Foundations of Statistical Inference

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Lecture 12 : Stein's paradox and the James-Stein estimator. Empirical Bayes

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- MLE
- MVUE
- Is it admissible? (for quadratic loss function say)

Recall that $\hat{\mu}$ is inadmissible if we can find $\tilde{\mu}$ such that

$$R(\mu, \hat{\mu}) \geq R(\mu, \tilde{\mu}),\ \forall \mu$$

with strict inequality for some μ.

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$\hat{\mu}$ is inadmissible for quadratic loss!

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Theorem

An estimator with lower risk is given by the James-Stein estimator

$$\hat{\mu}_{JSE} = \left(1 - \frac{p-2}{\sum_i X_i^2}\right) X$$

Implications of Stein's Paradox

Suppose we are interested in estimating

1. the weight of a randomly chosen loaf of bread from a supermarket.
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The James-Stein estimator tells us that we can get a better estimate (on average) for the vector of three parameters by simultaneously using the three unrelated measurements.

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Consider the alternative estimator

$$\hat{\mu}_{JSE} = \left(1 - \frac{a}{\sum_i X_i^2}\right) X \quad \text{(the James-Stein estimator)}$$

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We will show that if $a = p - 2$ then $R(\mu, \hat{\mu}_{JSE}) < R(\mu, \hat{\mu}_{MLE})$ for every $\mu \in R^n$, so that the MLE is inadmissable in this case.

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recognizing $\mathrm{Var}(X_i) = 1$.

Stein's Lemma

Lemma (Stein's Lemma)

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This can be shown by integrating by parts. Noting if $f(x) = -e^{-(x-\mu)^2/2}$ then $f'(x) = (x-\mu)e^{-(x-\mu_i)^2/2}$

$$\int (x_i - \mu)e^{-(x_i-\mu_i)^2/2}dx = -e^{-(x_i-\mu_i)^2/2}$$

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and thus we have

$$\int_{-\infty}^{\infty} (x_i - \mu_i) h(x) e^{-(x_i-\mu_i)^2/2} dx_i \quad = \quad -h(x) e^{-(x_i-\mu_i)^2/2} \Big|_{x_i=-\infty}^{x_i=\infty} + \int_{-\infty}^{\infty} \frac{\partial h(x)}{\partial x_i} e^{-(x_i-\mu_i)^2/2} dx_i$$

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The first term is zero if $h(x)$ (for eg) is bounded, giving the lemma.

Proof (continued)

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$$= \mathbb{E}\left(\frac{\sum_j X_j^2 - 2X_i^2}{(\sum_j X_j^2)^2}\right) = \mathbb{E}\left(\frac{1}{\sum_j X_j^2} - 2\frac{X_i^2}{(\sum_j X_j^2)^2}\right)$$

Proof (continued)

Putting the pieces together,

$$\sum_{i=1}^{p} \mathbb{E}(|\mu_i - \hat{\mu}_i|^2) \quad = \quad R(\mu, \hat{\mu}_{MLE}) - (2ap - 4a)\mathbb{E}\left(\frac{1}{\sum_j X_j^2}\right) + a^2\mathbb{E}\left(\frac{1}{\sum_j X_j^2}\right)$$

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and this is less than p if $2ap-4a-a^2>0$ and in particular at $a=p-2$, which minimizes the risk over $a\in R$.

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Further reading Young and Smith sec 3.4 which covers the James-Stein estimator is worth reading. It includes a nice example (sec 3.4.1) on the application of the estimator to estimation of baseball home run rates.

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If $\mu_i = \lambda \Rightarrow X_i = \lambda + Z_i$ where $Z_i \sim \mathrm{N}(0,1)$ and $\sum_j X_j^2 \sim \lambda^2 + \chi_p^2$

$\Rightarrow \mathbb{E}\left(\frac{1}{\sum_j X_j^2}\right) \to 0$ as $\lambda \to \infty$ so $R(\mu, \hat{\mu}_{JSE}) \to p$.

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So we get a smaller difference between $R(\mu, \hat{\mu}_{MLE})$ and $R(\mu, \hat{\mu}_{JSE})$ as $\mathbb{E}\left(\frac{1}{\sum_j X_j^2}\right)$ gets smaller.

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There is nothing special with the origin. Fix $\mu_0 \in R^p$ and define

$$\hat{\mu}_{JSE}^{(\mu_0)} = \mu_0 + \left(1 - \frac{p-2}{\|X - \mu_0\|^2}\right)(X - \mu_0).$$

As $R(\hat{\mu}_{JSE}^{(\mu_0)}, \mu - \mu_0) = R(\hat{\mu}_{JSE}, \mu)$, it is also strictly beter than X.

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Exercise A better estimator is $\bar{X}\mathbf{1}_p + \left(1 - \frac{a}{V}\right)(X - \bar{X}\mathbf{1}_p)$ where $V = \sum_{j-1}^{p}(X_j - \bar{X})^2$ and $\mathbf{1}_p$ is p-dimensional vector of 1's.

