Foundations of Statistical Inference

Julien Berestycki

Department of Statistics
University of Oxford

MT 2016

Lecture 11 : Finding minimax rules. Hypothesis testing with loss functions.

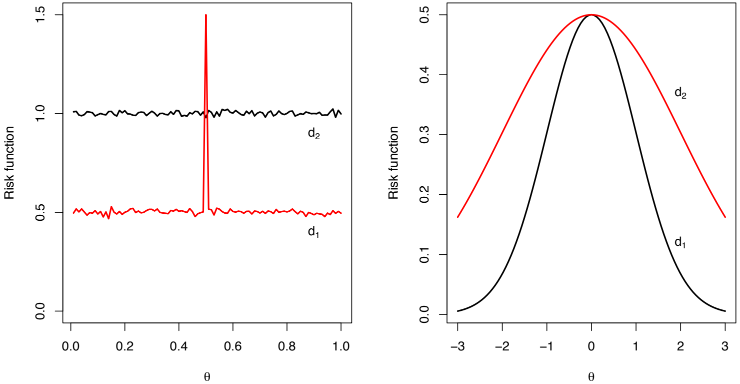
Minimax rules

Definition A rule δ is a minimax rule if $\max_\theta R(\theta, \delta) \leq \max_\theta R(\theta, \delta')$ for any other rule δ'. It minimizes the maximum risk.

Minimax rules

Definition A rule δ is a minimax rule if $\max_\theta R(\theta,\delta) \le \max_\theta R(\theta,\delta')$ for any other rule δ'. It minimizes the maximum risk.

Sometimes this doesn't produce a sensible choice of decision rule.



Finding minimax rules

Theorem 1 If δ is a Bayes rule for prior π, with $r(\pi, \delta) = C$, and δ_0 is a rule for which $\max_\theta R(\theta, \delta_0) = C$, then δ_0 is minimax.

Finding minimax rules

Theorem 1 If δ is a Bayes rule for prior π, with $r(\pi, \delta) = C$, and δ_0 is a rule for which $\max_\theta R(\theta, \delta_0) = C$, then δ_0 is minimax.

Proof If for some other rule δ', $\max_\theta R(\theta, \delta') = C - \epsilon$ for some $\epsilon > 0$ (so δ_0 is not minimax), then

$$r(\pi, \delta') = \int R(\theta, \delta')\pi(\theta) d\theta$$

Finding minimax rules

Theorem 1 If δ is a Bayes rule for prior π, with $r(\pi, \delta) = C$, and δ_0 is a rule for which $\max_\theta R(\theta, \delta_0) = C$, then δ_0 is minimax.

Proof If for some other rule δ', $\max_\theta R(\theta, \delta') = C - \epsilon$ for some $\epsilon > 0$ (so δ_0 is not minimax), then

$$\begin{aligned} r(\pi, \delta') &= \int R(\theta, \delta')\pi(\theta)d\theta \\ &\leq \int (C - \epsilon)\pi(\theta)d\theta \end{aligned}$$

Finding minimax rules

Theorem 1 If δ is a Bayes rule for prior π, with $r(\pi, \delta) = C$, and δ_0 is a rule for which $\max_\theta R(\theta, \delta_0) = C$, then δ_0 is minimax.

Proof If for some other rule δ', $\max_\theta R(\theta, \delta') = C - \epsilon$ for some $\epsilon > 0$ (so δ_0 is not minimax), then

$$\begin{aligned} r(\pi, \delta') &= \int R(\theta, \delta')\pi(\theta)d\theta \\ &\leq \int (C - \epsilon)\pi(\theta)d\theta \\ &= (C - \epsilon) \end{aligned}$$

Finding minimax rules

Theorem 1 If δ is a Bayes rule for prior π, with $r(\pi, \delta) = C$, and δ_0 is a rule for which $\max_\theta R(\theta, \delta_0) = C$, then δ_0 is minimax.

Proof If for some other rule δ', $\max_\theta R(\theta, \delta') = C - \epsilon$ for some $\epsilon > 0$ (so δ_0 is not minimax), then

$$\begin{aligned} r(\pi, \delta') &= \int R(\theta, \delta')\pi(\theta)d\theta \\ &\leq \int (C - \epsilon)\pi(\theta)d\theta \\ &= (C - \epsilon) \\ &< r(\pi, \delta) \end{aligned}$$

Finding minimax rules

Theorem 1 If δ is a Bayes rule for prior π, with $r(\pi, \delta) = C$, and δ_0 is a rule for which $\max_\theta R(\theta, \delta_0) = C$, then δ_0 is minimax.

Proof If for some other rule δ', $\max_\theta R(\theta, \delta') = C - \epsilon$ for some $\epsilon > 0$ (so δ_0 is not minimax), then

$$\begin{aligned} r(\pi, \delta') &= \int R(\theta, \delta')\pi(\theta)d\theta \\ &\leq \int (C - \epsilon)\pi(\theta)d\theta \\ &= (C - \epsilon) \\ &< r(\pi, \delta) \end{aligned}$$

so δ is not the Bayes rule for π, a contradiction

Finding minimax rules

Theorem 1 If δ is a Bayes rule for prior π, with $r(\pi, \delta) = C$, and δ_0 is a rule for which $\max_\theta R(\theta, \delta_0) = C$, then δ_0 is minimax.

Proof If for some other rule δ', $\max_\theta R(\theta, \delta') = C - \epsilon$ for some $\epsilon > 0$ (so δ_0 is not minimax), then

$$\begin{aligned} r(\pi, \delta') &= \int R(\theta, \delta')\pi(\theta)d\theta \\ &\leq \int (C - \epsilon)\pi(\theta)d\theta \\ &= (C - \epsilon) \\ &< r(\pi, \delta) \end{aligned}$$

so δ is not the Bayes rule for π, a contradiction. [*This is an informal treatment which assumes the min and max exist - see Y&S Ch 2 Sec 2.6*]

Finding minimax rules

Theorem 2 If δ is a Bayes rule for prior π with the property that $R(\theta, \delta)$ does not depend on θ, then δ is minimax.

