Foundations of Statistical Inference

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Lecture 8 : Hierarchical models. Bayesian Hypothesis Testing

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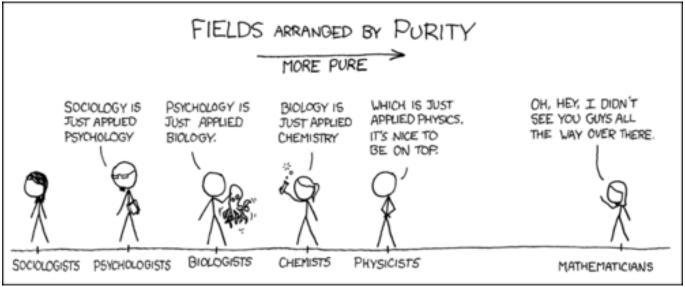

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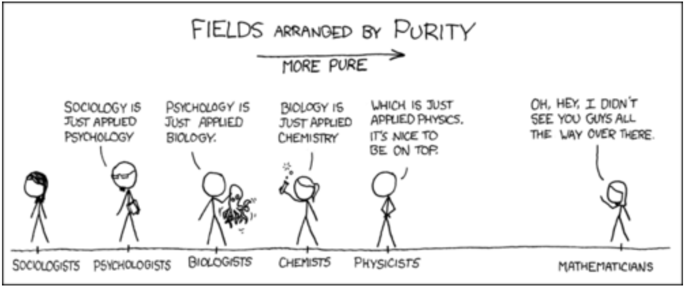

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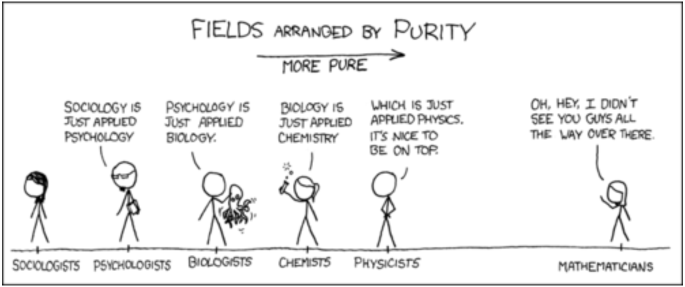

Example: study of the effectiveness of cardiac treatments. $\theta_j =$ survival proba. in hospital j. The θ_j are related. Model as sampled from some common distribution. The data $x_{i,j}$ (survival of patient i in hospital j) are naturally clustered. Other example : meta-analysis.

Example: Meta-analysis

To evaluate a drug for possible clinical application, a study is performed on rodents. For a particular study drawn from literature, the aim is to evaluate θ, the probability of tumor in a control population (no treatment). The data shows that 4 out of 14 rats developed a type of tumor.

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Previous experiments:

0/20	0/20	0/20	0/20	0/20	0/20	0/20	0/19	0/19	0/19
0/19	0/18	0/18	0/17	1/20	1/20	1/20	1/20	1/19	1/19
1/18	1/18	2/25	2/24	2/23	2/20	2/20	2/20	2/20	2/20
2/20	1/10	5/49	2/19	5/46	3/27	2/17	7/49	7/47	3/20
3/20	2/13	9/48	10/50	4/20	4/20	4/20	4/20	4/20	4/20
4/20	10/48	4/19	4/19	4/19	5/22	11/46	12/49	5/20	5/20
6/23	5/19	6/22	6/20	6/20	6/20	16/52	15/47	15/46	9/24

Current experiment:
4/14

Table 5.1 *Tumor incidence in historical control groups and current group of rats, from Tarone (1982). The table displays the values of $\frac{y_j}{n_j}$: (number of rats with tumors)/(total number of rats).*

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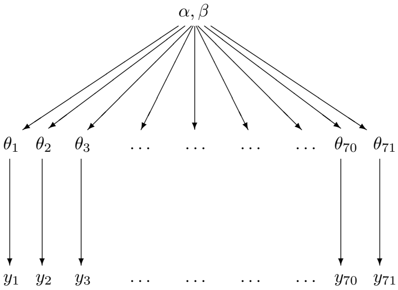

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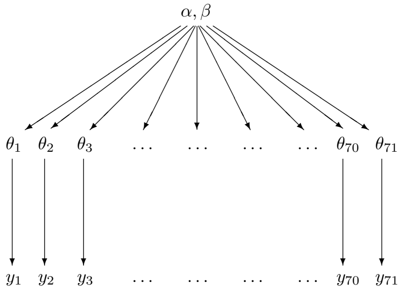

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Pick $\hat{\alpha}, \hat{\beta}$ to match empirical mean variance of the n_j/θ_j. Get $\hat{\alpha} = 1.4, \hat{\beta} = 8.6, p(\theta|y) \sim Beta(5.4, 18.6)$

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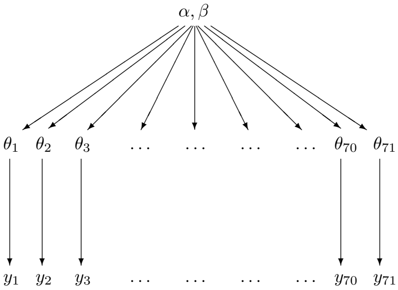

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Posterior mean is 0.223, lower than $4/14 = 0.286$. Current experiment has unusually high number of tumors.

