1. Let \((X_0, X_1, Y_1, \ldots, X_T, Y_T)\) be a discrete-state hidden Markov model, with \(X_i \in \{1, \ldots, K\}\) and \(Y_i \in \{1, \ldots, M\}\) with joint probability mass function (pmf)

\[
p(x_0, x_1, y_1, \ldots, x_T, y_T) := Pr(X_0 = x_0, X_1 = x_1, Y_1 = y_1, \ldots, X_T = x_T, Y_T = y_T)
\]

\[
= p(x_0) \prod_{t=1}^{T} p(x_t|x_{t-1})p(y_t|x_t)
\]

(i) Show carefully that, for any \(t = 0, \ldots, T-1\)

\[
p(x_t|x_{t+1}, \ldots, x_T) = p(x_t|x_{t+1})
\]

(ii) Deduce that the joint pmf admits the following alternative decomposition

\[
p(x_0, x_1, y_1, \ldots, x_T, y_T) = p(x_T) \prod_{t=T}^{1} p(x_{t-1}|x_t)p(y_t|x_t)
\]

(iii) Let \(\mu_i = Pr(X_T = i)\), \(B_{i,k} := Pr(Y_t = k|X_t = i)\) and \(C_{i,j} := Pr(X_{t-1} = j|X_t = i)\). Derive a backward recursion to calculate \(p(y_{t:T}, x_t)\) for \(t = T, T-1, \ldots, 1\). Justify carefully any conditional independence assumption.

(iv) What is the computational complexity of this recursion?

2. Suppose you have a state-space model of the form

\[
X_t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} X_{t-1} + \begin{pmatrix} V_{t,1} \\ 2V_{t,2} \end{pmatrix}
\]

\[
Y_t = \begin{pmatrix} 1 & 0 \end{pmatrix} X_t + \frac{1}{2}W_t
\]

where \(V_{t,1}, V_{t,2}\) and \(W_t\) are independent standard normal random variables for each \(t\). Assume that you have observed \(Y_1 = y_1, \ldots, Y_{10} = y_{10}\) and you have used the Kalman filter to calculate \(\mu_{10|10} := E[X_{10}|y_{1:10}]\) and \(\Sigma_{10|10} := cov[X_{10}|y_{1:10}]\), and obtain the values

\[
\mu_{10|10} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

\[
\Sigma_{10|10} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}
\]

(i) Calculate the state prediction \(\mu_{11|10}\) and its covariance \(\Sigma_{11|10}\)

(ii) Calculate the predicted observation \(\hat{y}_{11|10} := E[Y_{11}|y_{1:10}]\) and its variance \(S_{11} := Var(Y_{11}|y_{1:10})\)
3. Let \((X_0, X_1, Y_1, \ldots, X_T, Y_T)\) be a discrete-state hidden Markov model, with \(X_i \in \{1, \ldots, K\}\) and \(Y_i \in \{1, \ldots, M\}\) with joint probability mass function (pmf)

\[
p(x_0, x_1, y_1, \ldots, x_T, y_T) := \Pr(X_0 = x_0, X_1 = x_1, Y_1 = y_1, \ldots, X_T = x_T, Y_T = y_T)
\]

\[
= p(x_0) \prod_{t=1}^{T} p(x_t|x_{t-1})p(y_t|x_t)
\]

(i) Show that

\[
p(x_{t-1}, x_t | y_{1:T}) = \frac{p(x_{t-1}, y_{1:t-1})p(y_t|x_t)p(x_t|x_{t-1})p(y_{t+1:T}|x_t)}{p(y_{1:T})}
\]

(ii) Assume that you have used a forward-backward recursion to calculate \(\alpha_t(x_t) := p(x_t, y_{1:t})\) and \(\beta_t(x_t) := p(y_{t+1:T}|x_t)\) for each \(t = 1, \ldots, T\). Describe an algorithm to simulate \(N\) realizations \(x_{0:T}^{(i)}, i = 1, \ldots, N\), from the posterior distribution \(p(x_{0:T}|y_{1:T})\).

(iii) What is the overall complexity of simulating realizations from \(p(x_{0:T}|y_{1:T})\)?