.

Suppose that if $A, B \subset \Omega$ disjoint, and $(A \cup B)^c$ is non-empty, and we impose a unit voltage between $A$ and $B$, then if the entire network between $A$ and $B$ can be reduced to a single resistor by the Series Law, Parallel Law, and Gluing, then the value of this resistor must be the effective resistance $R(A, B)$.

2. Calculate $P_{\eta}(\tau_\sigma < \tau_\eta^+)$ for the reversible process on the following graph where each edge has conductance 1:

![Graph Diagram]

3. Let $G$ be a finite irreducible, connected directed graph, and let $(\gamma_n)_{n=0}^M$ be a simple random path on $G$ from $\eta$ to $\sigma$, with law $P$. Note the length $M$ is also random. That is $\gamma_i \in \mathcal{E}$ for each $0 \leq i \leq M$, and $\gamma_i^+ = \gamma_{i+1}^-$, where simple means each edge is used at most once. Show that

$$\theta(e) := \sum_{n \geq 0} (P(\gamma_n = e) - P(\gamma_n = -e))$$

defines a unit flow from $\eta$ to $\sigma$.

4. Let $G$ be a countably infinite connected graph with bounded vertex-degrees and labeled vertex $O$ called the origin. Let $H$ be a connected subgraph of $G$ containing $O$. Show that the simple random walk, starting at $O$, is recurrent on $H$ whenever it is recurrent on $G$.

5. Prove Kirchhoff’s Effective Resistance Formula (Lemma 4.8 in the notes), using Wilson’s algorithm, Question 8 from Sheet 1, together with the definition of the effective resistance and unit current flow.

6. Let $G_n$ be the ladder graph of height $n$ shown below. Suppose $T_n$ is a UST on $G_n$, determine $P$(rung 1 is in $T_n$) and it’s limit as $n \to \infty$.

![Ladder Graph Diagram]
7. Let $G$ be a finite network and $\eta \neq \sigma$ be two vertices, and let $i$ be the unit current flow from $\eta$ to $\sigma$. For every edge $e$ show that the probability the loop-erased random walk from $\eta$ to $\sigma$ crosses $e$ minus the probability it crosses $-e$ is equal to $i(e)$.

8. Let $G = (\Omega, E)$ be a connected subgraph of the finite connected graph $G'$. Let $T$ and $T'$ be uniform spanning trees on $G$ and $G'$ respectively. Show that, for any edge $e$ of $G$, $\mathbb{P}(e \in T) \geq \mathbb{P}(e \in T')$.

9. How efficient is Wilson’s algorithm? Assume that the only computation to take time is each step of the Markov process, which takes unit time to simulate. Firstly show that the time taken to generate a (directed) spanning tree rooted at $r$ for a finite-state, irreducible and reversible Markov chain is $\sum_{\eta \in \Omega} \sum_{e \in \partial \Lambda_n} c(e) R(\{\eta\}, \{r\})$.

Show that if we drop the reversibility condition, in general we have expected run time $\sum_{\eta \in \Omega} \pi(\eta) \left( \mathbb{E}_{\eta}[\tau] + \mathbb{E}_r[\tau_\eta] \right)$.

10. **Nash-Williams Inequality.** Consider a finite connected (weighted) graph $G$. For $A, B \subset \Omega$, disjoint, call a set $\Pi \subset E$ a cutset for $A$ and $B$ if every path with one end point in $A$ and the other end point in $B$ must include an edge in $\Pi$. If $\eta \neq \sigma \in \Omega$ separated by a sequence of disjoint cutsets $\Pi_1, \ldots, \Pi_n$, then $R(\{\eta\}, \{s\}) \geq \sum_{i=1}^{n} \left( \sum_{e \in \Pi_i} c(e) \right)^{-1}$.

**Hint: Consider just a single cut-set first**

11. Let $G$ be an infinite connected graph with bounded vertex degree and some vertex marked as the origin $O$. Let $\partial \Lambda_n$ be the set of vertices’s a graph distance $n$ from the origin, and $E_n$ the number of edges joining $\partial_n \Lambda_n$ and $\partial_{n+1} \Lambda_n$, show that a simple random walk on $G$ is recurrent if $\sum_n E_n^{-1} = \infty$. Interpret this result in the setting of Poly’as theorem.

12. **First and second moment methods.** Suppose $X$ is a random variable on $\mathbb{N}_0$, show that $\mathbb{P}(X \neq 0) \leq \mathbb{E}[X]$.

Let $X$ be a non-negative random variable, show that $\mathbb{P}(X > 0) \geq \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]}$.

Slightly more generally, suppose $X$ is a random variable with mean 1 and fix $t < 1$, show that $\mathbb{P}(X > t) \geq (1 - t)^2 / \mathbb{E}[X^2]$. [Hint: Recall the Cauchy-Schwarz inequality]

13. **Percolation on trees.** Suppose $T_d$ is a rooted tree in which each vertex has $d$ children, so all sites except the root have degree $d + 1$. Suppose each edge of the tree is open (present) independently with probability $p$. Let $p_c = \sup \{ p : \mathbb{P}_p(\text{There is an open path from the root to } \infty) = 0 \}$.

Show that $p_c = 1/d$.

[Hint: Recall basic results of branching processes.]
10. Suppose that the graph $\Gamma$ has a path $(\eta_i : 1 \leq i \leq |V|)$ that is a spanning tree. Let $(X_n)_{n \geq 0}$ be the simple random walk on $G$ as defined in the notes (each edge has equal conductance), and let $q_k = \mathbb{P}_{\eta_k} (\tau_k^+ > \tau_{(x_{k+1}, \ldots, x_n)})$. Show that the number of spanning trees of $G$ equals $\prod_{k<n} q_k \text{deg}_G(\eta_k)$, where $\text{deg}_G(\sigma)$ is the degree of the vertex $\sigma \in G$. 

The following is optional at this point, they will be saved for the final sheet containing longer questions.