Finding minimax rules

Theorem 2 If δ is a Bayes rule for prior π with the property that $R(\theta, \delta)$ does not depend on θ, then δ is minimax.

Proof (Y&S Ch 2) Let $R(\theta, \delta) = C$ (no θ dependence). This implies

$$r(\pi, \delta) = \int R(\theta, \delta)\pi(\theta)d\theta = C.$$

Finding minimax rules

Theorem 2 If δ is a Bayes rule for prior π with the property that $R(\theta, \delta)$ does not depend on θ, then δ is minimax.

Proof (Y&S Ch 2) Let $R(\theta, \delta) = C$ (no θ dependence). This implies

$$r(\pi, \delta) = \int R(\theta, \delta)\pi(\theta)d\theta = C.$$

If for some other rule δ', $\max_\theta R(\theta, \delta') = C - \epsilon$ for some $\epsilon > 0$ (so δ_0 is not minimax), then we have $r(\pi, \delta') \leq C - \epsilon$.

Finding minimax rules

Theorem 2 If δ is a Bayes rule for prior π with the property that $R(\theta, \delta)$ does not depend on θ, then δ is minimax.

Proof (Y&S Ch 2) Let $R(\theta, \delta) = C$ (no θ dependence). This implies

$$r(\pi, \delta) = \int R(\theta, \delta)\pi(\theta)d\theta = C.$$

If for some other rule δ', $\max_\theta R(\theta, \delta') = C - \epsilon$ for some $\epsilon > 0$ (so δ_0 is not minimax), then we have $r(\pi, \delta') \leq C - \epsilon$. But $r(\pi, \delta) = C$ so δ is not the Bayes rule for π, a contradiction.

Finding minimax rules

Theorem 2 tells us that

The Bayes estimator with constant risk is minimax.

Finding minimax rules

Theorem 2 tells us that

The Bayes estimator with constant risk is minimax.

This result is useful, as it gives an approach to finding minimax rules.

Finding minimax rules

Theorem 2 tells us that

The Bayes estimator with constant risk is minimax.

This result is useful, as it gives an approach to finding minimax rules.

Bayes rules are sometimes easy to compute, so if we find a prior that yields a Bayes rule with constant risk for all θ we have the minimax rule.

Finding minimax rules

Theorem 2 tells us that

The Bayes estimator with constant risk is minimax.

This result is useful, as it gives an approach to finding minimax rules.

Bayes rules are sometimes easy to compute, so if we find a prior that yields a Bayes rule with constant risk for all θ we have the minimax rule.

Example : minimax estimator for quadratic loss

X is Binomial (n, θ), and the prior $\pi(\theta)$ is a Beta (α, β) distribution. For a quadratic loss function, the Bayes estimator is $(\alpha + X)/(\alpha + \beta + n)$

Example : minimax estimator for quadratic loss

X is Binomial (n, θ), and the prior $\pi(\theta)$ is a Beta (α, β) distribution. For a quadratic loss function, the Bayes estimator is $(\alpha + X)/(\alpha + \beta + n)$

The risk function is

$$\mathbb{E}_{X|\theta}[(\widehat{\theta} - \theta)^2]$$

Example : minimax estimator for quadratic loss

X is Binomial (n, θ), and the prior $\pi(\theta)$ is a Beta (α, β) distribution. For a quadratic loss function, the Bayes estimator is $(\alpha + X)/(\alpha + \beta + n)$

The risk function is

$$\mathbb{E}_{X|\theta}[(\widehat{\theta} - \theta)^2] \quad = \quad \text{MSE}(\widehat{\theta}) = [\text{Bias}(\widehat{\theta})]^2 + \text{Var}[\widehat{\theta}]$$

Example : minimax estimator for quadratic loss

X is Binomial (n, θ), and the prior $\pi(\theta)$ is a Beta (α, β) distribution. For a quadratic loss function, the Bayes estimator is $(\alpha + X)/(\alpha + \beta + n)$

The risk function is

$$\begin{aligned} \mathbb{E}_{X|\theta}[(\widehat{\theta} - \theta)^2] &= \text{MSE}(\widehat{\theta}) = [\text{Bias}(\widehat{\theta})]^2 + \text{Var}[\widehat{\theta}] \\ &= \left[\theta - \mathbb{E}\left(\frac{\alpha + X}{\alpha + \beta + n}\right)\right]^2 + \text{Var}\left[\frac{\alpha + X}{\alpha + \beta + n}\right] \end{aligned}$$

Example : minimax estimator for quadratic loss

X is Binomial (n, θ), and the prior $\pi(\theta)$ is a Beta (α, β) distribution. For a quadratic loss function, the Bayes estimator is $(\alpha + X)/(\alpha + \beta + n)$

The risk function is

$$\begin{aligned}
\mathbb{E}_{X|\theta}[(\widehat{\theta} - \theta)^2] &= \text{MSE}(\widehat{\theta}) = [\text{Bias}(\widehat{\theta})]^2 + \text{Var}[\widehat{\theta}] \\
&= \left[\theta - \mathbb{E}\left(\frac{\alpha + X}{\alpha + \beta + n}\right)\right]^2 + \text{Var}\left[\frac{\alpha + X}{\alpha + \beta + n}\right] \\
&= \left[\theta - \left(\frac{\alpha + n\theta}{\alpha + \beta + n}\right)\right]^2 + \frac{n\theta(1 - \theta)}{(\alpha + \beta + n)^2}
\end{aligned}$$

Example : minimax estimator for quadratic loss

X is Binomial (n, θ), and the prior $\pi(\theta)$ is a Beta (α, β) distribution. For a quadratic loss function, the Bayes estimator is $(\alpha + X)/(\alpha + \beta + n)$

