University of Oxford

Statistical Methods

Autocorrelation

Decomposition and Smoothing

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Time Series Decomposition

1.1 Principles

Consider the following time series plots.

What is the simplest way that we can describe these series?

Are there similarities or differences between the three series?

Are the series completely random?
Figure 1.1: Time Series Plot of Sunspot Activity: 1749 - 1997
1. **TIME SERIES DECOMPOSITION**

<table>
<thead>
<tr>
<th>Year</th>
<th>AirPassengers</th>
</tr>
</thead>
<tbody>
<tr>
<td>1950</td>
<td>100</td>
</tr>
<tr>
<td>1952</td>
<td>200</td>
</tr>
<tr>
<td>1954</td>
<td>300</td>
</tr>
<tr>
<td>1956</td>
<td>400</td>
</tr>
<tr>
<td>1958</td>
<td>500</td>
</tr>
<tr>
<td>1960</td>
<td>600</td>
</tr>
</tbody>
</table>

![Graph showing Monthly US Air Passenger Numbers: 1949 - 1960](image)

Figure 1.2: Monthly US Air Passenger Numbers: 1949 - 1960
1. **TIME SERIES DECOMPOSITION**

![Time Series Plot of White Noise](image)

**Figure 1.3:** Time Series Plot of White Noise
1. TIME SERIES DECOMPOSITION

We propose to decompose each series into several components.

- \( TC_t \sim \) Trend-Cycle
- \( S_t \sim \) Seasonal
- \( R_t \sim \) Random

We will attempt to write the time series \( Y_t \) as a function of these components:

\[
Y_t = f(TC_t, S_t, R_t) \quad (1.1)
\]

The function \( f \) may take several forms:

**Additive** \( Y_t = TC_t + S_t + R_t \)

**Multiplicative** \( Y_t = TC_t \times S_t \times R_t \)

**Pseudo-Additive** \( Y_t = TC_t(S_t + R_t - 1) \)

By identifying the various components, our aim is to separate the Random component from the other components of the series. This way we hope to be able to eliminate the noise and isolate the true signal. Figure 1.4 illustrates how a series may be decomposed into three components.
1. TIME SERIES DECOMPOSITION

Figure 1.4: An Example of Seasonal Decomposition
The most common use of this decomposition is to isolate and remove the seasonal component.

In considering unemployment data we know that there are several factors that each year effect the unemployment levels during several months of the year. Consider the effect that the academic year has. In the Summer, when students are no longer in college, unemployment levels are traditionally higher than in the spring and they reduce again in the Autumn.

Suppose we were able to identify the separate components of the series then removing the seasonal component is simple:

In an additive model we find that the seasonally adjusted series $Z_t$ is

$$Z_t = Y_t - S_t = TC_t + R_t.$$  \hfill (1.2)

In the multiplicative model:

$$Z_t = \frac{Y_t}{S_t} = TC_t \times R_t$$ \hfill (1.3)
1.1 TIME SERIES DECOMPOSITION

1.2 Estimating the Trend-Cycle

Consider how we might isolate the Trend-Cycle. The Random component fluctuates from time period to time period. If we assume that this random component is equally likely to be positive as negative, then one method of removing this might be to average observations. We smooth the series to remove the random variation.

The Moving-Averages technique provides a simple method to smooth a time series. We replace an observation at time $t$ by the average of this observation and observations at times close to $t$.

1.2.1 Simple Moving Averages

The simple moving average technique can take different forms. We begin by choosing an order for the moving average $k$ where $k$ is an odd integer. Then the $k$ MA moving average is

$$MA_t = \frac{1}{k} \sum_{j=-m}^{m} Y_{t+j}$$  \hspace{1cm} (1.4)

where

$$m = \frac{(k - 1)}{2}$$  \hspace{1cm} (1.5)

1.2.2 Centered Moving Averages

While the definition of the $k$ MA works for odd values of $k$ it is less satisfactory for even values of $k$. Consider for instance when $k = 4$ then we have two alternative 4 MA moving averages:

$$MA_t^L = \frac{Y_{t-2} + Y_{t-1} + Y_{t} + Y_{t+1}}{4}$$  \hspace{1cm} (1.6)

$$MA_t^R = \frac{Y_{t-1} + Y_{t} + Y_{t+1} + Y_{t+2}}{4}$$  \hspace{1cm} (1.7)
1. **TIME SERIES DECOMPOSITION**

Which should we choose? Neither is completely satisfactory, therefore we actually choose to average the two.

