Foundations of Statistical Inference

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Lecture 13 : Empirical Bayes.

Empirical Bayes

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In Empirical Bayes, we use Bayesian reasoning to find estimators which can then be used in classical frequentist. EB uses a particular strategy to simplify hierarchical models.

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Recall the setup for Bayesian inference for hierarchical models.

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If we want minimim risk for quadratic loss (for eg) we should use

$$\hat{\theta} = \int\int \theta\pi(\theta,\psi|x)d\theta d\psi$$

Empirical Bayes

EB

The EB trick is to avoid doing ψ-integrals by replacing ψ with an estimate $\hat{\psi}$, derived from the data, and consider the model

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This EB approximation to the full posterior 'chops off' a layer of the hierarchy. The reduced model has posterior

$$\hat{\pi}(\theta|x) \propto L(\theta;x)\pi(\theta;\hat{\psi}),$$

and a Bayes estimator $\hat{\theta}_{EB}$ is calculated using $\hat{\pi}(\theta|x)$. For example, for quadratic loss,

$$\hat{\theta} = \int \theta \hat{\pi}(\theta|x) d\theta.$$

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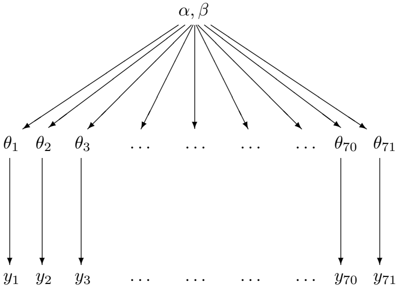
Method of moments estimators are also used.

Example: recall from lecture 8

Study of tumors in rodents:



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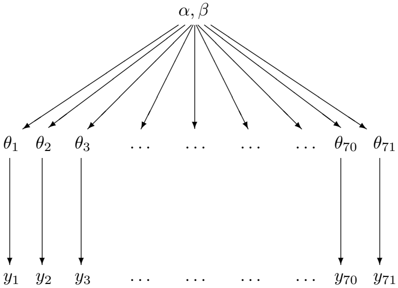

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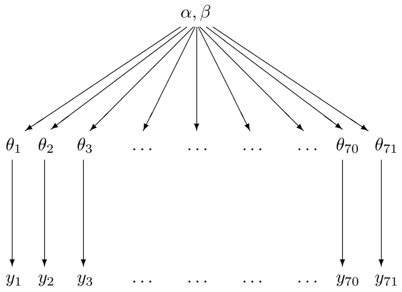

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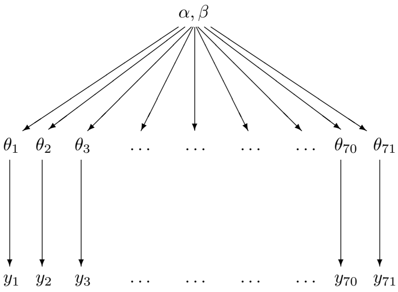

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$$p(\theta|y) = Beta(\alpha + 4, \beta + 10).$$

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Empirical Bayes

Previous experiments:

0/20	0/20	0/20	0/20	0/20	0/20	0/20	0/19	0/19	0/19
0/19	0/18	0/18	0/17	1/20	1/20	1/20	1/20	1/19	1/19
1/18	1/18	2/25	2/24	2/23	2/20	2/20	2/20	2/20	2/20
2/20	1/10	5/49	2/19	5/46	3/27	2/17	7/49	7/47	3/20
3/20	2/13	9/48	10/50	4/20	4/20	4/20	4/20	4/20	4/20
4/20	10/48	4/19	4/19	4/19	5/22	11/46	12/49	5/20	5/20
6/23	5/19	6/22	6/20	6/20	6/20	16/52	15/47	15/46	9/24

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4/14

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Moment estimators: Pick $\hat{\alpha}, \hat{\beta}$ to match empirical mean variance of the n_j/θ_j. Get $\hat{\alpha} = 1.4, \hat{\beta} = 8.6, p(\theta|y) \sim Beta(5.4, 18.6)$

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Posterior mean is 0.223, lower than $4/14 = 0.286$. Current experiment has unusually high number of tumors.

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If we knew τ we would have (completing the square - see Lecture 7 p16)

$$\theta_i | (x_i, \tau) \sim N\left(\frac{x_i \tau^2}{1+\tau^2}, \frac{\tau^2}{1+\tau^2}\right).$$

Example

To get an estimate for τ we compute the marginal distribution for X_i given τ, which is $X_i \sim N(0, \tau^2 + 1)$. The MLE for τ is then $\hat{\tau}^2 = \frac{1}{p} \sum_i X_i^2 - 1$, and this gives

$$\begin{aligned} \hat{\theta}_{EB,i} &= \frac{X_i \hat{\tau}^2}{1 + \hat{\tau}^2} \\ &= \left(1 - \frac{p}{\sum_i X_i^2}\right) X_i \end{aligned}$$

which is the James-Stein estimator

$$\hat{\theta}_{JS,i} = \left(1 - \frac{a}{\sum_i X_i^2}\right) X_i$$

with $a = p$. This isn't the minimum risk JS estimator for quadratic loss, (that is $a = p - 2$) but it already beats the MLE for all θ. (to get the best JS estimator (with $a = p - 2$) use a method of moments estimator for τ. See Young and Smith Section 3.5)

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Suppose the prior for θ_i's is iid Exponential(λ) i.e. $\pi(\theta_i|\lambda) = \lambda e^{-\lambda\theta_i}$.

$$p(x_i \mid \lambda) = \int_0^\infty \frac{e^{-\theta_i}\theta_i^{x_i}}{x_i!}\lambda e^{-\lambda\theta_i} d\theta_i$$

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The MLE of λ based on $x_1, \ldots, x_n$ is $\widehat{\lambda} = 1/\bar{x}$, where $\bar{x} = \frac{1}{n}\sum_1^n x_i$.