The risk function is

$$\begin{aligned}
\mathbb{E}_{X|\theta}[(\widehat{\theta} - \theta)^2] &= \text{MSE}(\widehat{\theta}) = [\text{Bias}(\widehat{\theta})]^2 + \text{Var}[\widehat{\theta}] \\
&= \left[\theta - \mathbb{E}\left(\frac{\alpha + X}{\alpha + \beta + n}\right)\right]^2 + \text{Var}\left[\frac{\alpha + X}{\alpha + \beta + n}\right] \\
&= \left[\theta - \left(\frac{\alpha + n\theta}{\alpha + \beta + n}\right)\right]^2 + \frac{n\theta(1-\theta)}{(\alpha + \beta + n)^2} \\
&= \frac{[\theta(\alpha + \beta) - \alpha]^2 + n\theta(1-\theta)}{[\alpha + \beta + n]^2}
\end{aligned}$$

Example : minimax estimator for quadratic loss

X is Binomial (n, θ), and the prior $\pi(\theta)$ is a Beta (α, β) distribution. For a quadratic loss function, the Bayes estimator is $(\alpha + X)/(\alpha + \beta + n)$

The risk function is

$$\begin{aligned}
\mathbb{E}_{X|\theta}[(\widehat{\theta} - \theta)^2] &= \text{MSE}(\widehat{\theta}) = [\text{Bias}(\widehat{\theta})]^2 + \text{Var}[\widehat{\theta}] \\
&= \left[\theta - \mathbb{E}\left(\frac{\alpha + X}{\alpha + \beta + n}\right)\right]^2 + \text{Var}\left[\frac{\alpha + X}{\alpha + \beta + n}\right] \\
&= \left[\theta - \left(\frac{\alpha + n\theta}{\alpha + \beta + n}\right)\right]^2 + \frac{n\theta(1-\theta)}{(\alpha + \beta + n)^2} \\
&= \frac{[\theta(\alpha + \beta) - \alpha]^2 + n\theta(1-\theta)}{[\alpha + \beta + n]^2}
\end{aligned}$$

The Bayes estimator with constant risk is minimax. This occurs when $\alpha = \beta = \sqrt{n}/2$, so the minimax estimator using quadratic loss is $(\alpha + x)/(\alpha + \beta + n) = (x + \sqrt{n}/2)/(n + \sqrt{n})$.

Hypothesis testing with loss functions

Suppose $X_i \sim f(x;\theta)$ iid for $i = 1, 2, ..., n$ and we wish to test $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$.

Hypothesis testing with loss functions

Suppose $X_i \sim f(x;\theta)$ iid for $i = 1, 2, ..., n$ and we wish to test $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$.

The decision rule is δ_C when we use a critical region C so $\delta_C(x) = H_1$ if $x \in C$ and otherwise $\delta_C(x) = H_0$.

Hypothesis testing with loss functions

Suppose $X_i \sim f(x;\theta)$ iid for $i = 1, 2, ..., n$ and we wish to test $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$.

The decision rule is δ_C when we use a critical region C so $\delta_C(x) = H_1$ if $x \in C$ and otherwise $\delta_C(x) = H_0$.

Loss function:

$$L_S(\theta, \delta_C(x)) = \begin{cases} a & \theta = \theta_0, x \in C \\ 0 & \theta = \theta_0, x \notin C \\ b & \theta = \theta_1, x \notin C \\ 0 & \theta = \theta_1, x \in C \end{cases}$$

so the loss for Type I error is a (H_0 holds and we accept H_1) and the loss for Type II error is b (H_1 holds and we accept H_0).

Hypothesis testing with loss functions

The risk function for the rule δ_C is

$$R(\theta_0; \delta_C) \quad = \quad \int L_S(\theta_0, \delta_C(x)) f(x; \theta_0) dx$$

Hypothesis testing with loss functions

The risk function for the rule δ_C is

$$\begin{aligned} R(\theta_0; \delta_C) &= \int L_S(\theta_0, \delta_C(x)) f(x; \theta_0) dx \\ &= \int a\mathbb{I}(x \in C) f(x; \theta_0) dx \end{aligned}$$

Hypothesis testing with loss functions

The risk function for the rule δ_C is

$$\begin{aligned} R(\theta_0; \delta_C) &= \int L_S(\theta_0, \delta_C(x)) f(x; \theta_0) dx \\ &= \int a \mathbb{I}(x \in C) f(x; \theta_0) dx \\ &= a\alpha \end{aligned}$$

Hypothesis testing with loss functions

The risk function for the rule δ_C is

$$\begin{aligned} R(\theta_0; \delta_C) &= \int L_S(\theta_0, \delta_C(x)) f(x; \theta_0) dx \\ &= \int a\mathbb{I}(x \in C) f(x; \theta_0) dx \\ &= a\alpha \end{aligned}$$

as $\alpha = P(X \in C | H_0)$ is the probability for Type I error.

Hypothesis testing with loss functions

The risk function for the rule δ_C is

$$\begin{aligned} R(\theta_0;\delta_C) &= \int L_S(\theta_0,\delta_C(x))f(x;\theta_0)dx \\ &= \int a\mathbb{I}(x\in C)f(x;\theta_0)dx \\ &= a\alpha \end{aligned}$$

as $\alpha = P(X \in C|H_0)$ is the probability for Type I error.

Similarly

$$R(\theta_1;\delta_C) = b\beta,$$

as $\beta = P(X \notin C|H_1)$ is the probability for a Type II error.

Hypothesis testing with loss functions

The risk function for the rule δ_C is

$$\begin{aligned} R(\theta_0; \delta_C) &= \int L_S(\theta_0, \delta_C(x)) f(x; \theta_0) dx \\ &= \int a \mathbb{I}(x \in C) f(x; \theta_0) dx \\ &= a\alpha \end{aligned}$$

as $\alpha = P(X \in C | H_0)$ is the probability for Type I error.

Similarly

$$R(\theta_1; \delta_C) = b\beta,$$

as $\beta = P(X \notin C | H_1)$ is the probability for a Type II error.

