Foundations of Statistical Inference

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Lecture 7 : Prior distributions. Predictive Distributions. Summarizing inference.

Constructing priors

Subjective Priors: Write down a distribution representing prior knowledge about the parameter before the data is available. If possible, build a model for the parameter. If different scientists have different priors or it is unclear how to represent prior knowledge as a distribution, then consider several different priors. Repeat the analysis and check that conclusions are insensitive to priors representing 'different points of view'.

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Non-Subjective Priors: Several approaches offer the promise of an 'automatic' and even 'objective' prior. We list some suggestions below (Uniform, Jeffreys, MaxEnt). In practice, if one of these priors conflicted prior knowledge, we wouldnt use it. These approaches can be useful to complete the specification of a prior distribution, once subjective considerations have been taken into account.

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Example $X \sim \text{Exp}(1/\mu)$ and $Y = \mathbb{I}(X < 1)$ yielding $Y = y$ with $y \in \{0, 1\}$. Suppose we observe $y = 0$. Now

$$L(\mu; y) = \exp(-1/\mu)$$

so if we take $\pi(\mu) \propto 1$ for $\mu > 0$ we have

$$\pi(\mu|y) \propto \exp(-1/\mu)$$

which is improper, as $\pi(\mu|y) \to 1$ as $\mu \to \infty$ so $\int_0^\infty \exp(-1/\mu) d\mu$ cannot exist.

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Now if $g(\psi) = \theta$ then

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and the rule is consistent in this respect.
It is sometimes desirable (on subjective grounds) to have a prior which is invariant under reparameterization.

Higher dimensions

If $\Theta \subset \mathbb{R}^k$, and $\ell(\theta; X) = \log(f(X;\theta))$, the Fisher information

$$[I_\theta]_{i,j} = -\mathbb{E}_\theta\left(\frac{\partial^2 \ell(\theta; X)}{\partial\theta_i \partial\theta_j}\right)$$

satisfies

$$-\mathbb{E}_\theta\left(\frac{\partial^2 \ell(\theta; X)}{\partial\theta_i \partial\theta_j}\right) = \mathbb{E}_\theta\left(\frac{\partial \ell(\theta; X)}{\partial\theta_i}\frac{\partial \ell(\theta; X)}{\partial\theta_j}\right)$$

subject to regularity conditions. A k-dimensional Jeffreys' prior

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Exercise Verify 1 to 1 $g(\psi) = \theta$ in R^k gives $\pi_\Psi(\psi) = \sqrt{I_{g(\psi)}}\left|\frac{\partial\theta^T}{\partial\psi}\right|$.

Maximum Entropy Priors

Choose a density $\pi(\theta)$ which maximizes the entropy

$$\phi[\pi] = -\int_{\Theta} \pi(\theta) \log \pi(\theta) d\theta$$

over *functions* $\pi(\theta)$ subject to constraints on π. This is a *Calculus of Variations problem*.

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Example The distribution π maximizing $\phi[\pi]$ over all densities π on $\Theta = R$, subject to

$$\int_0^\infty \pi(\theta) d\theta = 1, \quad \int_0^\infty \theta\pi(\theta) d\theta = \mu, \quad \text{and} \quad \int_0^\infty (\theta-\mu)^2 \pi(\theta) d\theta = \sigma^2,$$

(normalized with $\mathbb{E}\vartheta = \mu$ and $\mathrm{Var}(\vartheta) = \sigma^2$) is the normal density

$$\pi(\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\theta-\mu)^2/2\sigma^2}.$$

This is a special case of the following Theorem.

Theorem

The density $\pi(\theta)$ that maximizes $\phi(\pi)$, subject to

$$\mathbb{E}[t_j(\theta)] = t_j,\ j = 1, \dots, p$$

takes the p-parameter exponential family form

$$\pi(\theta) \propto \exp\left\{\sum_{i=1}^{p} \lambda_i t_i(\theta)\right\}$$

for all $\theta \in \Theta$, where $\lambda_1, \dots, \lambda_p$ are determined by the constraints.

(For the proof see Leonard and Hsu).

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(For the proof see Leonard and Hsu). Example In the normal case $t_1(\theta) = \theta$, $E(t_1) = \mu$, $t_2(\theta) = (\theta - \mu)^2$, $E(t_2) = \sigma^2$ gives $\pi(\theta) \propto \exp(\lambda_1\theta + \lambda_2(\theta - \mu)^2)$. Impose the constraints to get $\lambda_1 = 0$ and $\lambda_2 = -1/2\sigma^2$.

