A.1 Background and Electrical Networks

Hand in work by Wednesday 6pm in Statistics in week 2. Some of these questions will be revision of previous material.

1. Let \((X_n)_{n \geq 1}\) be a discrete time Markov chain on countable state space \(\Omega\) with transition matrix \(P\). Define the matrix \(L = P - I\) (notice \(-L\) looks like a discrete Laplacian matrix on the graph \(G\) from the lectures), this is sometimes called the generator of the discrete time process, in analogy with the continuous time \(Q\)-matrix/generator. Convince yourself, using the definition from the lectures, that \(f: \Omega \rightarrow \mathbb{R}\) is harmonic on \(A \subset \Omega\) if and only if
\[
Lf(\eta) = 0, \quad \forall \eta \in A.
\]
(a) Fix \(f_0: \Omega \setminus A \rightarrow \mathbb{R}\), show that \(f(\eta) = \mathbb{E}_\eta[f_0(X_{\tau_A})]\) satisfies the discrete Dirichlet problem;
\[
-Lf(\eta) = 0, \quad \forall \eta \in A, \quad \text{and} \quad f(\sigma) = f_0(\sigma), \quad \forall \sigma \in A^c.
\]
(b) Let \(h(\eta) = \mathbb{E}_\eta[\tau_{A^c}]\), show that
\[
-Lh(\eta) = 1, \quad \forall \eta \in A, \quad \text{and} \quad h(\sigma) = 0, \quad \forall \sigma \in A^c.
\]
Observe that you have essentially used both of the results above (without the obfuscation in notation) many times to calculate hitting probabilities and expected hitting times. Both results hold in continuous time with \(L\) replaced by the generator of the process (\(Q\)-matrix) \(\mathcal{L}\).

2. Let \((X_n)_{n \geq 1}\) be a discrete time Markov chain on finite state space \(\Omega\) with transition matrix \(P\), and let \(L = P - I\). We consider the Hilbert space \(\ell^2(\pi)\) with inner product
\[
\langle f, g \rangle_\pi = \sum_{\eta \in \Omega} f(\eta)g(\eta)\pi(\eta) \quad \text{for} \quad f, g: \Omega \rightarrow \mathbb{R}.
\]
(a) Show that \((X_n)_{n \geq 1}\) is reversible with respect to probability measure \(\pi\) if and only if
\[
\langle f, -Lg \rangle_\pi = \langle -Lf, g \rangle_\pi \quad \forall f, g: \Omega \rightarrow \mathbb{R},
\]
i.e. the generator is self adjoint in \(\ell^2(\pi)\).
(b) Now suppose that \((X_n)_{n \geq 1}\) is reversible with respect to \(\pi\), show that the definition of the energy (Dirichlet form) of a function \(f: \Omega \rightarrow \mathbb{R}\) in lectures corresponds to
\[
\mathcal{D}(f) = \langle f, -Lf \rangle_\pi.
\]
This will be important later when we look at relaxation and mixing of Markov chains (and in particular on spectral properties).

You should also convince yourself that the results of this question also apply if we have a continuous time chain with generator \(\mathcal{L}\).

3. Let \((X_n)_{n \geq 1}\) be a discrete time Markov chain on finite state space \(\Omega\) with transition matrix \(P\).
(a) Show that \(P\) is reversible with respect to \(\pi\) then \(\pi\) stationary.
Consider the weighted graph $G$ with transition matrix $P$. Fix $\eta, \sigma \in \Omega$, and let
\[
\pi(\eta) \prod_{i=1}^{n-1} P(\eta_i, \eta_{i+1}) = \pi(\eta_n) \prod_{i=1}^{n-1} \hat{P}(\eta_{n-i}, \eta_{n-i}).
\]
that is the chance of going round any loop in one direction is equal to the probability of going the other way. Show that this ‘loop-condition’ also implies reversibility.

Let $(X_n)_{n \geq 0}$ be a reversible Markov chain on finite graph $G$. Fix $\eta, \sigma \in \Omega$, and let
\[
\pi(\eta) G_{\tau_B}(\eta, \eta) = \pi(\sigma) G_{\tau_B}(\sigma, \eta).
\]
that is the chance of going round any loop in one direction is equal to the probability of going the other way. Show that this ‘loop-condition’ also implies reversibility.

Consider the weighted graph $G$ associated with a reversible, finite state, Markov chain with transition matrix $P$. Fix $a \neq b \in \Omega$. Show that if $i$ and $j$ are both flows from $\{a\}$ to $\{b\}$ with strength $|i| = |j|$ then $i \equiv j$ [Hint: One way is to use the result in the previous question].

Let $(X_n)$ be a random walk on $G$ that is absorbed at $b$, and for $(\eta, \sigma) \in E$ let $S_{\eta,\sigma}$ be the number of transitions from $\eta$ to $\sigma$ before $(X_n)$ is absorbed. Show that
\[
\mathbb{E}_a[\xi_{\eta,\sigma} - \xi_{\sigma,\eta}] = i_{\eta,\sigma}(\eta, \sigma).
\]