Hence we define the centered $2 \times 4$ MA smoother to be

$$MA''_t = \frac{MA^L_t + MA^R_t}{2}. \tag{1.8}$$
1. TIME SERIES DECOMPOSITION

1.2.3 Double Moving Averages

The centred moving average 2 × 4 introduced in the last section is a specific example of a concept known as the *Double Moving Average Smoother*.

- Firstly we applied a 4 MA moving average to the original data.
- Then we applied a 2 MA moving average to the resultant 4 MA smoothed series.

In fact this concept can be extended to any arbitrary integers $p$ and $q$:

- Take the original series $Y_t$ and smooth it using a $q$ MA smoother to get the series $MA_t$.
- This series is then smoothed again using a $p$ MA smoother to get $MA''_t$.
- The final series $MA''_t$ is the $p \times q$ MA moving average smoothed series.
1. TIME SERIES DECOMPOSITION

1.2.4 Weighted Moving Averages

Sometimes we may want to take a weighted average of the observations around \( Y_t \) rather than just taking a simple average as in equation 1.4.

In this case we define a weighted \( k \)-point moving average as:

\[
WMA_t = \sum_{j=-m}^{m} a_j Y_{t+j}
\]  \hspace{1cm} (1.9)

where the weights are denoted by \( a_j \) and just as in the simple moving average case we have:

\[
m = \frac{(k - 1)}{2}.
\]  \hspace{1cm} (1.10)

Note it is important that the weights \( a_j \) are symmetrical so that

\[
a_j = a_{-j}
\]

and also that the weights sum to one:

\[
\sum_{j=-m}^{m} a_j = 1.
\]
1. TIME SERIES DECOMPOSITION

1.2.5 Local Regression Smoothing and Loess Smoothing

In this section we will examine two further techniques which may be used to smooth a time series. These techniques share the same essential principle that was utilised by each of the moving average procedures.

That principle can be summarised as follows:

1. We are presented with an observation $Y_t$ at time $t$.

2. This observation is composed of trend-cycle $TC_t$, seasonal $S_t$ and random components $R_t$.

3. We wish to replace $Y_t$ by a new value which only contains $TC_t$ and $S_t$.

4. We incorporate information from the observations around $Y_t$ to provide a new estimate $Y_t'$.

What differentiates one smoothing procedure from another is step 4 above: the way in which the information from observations near $Y_t$ is combined to produce the smoothed estimate $Y_t'$.

In the moving average procedures $Y_t'$ is a either a simple or a weighted average of $Y_t$ itself and the $k - 1$ observations adjacent to $Y_t$. 
1. **TIME SERIES DECOMPOSITION**

**Local Regression Smoothing**

The principle behind Local Regression Smoothing is to make use of the fact that the time series $Y_t$ is dependent on time. So instead of replacing $Y_t$ with an average of adjacent values, we make use of these local values in a more sophisticated way explicitly incorporating the time dependence of the time series.

At each time $t$ we will fit a regression line using $Y_t$ itself and and the $k - 1$ observations adjacent to $Y_t$. So for each $t$ we estimate a different *local* regression equation:

$$Y_t' = a(t) + b(t)t.$$  \hfill (1.11)

By virtue of the fact that it incorporates an explicit time dependence, this new estimate $Y_t'$ should then be somewhat superior to the estimates provided by moving averages.

Note the time dependence of the intercept $a(t)$ and slope $b(t)$ indicate that we are not fitting one global regression line to the data but a series of local regression lines.

