1. Let \( X_1, \ldots, X_n \) be a random sample from the uniform distribution \( U[\theta - \frac{1}{2}, \theta + \frac{1}{2}] \). Show that the MLE of \( \theta \) is any value \( \hat{\theta} \) in the interval \([\max(X_i) - \frac{1}{2}, \min(X_i) + \frac{1}{2}]\). What is the method of moments estimator?

2. The random variable \( X \) has a discrete distribution such that \( P(X = r) = \theta^{-1} \) for \( r = 1, 2, \ldots, \theta \), where \( \theta \) is an unknown positive integer. Show that \( Y \), the maximum of a sample of \( n \) independent observations of \( X \), is a complete sufficient statistic for \( \theta \), and hence verify that
\[
\frac{Y^{n+1} - (Y - 1)^{n+1}}{Y^n - (Y - 1)^n}
\]
is a minimum-variance unbiased estimator for \( \theta \).

3. Consider a binomial experiment with probability of success \( p \) in which \( m \) fixed trials are conducted, resulting in \( R \) successes; a further set of trials is then conducted until \( s \) (fixed) further successes have occurred. The number of trials necessary in the second set is a random variable \( N \). By considering the function
\[
U(R, N) = \frac{R}{m} - \frac{s - 1}{N - 1}
\]
show that \((R, N)\) are jointly sufficient for \( p \), but not complete.

4. (GJJ 2.19) The random variables \( X_1, X_2, \ldots, X_n \) are iid with density \( f(x; \theta) = \theta x^{\theta-1} \) for \( 0 < x < 1 \) and \( \theta > 0 \) unknown.
   (i) Find a sufficient statistic \( T \) for \( \theta \).
   (ii) Given that \( -\log(X_1) \) is unbiased for \( \theta^{-1} \), find another unbiased estimator with smaller variance. Give a simple expression of this estimator involving \( T \).

5. Let \( X = (X_1, \ldots, X_n) \) be a random sample from a density \( f_X(x; \theta) \) belonging to a parametric family \( \mathcal{F} \). Let \( T = t(X) \) be a function of \( X \) and denote the density of \( T \) by \( f_T(t; \theta) \). Assuming statistical regularity, define \( i_X(\theta) \) to be the Fisher information about \( \theta \) in \( X \). Finally, let \( i_X|T(\theta) \) denote the Fisher information conditional on \( T = t \) and define
\[
i_X|T(\theta) = \int i_X|t(\theta)f_T(t; \theta)dt
\]
(a) Show that
\[
i_X(\theta) = i_X|T(\theta) + i_T(\theta)
\]
(b) Show that
\[
i_X(\theta) \geq i_T(\theta),
\]
with equality for all \( \theta \) if and only if \( T = t(X) \) is sufficient for \( \theta \).

*Hint: Use the factorization theorem for the density*
\[
f_X(x; \theta) = f_X|T(x \mid t; \theta)f_T(t; \theta).
\]
(c) Hence, or otherwise, determine the Fisher information about $\theta$ in the first $r$ order statistics

$$X_{(1)} < X_{(2)} < \cdots < X_{(r)}$$

of a sample of size $n$ from the density

$$f(x; \theta) = \theta \exp(-\theta x), \ x > 0$$

6. Suppose $T(x)$ is complete sufficient for $\theta$ given data $x$. Show that if a minimal sufficient statistic $S(x)$ for $\theta$ exists, then $T(x)$ is also minimal sufficient.