Example

Suppose prior probabilities are specified so that

$$P(a_{j-1} < \vartheta \leq a_j) = \phi_j, j = 1, \dots, p$$

with $\sum_j \phi_j = 1$ and

$$\vartheta \in (a_0, a_p),\ a_0 \leq a_1 \cdots \leq a_p \leq a_p.$$

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We find the maximum entropy distribution subject to these conditions. The conditions are equivalent to

$$\mathbb{E}[t_j(\vartheta)] = \phi_j, j = 1, \ldots, p$$

where $t_j(\vartheta) = \mathbb{I}[a_{j-1} < \vartheta \leq a_j]$. The posterior density of ϑ is

$$\pi(\theta) \propto \exp\left\{\sum_{j=1}^{p} \lambda_j \mathbb{I}[a_{j-1} < \theta \leq a_j]\right\},\ a_0 \leq \theta \leq a_p$$

where $\lambda_1, \ldots, \lambda_p$ are determined by the conditions. $\pi(\theta)$ is a histogram, with intervals $(a_0, a_1], (a_1, a_2], \ldots, (a_{p-1}, a_p]$.

Uninformative priors summary

- Uniform prior: $\pi(\theta) \propto \mathit{Cstt}$,
- Jeffrey's prior: $\pi(\theta) \propto \sqrt{I_\theta}$
- Entropy maximization: choose π to maximize $\Phi[\pi] = -\int_\Theta \pi(\theta) \log \pi(\theta) d\theta$ under constraints on π.

Priors for Exponential Families

Conjugate prior for an exponential family

$$f(x \mid \theta) = \exp\left\{ \sum_{j=1}^{k} A_j(\theta) \sum_{i=1}^{n} B_j(x_i) + \sum_{i=1}^{n} C(x_i) + nD(\theta) \right\}$$

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The prior distribution based on sufficient statistics

$$\pi(\theta) \propto \exp\{\tau_0 D(\theta) + \sum_{j=1}^{k} A_j(\theta)\tau_j\}$$

(where $(\tau_0, \ldots, \tau_k)$ are constant prior parameters) is conjugate.

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\begin{aligned}
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& \propto \exp\left\{ \sum_{j=1}^{k} A_j(\theta) \left[\sum_{i=1}^{n} B_j(x_i) + \tau_j \right] + (n + \tau_0) D(\theta) \right\}
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This is an updated form of the prior with

$$\begin{aligned} B_j'(x) &= \sum_{i=1}^{n} B_j(x_i) + \tau_j \\ n' &= n + \tau_0 \end{aligned}$$

Priors for Exponential Families: Example

$X_1, X_2, \ldots$ iid Poisson(θ).

$$p(y|\theta) \propto e^{-n\theta}\theta^{t(y)}, \qquad t(y) = \sum_{i=1}^{n} y_i.$$

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Gamma density with parameters (α, β).
Exercise: check that $p(\theta|y) \sim \text{Gamma}(\alpha + n\bar{y}, \beta + n)$.

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They are used also for model checking. Divide the data in two groups, $Y = (X_1, ..., X_a)$ and $Z = (X_{a+1}, ..., X_n)$. If we fit using Y and check that the 'reserved data' Z overlap $g(x_{n+1} \mid x)$ in distribution.

Poisson example cont'd

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which reduces to

$$p(y) = \binom{\alpha+y-1}{y}\left(\frac{\beta}{\beta+1}\right)^{\alpha}\left(\frac{1}{\beta+1}\right)^{y}, \quad y \sim \text{ Neg-bin } (\alpha,\beta).$$

In other words

$$\text{Neg-bin } (y|\alpha,\beta) = \int \text{Poisson}(y|\theta)\text{Gamma}(\theta|\alpha,\beta)d\theta.$$

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Therefore

$$p(y_{n+1}|y) = \int \text{Poisson}(y|\theta)\text{Gamma}(\theta|\alpha+n\bar{y},\beta+n)d\theta$$
$$\sim \text{ Neg-bin } (y|\alpha+n\bar{y},\beta+n).$$

Example : Normal with known variance

Data $X_1, \ldots, X_n$ are iid $N(\theta, \sigma^2)$ with σ^2 known and prior $\theta \sim N(\mu_0, \sigma_0^2)$.
Predict X_{n+1}.