To compute the estimate $Y_t'$ for a given value of $t$ we estimate $\hat{a}(t)$ and $\hat{b}(t)$ by minimizing the sum of squares

$$Q(a(t), b(t)) = \sum_{j=-m}^{m} (Y_{t+j} - a - b(t + j))^2,$$  \hfill (1.12)

where as before

$$m = \frac{(k - 1)}{2}.$$

We can generalize this procedure by minimizing the weighted sum of squares $Q_w$ instead of $Q$, where

$$Q_w(a(t), b(t)) = \sum_{j=-m}^{m} a_j(Y_{t+j} - a - b(t + j))^2.$$  \hfill (1.13)
1. **TIME SERIES DECOMPOSITION**

**LOESS**

Loess is a more advanced version of Local Regression Smoothing which was developed by Cleveland and his coworkers at AT&T Bell Labs in the late 1980’s [?].

- Loess is an iterative procedure which begins by applying the local regression smoother as described in previous section to produce a set of smoothed estimates $Y'_t$.

- We then compute the residuals $Y_t - Y'_t$.

- Next we compute the local regression again, having adjusted the weights $a_j$ so that reduced weight is placed on observations which produced large residuals $Y_t - Y'_t$ following the first step.

- We repeat this iterative procedure until the estimates $Y'_t$ converge.
1.3 Classical Decomposition

In this section we describe how to decompose a series into its various components.

1.3.1 Additive Decomposition

1. Estimate the Trend-Cycle component $TC_t$.
   - For monthly data, use a centered 12 MA smoother.
   - For quarterly data, use a centered 4 MA smoother.

2. Having estimated $TC_t$ we subtract this from the original series to get $Y_t - TC_t = S_t + R_t$.

3. We now assume that the seasonal variation is constant from one year to the next. Therefore we can estimate the seasonal component $S_t$ for a given month (quarter) by averaging all of the values of $Y_t - TC_t = S_t + R_t$ for the said month (quarter) over the years provided in the data set.

4. Finally we can estimate $R_t$ from

   \[ R_t = Y_t - TC_t - S_t \]  \hspace{1cm} (1.14)
1. **TIME SERIES DECOMPOSITION**

1.3.2 Multiplicative Decomposition

1. Estimate the Trend-Cycle component $TC_t$.

   - For monthly data, use a centered 12 MA smoother.
   - For quarterly data, use a centered 4 MA smoother.

2. Having estimated $TC_t$ we divide this into the original series to get $\frac{Y_t}{TC_t} = S_t \times R_t$

3. We now assume that the seasonal variation is constant from one year to the next. Therefore we can estimate the seasonal component $S_t$ for a given month (quarter) by averaging all of the values of $\frac{Y_t}{TC_t} = S_t \times R_t$ for the said month (quarter) over the years provided in the data set.

4. Finally we can estimate $R_t$ from

   $$R_t = \frac{Y_t}{TC_t \times S_t} \quad (1.15)$$
1. TIME SERIES DECOMPOSITION

1.3.3 Improvements upon Classical Decomposition

- Sometimes it may be that the set of de-trended observations for a given month or quarter contain some outliers. Hence in computing the seasonal component by using a simple average we are including these outliers in our calculations. An improvement may be to compute a trimmed mean instead of a simple mean. So for a given month (quarter) we remove any outliers and then compute the mean of the remaining values. The result then gives our estimate of $S_t$.

- It may also be that the seasonal component varies over time. So from year to year we should compute a different seasonal component $S_t$. Using a simple average will not suffice in this case. Therefore to estimate the seasonal component for a given period $t$ it may be preferable to compute a moving average over the years in the data set instead.
1. TIME SERIES DECOMPOSITION

1.4 US Census Bureau Techniques

Since 1955 the US Census Bureau have developed more and more detailed decomposition techniques. The first method was known as Census II and in 1967 this became X-11 which was further developed by the Canadian statistical office Statistics Canada into X-11-Arima. The current variation is known as X-12-Arima.

To find out about these techniques students are advised to make use of the internet and the library. You may find it helpful to consult the following paper:


Derivation and Properties of the X11-Arima and Census II Linear Filters.