Note α and β are functions of the critical region C.

Hypothesis testing with loss functions

To calculate the Bayes risk we need a prior. Let $\pi(\theta_0) = p_0$ and $\pi(\theta_1) = p_1$ be the prior probabilities that H_0 and H_1 hold.

Hypothesis testing with loss functions

To calculate the Bayes risk we need a prior. Let $\pi(\theta_0) = p_0$ and $\pi(\theta_1) = p_1$ be the prior probabilities that H_0 and H_1 hold.

The Bayes risk is

$$r(\pi, \delta_C) = \sum_{\theta \in \{\theta_0, \theta_1\}} R(\theta; \delta_C)\pi(\theta)$$

Hypothesis testing with loss functions

To calculate the Bayes risk we need a prior. Let $\pi(\theta_0) = p_0$ and $\pi(\theta_1) = p_1$ be the prior probabilities that H_0 and H_1 hold.

The Bayes risk is

$$\begin{aligned} r(\pi, \delta_C) &= \sum_{\theta \in \{\theta_0, \theta_1\}} R(\theta; \delta_C)\pi(\theta) \\ &= p_0 a \alpha(C) + p_1 b \beta(C). \end{aligned}$$

The Bayes test chooses the critical region C to minimize the Bayes risk.

Hypothesis testing with loss functions

To calculate the Bayes risk we need a prior. Let $\pi(\theta_0) = p_0$ and $\pi(\theta_1) = p_1$ be the prior probabilities that H_0 and H_1 hold.

The Bayes risk is

$$\begin{aligned} r(\pi, \delta_C) &= \sum_{\theta \in \{\theta_0, \theta_1\}} R(\theta; \delta_C)\pi(\theta) \\ &= p_0 a\alpha(C) + p_1 b\beta(C). \end{aligned}$$

The Bayes test chooses the critical region C to minimize the Bayes risk.

Question How do we do this?

Hypothesis testing with loss functions

To answer this question, let's revise what we know about Frequentist hypothesis testing.

Hypothesis testing with loss functions

To answer this question, let's revise what we know about Frequentist hypothesis testing.

Neyman-Pearson lemma The best test of size α of H_0 vs H_1 is a likelihood ratio test with critical region

$$C' = \left\{ x; \frac{L(\theta_1; x)}{L(\theta_0; x)} \geq A \right\}$$

for some constant $A > 0$ chosen so that $P(X \in C' | H_0) = \alpha$.

Hypothesis testing with loss functions

To answer this question, let's revise what we know about Frequentist hypothesis testing.

Neyman-Pearson lemma The best test of size α of H_0 vs H_1 is a likelihood ratio test with critical region

$$C' = \left\{ x; \frac{L(\theta_1; x)}{L(\theta_0; x)} \geq A \right\}$$

for some constant $A > 0$ chosen so that $P(X \in C' | H_0) = \alpha$.

By 'best test' we mean the test with the highest power, where Power = 1 - Type II error.

Hypothesis testing with loss functions

To answer this question, let's revise what we know about Frequentist hypothesis testing.

Neyman-Pearson lemma The best test of size α of H_0 vs H_1 is a likelihood ratio test with critical region

$$C' = \left\{ x; \frac{L(\theta_1; x)}{L(\theta_0; x)} \geq A \right\}$$

for some constant $A > 0$ chosen so that $P(X \in C' | H_0) = \alpha$.

By 'best test' we mean the test with the highest power, where Power = 1 - Type II error.

In frequentist statistics we fix the Type I error (α) and this determines the value of A, thus defining the critical region for the test.

Bayes tests

The following theorem tells us how to choose the critical region to minimize the Bayes risk, in the same way that the Neyman-Pearson lemma tells us how to maximize the power at fixed size.

Bayes tests

The following theorem tells us how to choose the critical region to minimize the Bayes risk, in the same way that the Neyman-Pearson lemma tells us how to maximize the power at fixed size.

Theorem The critical region for the Bayes test is the critical region for a LR test with

$$A = \frac{p_0 a}{p_1 b}$$

Every LR test is a Bayes test for some p_0, p_1.

Proof

The Bayes test minimizes

$$p_0 a\alpha + p_1 b\beta \quad = \quad p_0 a P(X \in C | H_0) + p_1 b P(X \in C' | H_1)$$

Proof

The Bayes test minimizes

$$\begin{aligned} p_0 a\alpha + p_1 b\beta &= p_0 a P(X \in C | H_0) + p_1 b P(X \in C' | H_1) \\ &= p_0 a \int_C L(\theta_0; x) dx + p_1 b \int_{C'} L(\theta_1; x) dx \end{aligned}$$

Proof

The Bayes test minimizes

$$\begin{aligned} p_0 a\alpha + p_1 b\beta &= p_0 a P(X \in C|H_0) + p_1 b P(X \in C'|H_1) \\ &= p_0 a \int_C L(\theta_0; x)dx + p_1 b \int_{C'} L(\theta_1; x)dx \\ &= p_0 a \int_C L(\theta_0; x)dx + p_1 b \Big[1 - \int_C L(\theta_1; x)dx \Big] \end{aligned}$$

Proof

The Bayes test minimizes

$$\begin{aligned}
p_0 a\alpha + p_1 b\beta &= p_0 a P(X \in C | H_0) + p_1 b P(X \in C' | H_1) \\
&= p_0 a \int_C L(\theta_0; x) dx + p_1 b \int_{C'} L(\theta_1; x) dx \\
&= p_0 a \int_C L(\theta_0; x) dx + p_1 b \Big[1 - \int_C L(\theta_1; x) dx \Big] \\
&= p_1 b + \int_C \Big[p_0 a L(\theta_0; x) - p_1 b L(\theta_1; x) \Big] dx
\end{aligned}$$