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Setting: J exepriments, observations $y_1, \dots, y_J$ with likelihoods $L(y_j, \theta_j)$. key: specify a full probabilistic model for the θ_j.

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Remark: De Finetti's Theorem states that all exchangeable distributions are of this form in the large sample limit.

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To analyze a hierarchical model :

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For the last step, observe that

$$p(\psi|\theta) = \frac{p(\theta, \psi|y)}{p(\theta|\psi, y)}.$$

Careful about normalizing factors.

Example: Risk of tumors in rats cont'd

Full probability model:

- the y_j are independent with $y_j \sim B(n_j, \theta_j)$.
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$$\propto p(\alpha, \beta) \prod_{j=1}^{J} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta_j^{\alpha-1}(1-\theta_j)^{\beta-1} \prod_{j=1}^{J} \theta_j^{y_j}(1-\theta_j)^{n_j-y_j}.$$

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Step2: Posterior density of θ given (α, β)

$$p(\theta| \alpha, \beta, y) = \prod_{j=1}^{J} \frac{\Gamma(\alpha+\beta+n_j)}{\Gamma(\alpha+y_j)\Gamma(\beta+n_j-y_j)} \theta_j^{\alpha+y_j-1}(1-\theta_j)^{\beta+n_j-y_j-1}$$

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Step3: Posterior of α, β using $p(\phi|y) = p(\theta, \phi|y)/p(\theta|\phi, y)$

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Possible noninformative prior for α, β

- Uniform in α, β ($\rightarrow$ proper posterior?)
- recall that mean is α/β and that $\alpha+\beta$ is 'sample size'. Take log to put on a $(-\infty, \infty)$ scale and then uniform on $(\log(\alpha/\beta), \log(\alpha+\beta))$.

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- reasonable choice of diffuse hyperprior density is uniform on $(\alpha/(\alpha+\beta), (\alpha+\beta)^{-1/2})$ which translates to $p(\alpha, \beta) \propto (\alpha+\beta)^{-5/2}$.

Example: Normal data

For $i = 1, 2, ..., k$ we make n_i observations $X_{i,1}, X_{i,2}, ..., X_{i,n_i}$ on population i, with $X_{ij} \sim N(\theta_i, \sigma^2)$. The θ_i are the unknown means for observations on the i'th population but σ^2 is known.

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- Simple natural idea: $\hat{\theta}_j = \bar{y}_{\cdot j} = \frac{1}{n_j} \sum_{i=1}^{n_j} y_{ij}$
- If the J experiment are very close might prefer $\hat{\theta}_j = \hat{\theta} = \bar{y}_{\cdot\cdot} = \frac{1}{N} \sum_{i,j=1}^{n_j, J} y_{ij}$

To decide which to use, usually ANOVA F-test.

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1. The unpooled estimate $\hat{\theta}_j = \bar{y}_{\cdot j}, \lambda_j = 1$ corresponds to θ_j having independent uniform priors
2. The pooled estimate $\lambda_j = 0$ corresponds to the θ_j restricted to be equal with uniform prior.
3. The weighted estimates $\lambda_j \in (0, 1)$ corresponds to the case where the θ_j are iid normal.

Example: Hierarchical model for normal data

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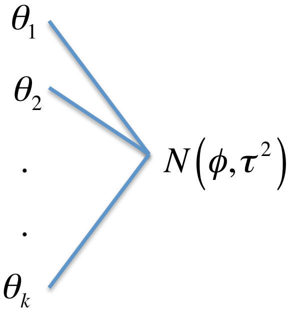
$$X_{1,1}, \ldots, X_{1,n_1} \sim N\left(\theta_1, \sigma^2\right)$$

$$X_{2,1}, \ldots, X_{2,n_2} \sim N\left(\theta_2, \sigma^2\right)$$

$$\vdots$$

$$X_{k,1}, \ldots, X_{k,n_k} \sim N\left(\theta_k, \sigma^2\right)$$



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Now we need a prior for ϕ and τ^2. Suppose we take

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The joint posterior of the parameters is

$$\pi(\theta, \psi | x) \propto f(x; \theta) \pi(\theta|\psi) g(\psi)$$

$$\propto g(\psi) \prod_{i=1}^{J} N(\theta_j | \phi, \tau^2) \prod_{j=1}^{J} N(\bar{y}_{\cdot j} | \theta_j, \sigma_j^2)$$

where $\sigma_j^2 = \sigma^2 / n_j$

Example: Hierarchical model for normal data

Step 2: Now we want to fix ψ and write the conditional posterior of θ. Because conditionally on ψ the θ_j are iid we can treat each θ_j in turn

$$\theta_j | \phi, \tau^2, y \sim N(\hat{\theta}_j, V_j)$$

with

$$\hat{\theta}_j = \frac{\sigma_j^{-2} \bar{y}_{.j} + \tau^{-2} \phi}{\sigma_j^{-2} + \tau^{-2}} \text{ and } V_j = \left(\sigma_j^{-2} + \tau^{-2}\right)^{-1}.$$