Example: Poisson

Now, under the EB simplification, set $\lambda = \hat{\lambda}$, so that

$$\hat{\pi}(\theta|x) \propto L(\theta; x)\pi(\theta|\hat{\lambda}) \; = \; \prod_i e^{-\theta_i} \theta_i^{x_i} \hat{\lambda} e^{-\hat{\lambda}\theta_i}$$

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We can rewrite this

$$\hat{\theta}_{EB,i} = x_i + \frac{\bar{x}}{\bar{x} + 1}(\bar{x} - x_i)$$

showing that this EB estimator shrinks the estimates towards the common mean.

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Suppose $X \sim \text{Poisson}(\theta)$ and $\theta \sim \Gamma(\alpha, \beta)$. Using

$$\begin{aligned} E[X] &= E_\psi[E_\theta(X|\theta)] \\ V(X) &= E_\psi[V_\theta(X|\theta)] + V_\psi[E_\theta(X|\theta)] \end{aligned}$$

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Hence, if $\bar{x}$ and s^2 are the sample mean and variance, the method of moments yields

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\begin{aligned}
\hat{\alpha} &= \bar{x}^2/(s^2 - \bar{x}) \\
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Estimate p from data $x_1, \dots, x_k$ by maximising

$$L(p; x) = \prod_{i=1}^{k} \{pf(x_i, \theta_0^*) + (1-p)f(x_i, \theta_1^*)\}$$

so $\bar{p}$ solves

$$\sum_{i=1}^{k} \frac{f(x_i, \theta_0^*) - f(x_i, \theta_1^*)}{pf(x_i, \theta_0^*) + (1-p)f(x_i, \theta_1^*)} = 0.$$

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Exercise: If $E[X] = \theta$ then the method of moments yields that $\bar{p}$ solves

$$\bar{x} = p\theta_0^* + (1 - p)\theta_1^*.$$

Hypothesis testing

For a, b loss function reject if

$$\frac{L(\theta_1^*, x_k)}{L(\theta_0^*, x_k)} \geq \frac{\hat{p}a}{(1-\hat{p})b}.$$

Example: For $i = 1, 2, \dots$ suppose $X_i \sim N(\theta_i, \sigma^2)$ with σ^2 known. $H_0 : \theta_k = \mu_0$ against $H_1 : \theta_k = \mu_1$ with $\mu_1 > \mu_0$. We reject H_0 if

$$x_k > \frac{1}{2}(\mu_0 + \mu_1) + \frac{\sigma^2}{\mu_1 - \mu_2} / \log\left(\frac{\hat{p}a}{(1-\hat{p})b}\right)$$

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$$\begin{aligned}
\hat{\theta}_i(y) &= E[\theta_i|\underline{y}] = E[\theta_i|y_i] \\
&= \frac{\int u^{y_i+1} e^{-u} \pi(du)}{\int u^{y_i} e^{-u} \pi(du)} \\
&= \frac{(y_i+1)p(y_i+1)}{p(y_i)}
\end{aligned}$$

Non-parametric EB

Assume only that the θ_i are iid from some distribution π. Use the data to estimate the prior or the marginal distribution directly.
(pioneered/championed by Robbins 1950âs; actually older than PEB)

Model: $y_i|\theta_i \sim f(y_i|\theta_i) = \text{Poi.}(\theta_i)$ and $\theta_i \overset{iid}{\sim} \pi(\cdot)$
Square error loss $\Rightarrow$ Bayes estimator = post. mean

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The Robins miracle: $\hat{\theta}_i$ is directly estimable as

$$\hat{\theta}_i = \frac{(y_i+1)\hat{p}(y_i+1)}{\hat{p}(y_i)} = \frac{(y_i+1)[\#y's = (y_i+1)]}{[\#y's = y_i])}$$

Non-parametric EB

Intermediate approach: let θ have a pmf

$$\pi(\theta = \phi_j) = p_j$$

for $j = 1, \ldots, m$ with $\sum p_j = 1$.

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As $m \to \infty$ the cdf converges to the thrue cdf of θ.
When the p_j are fixed, can use the EM algorithm to determine $\phi_1 \leq \phi_2 \leq \ldots \leq \phi_m$.

$$L(\phi, x) = \prod_{i=1}^{k} \left\{ \sum_{j=1}^{m} f(x_i; \phi_j) p_j \right\}$$

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2. Non-parametric EB: suppose θ_i iid $\pi(\cdot)$ and estimate $\hat{\pi}$ fom data.