So choose C such that $x \in C$ iff $p_0 a L(\theta_0; x) - p_1 b L(\theta_1; x) \leq 0$, i.e. when

$$\frac{L(\theta_1; x)}{L(\theta_0; x)} \geq \frac{p_0 a}{p_1 b}$$

Example

Let $X_1, \ldots, X_n$ be N(μ, σ^2) with σ^2 known, and we want to test $H_0 : \mu = \mu_0$ vs $H_1 : \mu = \mu_1$, with $\mu_1 > \mu_0$. The critical region for the LR test

$$\frac{L(\theta_1; x)}{L(\theta_0; x)} \geq A$$

becomes

$$\bar{x} \geq \frac{\sigma^2 \log(A)}{n(\mu_1 - \mu_0)} + \frac{1}{2}(\mu_0 + \mu_1) = B(\text{ say})$$

Example

Let $X_1, \ldots, X_n$ be N(μ, σ^2) with σ^2 known, and we want to test $H_0 : \mu = \mu_0$ vs $H_1 : \mu = \mu_1$, with $\mu_1 > \mu_0$. The critical region for the LR test

$$\frac{L(\theta_1; x)}{L(\theta_0; x)} \geq A$$

becomes

$$\bar{x} \geq \frac{\sigma^2 \log(A)}{n(\mu_1 - \mu_0)} + \frac{1}{2}(\mu_0 + \mu_1) = B(\text{ say})$$

In the classical case we ignore the exact value of B, but in a Bayes test $A = p_0 a/(p_1 b)$ and we substitute into B.

Example

Let $X_1, \ldots, X_n$ be N(μ, σ^2) with σ^2 known, and we want to test $H_0 : \mu = \mu_0$ vs $H_1 : \mu = \mu_1$, with $\mu_1 > \mu_0$. The critical region for the LR test

$$\frac{L(\theta_1; x)}{L(\theta_0; x)} \geq A$$

becomes

$$\bar{x} \geq \frac{\sigma^2 \log(A)}{n(\mu_1 - \mu_0)} + \frac{1}{2}(\mu_0 + \mu_1) = B(\text{ say})$$

In the classical case we ignore the exact value of B, but in a Bayes test $A = p_0 a/(p_1 b)$ and we substitute into B. As an example take

$$\mu_0 = 0, \mu_1 = 1, \sigma^2 = 1, n = 4, a = 2, b = 1, p_0 = 1/4, p_1 = 3/4$$

Then the Bayes test has critical region

$$\bar{x} \geq \frac{1}{4}\log(\frac{1}{3} \times \frac{2}{1}) + \frac{1}{2} = \frac{1}{4}\log(\frac{2}{3}) + \frac{1}{2} = 0.399$$

For this test (using the fact that $\bar{X}$ is $N(\mu, 1/4)$)

$$\alpha = P(\bar{X} \geq 0.3999 \mid \mu = 0, \sigma^2/n = 1/4) = 0.212$$

and

$$\beta = P(\bar{X} < 0.3999 \mid \mu = 1, \sigma^2/n = 1/4) = 0.115$$

In a classical approach fixing $\alpha = 0.05$, $B = 1.645\sqrt{1/4} = 0.822$, so

$$\beta = P(\bar{X} < 0.822 \mid \mu = 1, \sigma^2 = 1/4) = 0.363$$

In the Bayes test α has been increased and β decreased.

Example

Let $X_1, \ldots, X_{10}$ be i.i.d. Bernoulli(p). $H_0 : \; p = 0.5$ and $H_1 : p = 0.8$ with respective prior probabilities π_0 and π_1. As usual

$$\alpha = P(\text{reject } H_0 | H_0) \text{ and } \beta = P(\text{acept } H_0 | H_1).$$

Example

Let $X_1, \ldots, X_{10}$ be i.i.d. Bernoulli(p). $H_0 : \; p = 0.5$ and $H_1 : p = 0.8$ with respective prior probabilities π_0 and π_1. As usual

$$\alpha = P(\text{reject } H_0 | H_0) \text{ and } \beta = P(\text{acept } H_0 | H_1).$$

The loss function is specified by

	H_0 holds	H_1 holds
Accept H_0	0	b
Accept H_1	a	0

Example

Let $X_1, \dots, X_{10}$ be i.i.d. Bernoulli(p). $H_0 :\ p = 0.5$ and $H_1 : p = 0.8$ with respective prior probabilities π_0 and π_1. As usual

$$\alpha = P(\text{reject } H_0 | H_0) \text{ and } \beta = P(\text{acept } H_0 | H_1).$$

The loss function is specified by

	H_0 holds	H_1 holds
Accept H_0	0	b
Accept H_1	a	0

Let δ_B be the test with critical region $S = \sum X_i \geq B$. The Bayes risk is

Example

Let $X_1, \ldots, X_{10}$ be i.i.d. Bernoulli(p). $H_0 : \ p = 0.5$ and $H_1 : p = 0.8$ with respective prior probabilities π_0 and π_1. As usual

$$\alpha = P(\text{reject } H_0 | H_0) \text{ and } \beta = P(\text{acept } H_0 | H_1).$$

The loss function is specified by

	H_0 holds	H_1 holds
Accept H_0	0	b
Accept H_1	a	0

Let δ_B be the test with critical region $S = \sum X_i \geq B$. The Bayes risk is

$$p_0 a\alpha + p_1 b\beta$$

and the Bayes test minimizes this expression.

Example

Suppose $a = 2, b = 1$.

B	0	1	2	3	4	5
$a\alpha$	2.000	1.980	1.978	1.890	1.656	1.246
$b\beta$	0.000	0.000	0.000	0.000	0.001	0.006
B	6	7	8	9	10	11
$a\alpha$	0.754	0.344	0.110	0.022	0.002	0.000
$b\beta$	0.033	0.121	0.322	0.624	0.893	1.000

Let δ_B be the test with critical region $S = \sum X_i \geq B$. What is δ_{11} and can it be the Bayes test?

Example

Suppose $a = 2, b = 1$.

