

Foundations of Statistical Inference, BS2a, Exercises 2

1. Let $X = (X_1, \dots, X_n)$ be a random sample from a density $f_X(x; \theta)$ belonging to a parametric family \mathcal{F} . Let $T = t(X)$ be a function of X and denote the density of T by $f_T(t; \theta)$. Assuming statistical regularity, define $i_X(\theta)$ to be the Fisher information about θ in X . Finally, let $i_{X|t}(\theta)$ denote the Fisher information conditional on $T = t$ and define

$$i_{X|T}(\theta) = \int i_{X|t}(\theta) f_T(t; \theta) dt$$

(a) Show that

$$i_X(\theta) = i_{X|T}(\theta) + i_T(\theta)$$

(b) Show that

$$i_X(\theta) \geq i_T(\theta),$$

with equality for all θ if and only if $T = t(X)$ is sufficient for θ .

Hint: Use the factorization theorem for the density

$$f_X(x; \theta) = f_{X|T}(x | t; \theta) f_T(t; \theta).$$

(c) Hence, or otherwise, determine the Fisher information about θ in the first r order statistics

$$X_{(1)} < X_{(2)} < \dots < X_{(r)}$$

of a sample of size n from the density

$$f(x; \theta) = \theta \exp(-\theta x), \quad x > 0$$

2. Suppose $T(x)$ is complete sufficient for θ given data x . Show that if a minimal sufficient statistic $S(x)$ for θ exists, then $T(x)$ is also minimal sufficient.

3. The number of phone calls a man receives in a week follows a Poisson distribution with mean θ . At the start of week 1, the man's opinion about the value of θ corresponds to the gamma distribution

$$\pi(\theta) = \frac{1}{54} \theta^2 e^{-\theta/3}, \quad \theta > 0.$$

In the 4 weeks following the start of week 1, the man received 3, 7, 6, and 10 phone calls, respectively. Determine the posterior distribution of θ and the predictive distribution for the number of calls that he will receive in week 5.

4. In order to measure the intensity, θ , of a source of radiation in a noisy environment a measurement X_1 is taken without the source present and a second, independent measurement X_2 is taken with it present. It is known that X_1 is $N(\mu, 1)$ and X_2 is $N(\mu + \theta, 1)$, where μ is the mean noise level. The prior distribution for μ is $N(\mu_0, 1)$ while the prior for θ is constant.

(a) Write down the joint posterior distribution of μ and θ up to a constant of proportionality.

(b) Hence obtain the posterior marginal distribution of θ .

(c) The usual estimate of θ is $x_2 - x_1$ explain why $\frac{1}{2}(2x_2 - x_1 - \mu_0)$ might be better.

5. A function $f(\theta)$ of the parameter is said to be *proper* if its integral is finite

$$\int f(\theta)d\theta < \infty$$

and can thus be normalized to be a probability distribution. Otherwise it is said to be *improper*.

Let X_1, \dots, X_n be a random sample from a normal distribution with unknown mean μ and unknown variance λ^{-1} , with the improper prior

$$\pi_0(\mu, \lambda) = \lambda^{-1}, \lambda > 0, -\infty < \mu < \infty.$$

(a) Find the joint posterior density $\pi(\mu, \lambda|x)$ of (μ, λ) .

(b) Find the marginal posterior density $\pi(\mu|x)$ of μ .

(c) Show that the joint posterior density in (a) is proper.

6. Let $E(a, b)$ be the distribution of the shifted exponential with density

$$\frac{1}{b}e^{-(x-a)/b}, \quad x > a$$

where $a \in \mathbb{R}, b > 0$ are parameters. Let X_1, \dots, X_n be a random sample from the distribution $E(a, b)$.

1. In this part suppose that a is fixed and known.

- (a) Let T be a statistic from the random sample. Write down the definition of *sufficiency, minimality and completeness* of T for a parameter θ .
- (b) Show that this is a one parameter exponential family. Without calculation, give a sufficient and complete statistic for b .
- (c) Find the minimal variance unbiased estimator (MVUE) \hat{b} of b . (hint: remember that if $G \sim \Gamma(u, v)$ then $E(G) = u/v$ and that a sum of independent exponential variables with same means is Gamma distributed)
- (d) Compute the Fisher information for b and compare to the variance of \hat{b} . Does \hat{b} attain the Cramer-Rao lower bound? How could you have inferred this directly from the log-likelihood ℓ ? (Hint: if $G \sim \Gamma(u, v)$ then $Var(G) = u/v^2$).
2. Now we suppose that it is b which is known and a that we wish to estimate.
- (a) Can we still say that the distribution $E(a, b)$ with b known is an exponential family? Briefly justify your answer.
- (b) Find the distribution of $X_{(1)} = \min_{i=1, \dots, n} X_i$?
- (c) Show that $X_{(1)}$ is sufficient and complete.
- (d) Hence deduce the MVUE of a .
3. Suppose now that both a and b are unknown. Define $T_1(X) = X_{(1)}$ and $T_2(X) = \sum_{i=1}^n X_i - X_{(1)}$.
- (a) Fix $i \in \{1, \dots, n\}$. Describe the conditional distribution of $X_i - X_{(1)}$ given $X_{(1)}$.
- (b) Deduce from c.(i) that $T_2(X)$ has a Gamma distribution and give its parameters.
- (c) Show that (T_1, T_2) forms a complete sufficient statistic for (a, b) and thus obtain an MVUE for (a, b) (Hint: you can assume without proof that T_1 and T_2 are independent).

7. From 2015.

1. Let $X = (X_1, \dots, X_n)$ be a sample from a continuous distribution with probability density function $f(x; \theta)$ where θ is an unknown parameter. Let $L(\theta; X)$ be the likelihood function and $\ell(\theta) = \log L(\theta; X)$

- (a) Prove that $\mathbb{E}[\frac{\partial \ell}{\partial \theta}] = 0$.
- (b) If we define $I_\theta = -\mathbb{E}[\frac{\partial^2 \ell}{\partial \theta^2}]$ show that $I_\theta = \mathbb{E}[(\frac{\partial \ell}{\partial \theta})^2] = \text{Var}(\frac{\partial \ell}{\partial \theta})$.
- (c) Show that the variance of an unbiased estimator of θ , denoted $\hat{\theta}(X)$, satisfies the Cramér-Rao inequality

$$\text{Var}[\hat{\theta}(X)] \geq I_\theta^{-1},$$

and provide a brief statement about why regularity conditions are needed for this result to hold.

- (d) Show that there exists an unbiased estimator $\hat{\theta}$ which attains the Cramér-Rao lower bound (under regularity conditions) if and only if

$$\frac{\partial \ell}{\partial \theta} = I_\theta(\hat{\theta} - \theta)$$

- (e) If $\hat{g}(X)$ is an unbiased estimator of the function $g(\theta)$ derive a lower bound for the variance of the estimator.

2. [10 marks] Suppose that

$$f(x; \theta) = \frac{\theta^3 x(x+1)}{\theta+2} e^{-\theta x}, \quad x \geq 0, \quad \theta > 0.$$

Find an unbiased estimator of $(\theta^2 - 6)/[\theta(\theta + 2)]$ whose variance attains the Cramér-Rao lower bound.