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Step 3 Now we go full Bayesian on the hyperparameters.

$$p(\phi, \tau|y) \propto g(\phi, \tau)p(y|\phi, \tau).$$

In general this expression is no help because $p(y|\phi, \tau)$ doesn't have a closed form.

Example: Hierarchical model for normal data

Step 2: Now we want to fix ψ and write the conditional posterior of θ. Because conditionally on ψ the θ_j are iid we can treat each θ_j in turn

$$\theta_j|\phi,\tau^2,y \sim N(\hat{\theta}_j, V_j)$$

with

$$\hat{\theta}_j = \frac{\sigma_j^{-2}\bar{y}_{\cdot j} + \tau^{-2}\phi}{\sigma_j^{-2} + \tau^{-2}} \text{ and } V_j = \left(\sigma_j^{-2} + \tau^{-2}\right)^{-1}.$$

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$$p(\phi,\tau|y) \propto g(\phi,\tau)\prod_{j=1}^{J} N(\bar{y}_{\cdot j}|\phi,\tau^2+\sigma^2).$$

Example: Hierarchical model for normal data

$$p(\phi, \tau | y) \propto g(\phi, \tau) \prod_{j=1}^{J} N(\bar{y}_{\cdot j} | \phi, \tau^2 + \sigma^2).$$

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Start by fixing τ and compute $p(\phi|\tau, y)$. Using that $g(\phi, \tau^2) \propto p(\tau)$ we see that $\log p(\phi|\tau, y)$ is quadratic in ϕ and thus

$$\phi|\tau, y \sim N(\hat{\phi}, V_\phi) \quad \text{where} \quad \hat{\phi} = \frac{\sum_{j=1}^{J} \frac{\bar{y}_{\cdot j}}{\sigma_j^2 + \tau^2}}{\sum_{j=1}^{J} \frac{1}{\sigma_j^2 + \tau^2}} \quad V_\phi^{-1} = \left(\sum_{j=1}^{J} \frac{1}{\sigma_j^2 + \tau^2} \right)^{-1}$$

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Example: Hierarchical model for normal data

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Using $p(\phi, \tau|y) = p(\phi|\tau, y)p(\tau|y)$ we get

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$$p(\tau|y) \propto p(\tau) V_\phi^{1/2} \prod_{j=1}^{J} (\tau^2 + \sigma_j^2)^{-1/2} \exp\left\{ -\frac{(\bar{y}_{\cdot j} - \hat{\phi})^2}{2(\sigma_j^2 + \tau^2)} \right\}$$

Both $\hat{\phi}$ and V_ϕ are functions of τ.

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Using the second approach

$$\begin{aligned}\pi(\theta,\phi,\tau^2|x) \quad &\propto \quad \left[\prod_{j=1}^{J} \exp\left\{-\frac{1}{2\sigma_j^2}(\bar{y}_{\cdot j}-\theta_j)^2\right\}\right] \\ &\times \quad \left[\prod_{j=1}^{J} \tau^{-1}\exp\left\{-\frac{1}{2\tau^2}(\theta_j-\phi)^2\right\}\right]\end{aligned}$$

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Integrate out wrt ϕ and τ^2 to obtain the marginal posterior distribution of θ.

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Exercise Then the integral wrt τ gives a term proportional to

$$\left[\sum (\theta_j - \bar{\theta})^2 \right]^{1-J/2}$$

Example: Hierarchical model for normal data

Thus the posterior distribution of θ is

$$\pi(\theta|x) \quad \propto \quad \left[\prod_{i=1}^{J} \exp\left\{-\frac{1}{2\sigma_j^2}(\bar{y}_{\cdot j} - \theta_j)^2\right\}\right] \cdot \left[\sum(\theta_j - \bar{\theta})^2\right]^{1-J/2}$$

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Let $\widehat{\theta}_j$ be the MAP estimate for θ_j (posterior mode) and put $\widehat{\theta}^* = \sum \widehat{\theta}_j / k$.

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$$\widehat{\theta}_j = (\sigma_j^{-2}\bar{x}_j + \nu\widehat{\theta}^*)/(\sigma_j^{-2} + \nu),$$

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where $\nu = \left[\sum(\widehat{\theta}_i - \widehat{\theta}^*)^2/(J-2)\right]^{-1}$.
If the θ_j were unrelated then $\hat{\theta}_j = \bar{x}_j$. The model modifies the estimate by pulling it towards the mean of the estimated θ_is. Another kind of interpolation model