B	0	1	2	3	4	5
$a\alpha$	2.000	1.980	1.978	1.890	1.656	1.246
$b\beta$	0.000	0.000	0.000	0.000	0.001	0.006
B	6	7	8	9	10	11
$a\alpha$	0.754	0.344	0.110	0.022	0.002	0.000
$b\beta$	0.033	0.121	0.322	0.624	0.893	1.000

Let δ_B be the test with critical region $S = \sum X_i \geq B$. What is δ_{11} and can it be the Bayes test? Always accept H_0 and yes if p_0 is very close to 1 (in fact as soon as $p_0 \geq 0.982$).

Example

Suppose $a = 2, b = 1$.

B	0	1	2	3	4	5
$a\alpha$	2.000	1.980	1.978	1.890	1.656	1.246
$b\beta$	0.000	0.000	0.000	0.000	0.001	0.006
B	6	7	8	9	10	11
$a\alpha$	0.754	0.344	0.110	0.022	0.002	0.000
$b\beta$	0.033	0.121	0.322	0.624	0.893	1.000

Let δ_B be the test with critical region $S = \sum X_i \geq B$. What is δ_{11} and can it be the Bayes test? Always accept H_0 and yes if p_0 is very close to 1 (in fact as soon as $p_0 \geq 0.982$).
Similarly, δ_7 is optimal when $p_0 \in (0.177, 0.462)$.

Example

Suppose $a = 2, b = 1$.

B	0	1	2	3	4	5
$a\alpha$	2.000	1.980	1.978	1.890	1.656	1.246
$b\beta$	0.000	0.000	0.000	0.000	0.001	0.006
B	6	7	8	9	10	11
$a\alpha$	0.754	0.344	0.110	0.022	0.002	0.000
$b\beta$	0.033	0.121	0.322	0.624	0.893	1.000

Let δ_B be the test with critical region $S = \sum X_i \geq B$. What is δ_{11} and can it be the Bayes test? Always accept H_0 and yes if p_0 is very close to 1 (in fact as soon as $p_0 \geq 0.982$).
Similarly, δ_7 is optimal when $p_0 \in (0.177, 0.462)$. Graphical interpretation?

Example

Suppose $a = 2, b = 1$.

B	0	1	2	3	4	5
$a\alpha$	2.000	1.980	1.978	1.890	1.656	1.246
$b\beta$	0.000	0.000	0.000	0.000	0.001	0.006
B	6	7	8	9	10	11
$a\alpha$	0.754	0.344	0.110	0.022	0.002	0.000
$b\beta$	0.033	0.121	0.322	0.624	0.893	1.000

Let δ_B be the test with critical region $S = \sum X_i \geq B$. What is δ_{11} and can it be the Bayes test? Always accept H_0 and yes if p_0 is very close to 1 (in fact as soon as $p_0 \geq 0.982$).
Similarly, δ_7 is optimal when $p_0 \in (0.177, 0.462)$. Graphical interpretation?What is the minimax test? (first only for non random rules and then in general)

Sequential procedure

With sequential procedure, sample items are drawn one at a time. After each sample there is a choice of stopping and making decision/estimation or to continue.

Sequential procedure

With sequential procedure, sample items are drawn one at a time. After each sample there is a choice of stopping and making decision/estimation or to continue. In general a cost is associated to sampling.

Sequential procedure

With sequential procedure, sample items are drawn one at a time. After each sample there is a choice of stopping and making decision/estimation or to continue. In general a cost is associated to sampling.

.

Example: Suppose the X_i are iid Bernoulli(θ) and we want to test $H_0 : \theta = 0.10$ against $H_1 : \theta = 0.20$. If we choose in advance the sample size n, it might become clear in hindsight that the sample was unnecessarily large or small.

- $n = 100$, $S_n = \sum_{i=1}^n X_i = 25$. The sample was probably too large. The evidence was clear before the 100th draw.
- $n = 100, S_n = 15$. We need a larger sample.

Sequential procedure

A sequential decision rule involves

1. A stopping rule (when to stop)
2. and a terminal decision rule (upon stopping, what decision do we take).

Sequential procedure

A sequential decision rule involves

1. A stopping rule (when to stop)
2. and a terminal decision rule (upon stopping, what decision do we take).

Notation:

- $X_1, X_2, \dots$ stream of observations, supposed iid $\sim f(x, \theta)$, and write $\underline{x}_N = (x_1, \dots, x_n)$ for the vector of the first n observations.
- The stopping rule S is a stopping time in the filtration of the X_i. i.e. the event $S = n$ depend only on $\underline{x}_n$.
- The terminal decision rule is a family of decisions rule indexed by n: $d_n(\underline{x}_n)$
- c_i = cost of observation i
- the total loss function is

$$L(\theta, d_S(\underline{x}_S) + \sum_{i=1}^{S} c_i.$$

Sequential procedure

The Risk of a sequential procedure is thus

$$\begin{aligned} R(\theta, S, d_S) &= E_{\bar{x}|\theta}\left[L(\theta, d_S(\underline{x}_S) + \sum_{i=1}^{S} c_i\right] \\ &= E\left[\sum_n 1_{\{S=n\}}\left(L(\theta, d_n(\underline{x}_n) + \sum_{i=1}^{n} c_i\right)\right] \end{aligned}$$

Sequential procedure

The Risk of a sequential procedure is thus

$$R(\theta, S, d_S) = E_{\bar{x}|\theta}\left[L(\theta, d_S(\underline{x}_S) + \sum_{i=1}^{S} c_i\right]$$
$$= E\left[\sum_n 1_{\{S=n\}}\left(L(\theta, d_n(\underline{x}_n) + \sum_{i=1}^{n} c_i\right)\right]$$

The Bayes risk integrates this quantity against the prior $\pi(\theta)$

$$B(S, d_S) = E_\theta\left[E_{\underline{X}|\theta}\left[L(\theta, d_S(\underline{x}_S) + \sum_{i=1}^{S} c_i\right]\right]$$
$$= E_{\underline{X}}\left[E_{\theta|\underline{x}_S}\left[L(\theta, d_S(\underline{x}_S) + \sum_{i=1}^{S} c_i\right]\right]$$
$$= E_{\underline{X}}\left[\sum 1_{\{S=n\}} E_{\theta|\underline{x}_n}\left[L(\theta, d_n(\underline{x}_n) + \sum^{n} c_i\right]\right]$$

Sequential procedure

$$B(S, d_S) = E_{\underline{X}}\left[\sum_n 1_{\{S=n\}} E_{\theta|\underline{x}_n}\left[L(\theta, d_n(\underline{x}_n) + \sum_{i=1}^n c_i\right]\right]$$

Therefore for each fixed n we need to minimize $E_{\theta|\underline{x}_n}[L(\theta, d_n(\underline{x}_n)]$ and thus for each n fixed d_n must be the Bayes rule.