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Note that the shrinkage factor becomes negative when $\|X - \mu_0\|^2 < p - 2$. It can be shown that

$$\hat{\mu}_{JSE+}^{(\mu_0)} = \mu_0 + \left(1 - \frac{p-2}{\|X - \mu_0\|^2}\right)_+ (X - \mu_0).$$

dominates strictly $\hat{\mu}_{JSE}^{(\mu_0)}$.

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Although Stein's result is very clean to state and prove, it may seem somewhat removed from practical statistical problems. Nevertheless, the idea at the heart of Stein's proposal, namely that of employing shrinkage to reduce variance (at the expense of introducing bias) turns out to be a very powerful one that has had a huge impact on statistical methodology.

The baseball example

Player	n_i	Z_i	π_i
Baines	415	0.284	0.289
Barfield	476	0.246	0.256
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To see this

$$
\begin{aligned}
X_i - \theta_i &= g(Z_i) - g(\pi_i) \simeq g'(\pi_i)(Z_i - \pi_i) \\
&= = \frac{\sqrt{n_i}(Z_i - \pi_i)}{\sqrt{\pi_i(1-\pi_i)}}.
\end{aligned}
$$

The baseball example

So $\|X - \theta\|^2 = 2.56 < 9$.

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Using $\theta_0 = \sqrt{\bar{n}}\sin^{-1}(2\pi_0 - 1)$ with $\pi_0 = 0.275$ we get

$$\|\hat{\theta}^{(\theta_0)}_{JSE+} - \theta\|^2 = 1.50.$$

The baseball example 2

	Y_i	n_i	p_i	AB	X_i	JS_i	μ_i	HR	$\hat{HR}$	$\hat{HR}_s$
McGwire	7	58	0.138	509	−6.56	−7.12	−6.18	70	61	50
Sosa	9	59	0.103	643	−5.90	−6.71	−7.06	66	98	75
Griffey	4	74	0.089	633	−9.48	−8.95	−8.32	56	34	43
Castilla	7	84	0.071	645	−9.03	−8.67	−9.44	46	54	61
Gonzalez	3	69	0.074	606	−9.56	−9.01	−8.46	45	26	35
Galaragga	6	63	0.079	555	−7.49	−7.71	−7.94	44	53	48
Palmeiro	2	60	0.070	619	−9.32	−8.86	−8.04	43	21	28
Vaughn	10	54	0.066	609	−5.01	−6.15	−7.73	40	113	78
Bonds	2	53	0.067	552	−8.59	−8.40	−7.62	37	21	24
Bagwell	2	60	0.063	540	−9.32	−8.86	−8.23	34	18	24
Piazza	4	66	0.057	561	−8.72	−8.48	−8.84	32	34	38
Thome	3	66	0.068	440	−9.27	−8.83	−8.47	30	20	25
Thomas	2	72	0.050	585	−10.49	−9.59	−9.52	29	16	28
T. Martinez	5	64	0.053	531	−8.03	−8.05	−8.86	28	41	41
Walker	3	42	0.051	454	−6.67	−7.19	−7.24	23	32	24
Burks	2	38	0.042	504	−6.83	−7.29	−7.15	21	27	19
Buhner	6	58	0.062	244	−6.98	−7.38	−8.15	15	25	21

$Y_i = \#$ home runs in pre-season, $n_i = \#$ times at bat, $p_i =$ true full-season strike rate.

The baseball example 2

	Y_i	n_i	p_i	AB	X_i	JS_i	μ_i	HR	$\hat{HR}$	$\hat{HR}_s$
McGwire	7	58	0.138	509	−6.56	−7.12	−6.18	70	61	50
Sosa	9	59	0.103	643	−5.90	−6.71	−7.06	66	98	75
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$Y_i = \#$ home runs in pre-season, $n_i = \#$ times at bat, $p_i =$ true full-season strike rate.
Naive estimator is $\hat{p}_i = Y_i/n_i$.

The baseball example 2

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As before define $f_n(y) = n^{1/2} \sin^{-1}(2y - 1)$ and $X_i = f_{n_i}(Y_i/n_i), \theta_i = f_{n_i}(p_i)$. so that $X_i \sim N(\theta_i, 1)$.

The baseball example 2

Use the estimator

$$JS_i = \bar{X} + (1 - (p-3)/V)(X_i - \bar{X})$$

where $V = \|X - \bar{X}\|^2 = \sum(X_i - \bar{X})^2$ and $\bar{X} = \frac{1}{p}\sum X_i$. The true θ_i must be clustered more closely around their mean than the X_i.

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$\sum(X_i - \theta_i)^2 = 19.68$ compared with $\sum(JS_i - \theta_i)^2 = 8.07$.

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HR is actual # of home runs in the whole season, $\hat{HR}$ is just the extrapolation from the pre-season, $\hat{HR}_s$ is the prediction based on the JS estimator.

The baseball example 2

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HR is actual $\#$ of home runs in the whole season, $\hat{HR}$ is just the extrapolation from the pre-season, $\hat{HR}_s$ is the prediction based on the JS estimator. does better on aggregate