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Complete the squares to obtain

$$p(\theta|y) = p(\theta|\bar{y}) = N(\theta|\mu_n, \sigma_n^2)$$

where

$$\mu_n = \frac{\sigma_0^{-2}\mu_0 + n\sigma^{-2}\bar{y}}{\sigma_0^{-2} + n\sigma^{-2}} \text{ and } \sigma_n^{-2} = \sigma_0^{-2} + n\sigma^{-2}.$$

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Observe 1) that if $\sigma_0^2 = \sigma^2$ then the prior has same weight as one extra observation! 2) If n large then $p(\theta|y) \approx N(\theta|\bar{y}, \sigma^2/n)$.

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In order to calculate the posterior predictive density for X_{n+1} we need to evaluate

$$g(x_{n+1} \mid x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\theta)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{(\theta-\mu_n)^2}{2\sigma_1^2}} d\theta$$

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$$\theta = \mu_n + \sigma_n Z + \sigma Y.$$

It follows that $X_{n+1} \sim N(\mu_n, \sigma^2 + \sigma_n^2)$ is the posterior predictive density for $X_{n+1}|X_1, ..., X_n$.

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The posterior $p(\theta|y)$ contains all current information.

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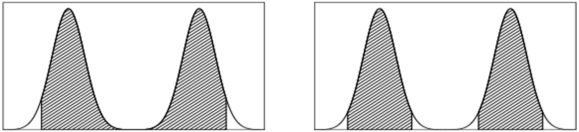
- Standard deviation
- Central interval, highest posterior density interval (HPD).



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$$p(\theta|y) \approx N(\hat{\theta}, [I(\hat{\theta})]^{-1})$$

Where $I(\theta)$ is the observed information

$$I(\theta) = -\frac{d^2}{d\theta^2} \log p(\theta|y)$$

positive definite if $\hat{\theta}$ in the interior of Θ.

Normal approximation: example

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Observe that

$$\log p(\mu, \log \sigma | y) = -n \log \sigma - \frac{1}{2\sigma^2}((n-1)s^2 + n(\bar{y} - \mu)^2) + cstt$$

so that

$$\frac{d}{d\mu} \log p(\mu, \log \sigma | y) = \frac{n(\bar{y} - \mu)}{\sigma^2}$$

$$\frac{d}{d(\log \sigma)} \log p(\mu, \log \sigma | y) = -n + \frac{(n-1)s^2 + n(\bar{y} - \mu)^2}{\sigma^2}$$

so the posterior mode is

$$(\hat{\mu}, \log \hat{\sigma}) = \left(\bar{y}, \log \left(\sqrt{\frac{n-1}{n}} s \right) \right).$$

The second derivatives are

$$\frac{d^2}{d\mu^2}\log p(\mu,\log\sigma|y) = -\frac{n}{\sigma^2}$$

$$\frac{d^2}{d\mu d(\log\sigma)}\log p(\mu,\log\sigma|y) = -2n\frac{(\bar{y}-\mu)}{\sigma^2}$$

$$\frac{d^2}{d(\log\sigma)^2}\log p(\mu,\log\sigma|y) = -\frac{2}{\sigma^2}((n-1)s^2 + n(\bar{y}-\mu)^2)$$

and thus

$$p(\mu,\log\sigma|y) \approx N\left(\left(\begin{array}{c}\mu\\ \log\sigma\end{array}\right)\Big|\left(\begin{array}{c}\bar{y}\\ \log(\hat{\sigma})\end{array}\right),\left(\begin{array}{cc}\hat{\sigma}^2/n & 0\\ 0 & 1/(2n)\end{array}\right)\right)$$

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Exercise: Check that if we had conducted the analysis in terms of $p(\mu,\sigma^2)$ the second derivative matrix would be multiplied by the Jacobian of $\log\sigma \mapsto \sigma^2$ yielding a mode $\tilde{\sigma}^2 = \frac{n}{n+2}\hat{\sigma}^2$, the approx. post. dist would still have independent components with $p(\sigma^2|y) \approx N(\sigma^2|\tilde{\sigma}^2, 2\tilde{\sigma}^4/(n+2))$.

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Furtthermore

$$p(\theta|y) \approx N(\theta_0, (nJ(\theta_0))^{-1})$$

where J is the Fisher information.