Sequential procedure

$$B(S, d_S) = E_{\underline{X}}\left[\sum_n 1_{\{S=n\}} E_{\theta|\underline{x}_n}\left[L(\theta, d_n(\underline{x}_n)) + \sum_{i=1}^n c_i\right]\right]$$

Therefore for each fixed n we need to minimize $E_{\theta|\underline{x}_n}[L(\theta, d_n(\underline{x}_n))]$ and thus for each n fixed d_n must be the Bayes rule.

The hard part is to decide when to stop.

Sequential procedure

$$B(S, d_S) = E_{\underline{X}}\left[\sum_n 1_{\{S=n\}} E_{\theta|\underline{x}_n}\left[L(\theta, d_n(\underline{x}_n) + \sum_{i=1}^n c_i\right]\right]$$

Therefore for each fixed n we need to minimize $E_{\theta|\underline{x}_n}[L(\theta, d_n(\underline{x}_n)]$ and thus for each n fixed d_n must be the Bayes rule.

The hard part is to decide when to stop. In general: stop iff current Bayes risk of making the decision now is less that expected Bayes risk of taking one more observation.

Sequential procedure

$$B(S, d_S) = E_{\underline{X}} \left[\sum_n 1_{\{S=n\}} E_{\theta|\underline{x}_n} \left[L(\theta, d_n(\underline{x}_n) + \sum_{i=1}^{n} c_i \right] \right]$$

Therefore for each fixed n we need to minimize $E_{\theta|\underline{x}_n}[L(\theta, d_n(\underline{x}_n)]$ and thus for each n fixed d_n must be the Bayes rule.

The hard part is to decide when to stop. In general: stop iff current Bayes risk of making the decision now is less that expected Bayes risk of taking one more observation.

Generally hard, except when:

1. For each n, the Bayes risk does not depend on the observation $\underline{x}_n$
2. When there is a fixed upper bound for the number of observation (using backward induction).

Example: Backward induction

Max # of obs.= 3. $X_i \sim Ber(\theta)$ we want to test $H_0 : \theta = 1/4$ against $H_1 : \theta = 1/2$ with uniform prior $\pi_0 = \pi_1 = 1/2$. The loss function is specified by $a = b = 30$ and $c_i = 1$. Then

$$\begin{aligned} P(H_0|S_n) &= \frac{P(S_n|H_0)P(H_0)}{P(S_n|H_0)P(H_0) + P(S_n|H_1)P(H_1)} \\ &= \frac{\binom{n}{s}4^{-s}(3/4)^{n-s}}{\binom{n}{s}4^{-s}(3/4)^{n-s} + \binom{n}{s}2^{-n}} \\ &= \frac{3^{n-s}}{3^{n-s}+2^n} \end{aligned}$$

and $P(H_0|S_n) = \frac{2^n}{3^{n-s}+2^n}$. The Bayes rule choses H_0 when

$$3^{n-s} < 2^n.$$

Example: Backward induction

Max # of obs.= 3. $X_i \sim Ber(\theta)$ we want to test $H_0 : \theta = 1/4$ against $H_1 : \theta = 1/2$ with uniform prior $\pi_0 = \pi_1 = 1/2$. The loss function is specified by $a = b = 30$ and $c_i = 1$. Then

$$\begin{aligned} P(H_0|S_n) &= \frac{P(S_n|H_0)P(H_0)}{P(S_n|H_0)P(H_0) + P(S_n|H_1)P(H_1)} \\ &= \frac{\binom{n}{s}4^{-s}(3/4)^{n-s}}{\binom{n}{s}4^{-s}(3/4)^{n-s} + \binom{n}{s}2^{-n}} \\ &= \frac{3^{n-s}}{3^{n-s} + 2^n} \end{aligned}$$

and $P(H_0|S_n) = \frac{2^n}{3^{n-s}+2^n}$. The Bayes rule choses H_0 when

$$3^{n-s} < 2^n.$$

Expected risk from terminal decision is

$$30(3^{n-s}, 2^n)/(3^{n-s} + 2^n)$$

Example: Backward induction

After $n = 3$ obs.

S	0	1	2	3
Bayesrisk	9.857	17.118	11.182	6.333

Example: Backward induction

After $n = 3$ obs.

S	0	1	2	3
Bayesrisk	9.857	17.118	11.182	6.333

Suppose now that $n = 2, S_2 = 1$. Bayes risk of decision at this stage is 12.857, so 14.857 with costs .

Example: Backward induction

After $n = 3$ obs.

S	0	1	2	3
Bayesrisk	9.857	17.118	11.182	6.333

Suppose now that $n = 2, S_2 = 1$. Bayes risk of decision at this stage is 12.857, so 14.857 with costs .

$$P(X_3 = 1 | S_2 = 1) = \frac{1}{4} P(H_0 | S_2 = 1) + \frac{1}{2} P(H_1 | S_2 = 1) = \frac{1}{4}\frac{3}{7} + \frac{1}{2}\frac{4}{7} = 0.392$$

So expected Bayes risk of taking one more observation is

$$0.3929 \times 11.182 + (1 - 0.3929) \times 17.118 = 14.786$$

So should take the third observation.

Example: Backward induction

After $n = 3$ obs.

S	0	1	2	3
$Bayesrisk$	9.857	17.118	11.182	6.333

Suppose now that $n = 2, S_2 = 1$. Bayes risk of decision at this stage is 12.857, so 14.857 with costs .

$$P(X_3 = 1|S_2 = 1) = \frac{1}{4}P(H_0|S_2 = 1) + \frac{1}{2}P(H_1|S_2 = 1) = \frac{1}{4}\frac{3}{7} + \frac{1}{2}\frac{4}{7} = 0.392$$

So expected Bayes risk of taking one more observation is

$$0.3929 \times 11.182 + (1 - 0.3929) \times 17.118 = 14.786$$

So should take the third observation. The Bayes risk after $n = 2$ obs. as a function of s is

S	0	1	2
$Bayesrisk$	11.23	14.786	8.0

Sequential probability ratio test

Can be viewed both as frequentist or bayesian.

$X_1, X_2, \ldots$ stream of iid observations $\sim f(x, \theta)$.

Hypothesis	H_0	H_1
	$\theta = \theta_0$	$\theta = \theta_1$

Sequential probability ratio test

Can be viewed both as frequentist or bayesian.
$X_1, X_2, \ldots$ stream of iid observations $\sim f(x, \theta)$.

Hypothesis	H_0	H_1
	$\theta = \theta_0$	$\theta = \theta_1$

Frequentist approach

$$LR(n) = \frac{L(\theta_1, \underline{x}_n)}{L(\theta_0, \underline{x}_n)} = \frac{\prod_{i=1}^n f(\theta_1, x_i)}{\prod_{i=1}^n f(\theta_0, x_i)}$$

Test procedure: take $0 < K_1 < K_2 < \infty$.

- If $LR(n) \leq K_1$ stop and accept H_0,
- If $LR(n) \geq K_2$ stop and accept H_1,
- If $LR(n) \in (K_1, K_2)$ take one more observation.

Sequential probability ratio test

Bayesian approach Each obs. cost c.

Hypothesis	H_0	H_1
	$\theta = \theta_0$	$\theta = \theta_1$
Prior probability	π_0	π_1
incorrect rejection cost	a	b

Sequential probability ratio test

Bayesian approach Each obs. cost c.

Hypothesis	H_0	H_1
	$\theta = \theta_0$	$\theta = \theta_1$
Prior probability	π_0	π_1
incorrect rejection cost	a	b

Then

Lemma

For a good choice of $0 < K_1 < K_2 < \infty$.

- *If* $LR(n) \leq K_1$ *stop and accept* H_0,
- *If* $LR(n) \geq K_2$ *stop and accept* H_1,
- *If* $LR(n) \in (K_1, K_2)$ *take one more observation.*

is the Bayes procedure.

Sequential probability ratio test

At stage *n* of sequential sampling, decision (stop and decide or continue) depends on

1. cost *c* of future obs.

Sequential probability ratio test

At stage n of sequential sampling, decision (stop and decide or continue) depends on

1. cost c of future obs.
2. Loss of decision a, b

Sequential probability ratio test

At stage n of sequential sampling, decision (stop and decide or continue) depends on

1. cost c of future obs.
2. Loss of decision a, b
3. The amount of information about θ that future obs. can provide

Sequential probability ratio test

At stage n of sequential sampling, decision (stop and decide or continue) depends on

1. cost c of future obs.
2. Loss of decision a, b
3. The amount of information about θ that future obs. can provide
4. The current info about θ given by $p = P(\theta = \theta_0 | \underline{x}_n)$

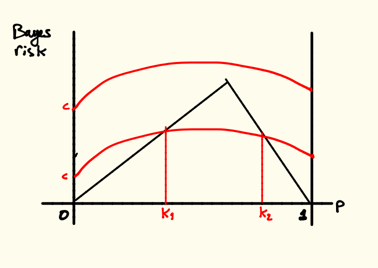
If we stop, expected losses are $b(1-p)$ if we chose H_0 and ap if we chose H_1.

Sequential probability ratio test

At stage n of sequential sampling, decision (stop and decide or continue) depends on

1. cost c of future obs.
2. Loss of decision a, b
3. The amount of information about θ that future obs. can provide
4. The current info about θ given by $p = P(\theta = \theta_0 | \underline{x}_n)$

If we stop, expected losses are $b(1-p)$ if we chose H_0 and ap if we chose H_1. So decision is $H_0 \Leftrightarrow p > b/(a+b)$.



$B^*(p) = \min_\delta B(\delta, p)$ where δ any rule that take more obs. Is Bayes risk.
$B^*(p) \geq c$, $B^*(0) = B^*(1) = c$
$p \mapsto B^*(p)$ is concave.

Sequential probability ratio test

The rule is then

- If $LR(n) \leq K_1$ stop and accept H_0,
- If $LR(n) \geq K_2$ stop and accept H_1,
- If $LR(n) \in (K_1, K_2)$ take one more observation.

Observe that

$$p = \frac{P(H_0)}{P(H_0)} + LR(n)P(H_1)$$

so we continue sampling if

$$\begin{aligned} k_1 < p < k_2 \quad &\equiv \quad \frac{1}{k_2} < \frac{1}{p} < \frac{1}{k_1} \\ &\equiv \quad \frac{1}{k_2} < 1 + LR(n)P(H_1)/P(H_0) < \frac{1}{k_1} \\ &\equiv \quad \frac{1-k_2}{k_2}\frac{P(H_0)}{P(H_1)} < LR(n) < \frac{1-k_1}{k_1}\frac{P(H_0)}{P(H_1)} \end{aligned}$$

Agree with SPRT WITH $K_1 = \frac{1-k_1}{k_1}\frac{P(H_0)}{P(H_1)}, K_2 = \frac{1-k_2}{k_2}\frac{P(H_0)}{P(H_1)}$