# Foundations of Statistical Inference

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# Course arrangements

- **Lectures** Mon. 3 pm and Tue.11 weeks 1-8
- **Classes** (1) & (2) Weeks 3,5,7,8 (J. Berestycki) Wed. 10am and 11:30am, LG04 (3)& (4) Weeks 3,4,7,8 (S. Filippi) Thur. 9am and 12 am.
- Hand in solutions by noon Monday of the week. Class Tutors : Julien Berestycki and Sarah Filippi
- Notes and Problem sheets will be available at `www.stats.ox.ac.uk\~berestyc\SB2a.html`
- **Books**
  - Garthwaite, P. H., Jolliffe, I. T. and Jones, B. (2002) Statistical Inference, Oxford Science Publications
  - Leonard, T., Hsu, J. S. (2005) Bayesian Methods, Cambridge University Press.
  - D. R. Cox (2006) Principals of Statistical Inference
- This course builds on notes from Bob Griffiths, Geoff Nicholls and Jonathan Marchini

# Part A Statistics

The majority of the statistics that you have learned up to now falls under the philosophy of classical (or Frequentist) statistics. This theory makes the assumption that we can randomly take repeated samples of data from the same population.

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- Hypothesis testing (marginal likelihoods, Bayes Factors)

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- How can we find good estimators of a parameter $\theta$? $\Rightarrow$Rao-Blackwell Theorem and Lehmann-Scheffé Theorem.

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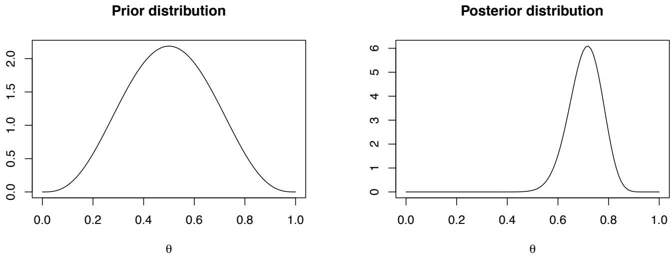
Quick Example Suppose I give a coin and tell you that it is bit biased. We might use a Beta(4,4) distribution to represent our beliefs about the $\theta$. If we observe 30 heads and 10 tails we can use probability theory to infer a posterior distribution for $\theta$ of Beta(34, 14).



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  - Bayesian Information Criterion (BIC)
- The EM algorithm NEW
  - useful in Frequentist and Bayesian inference of missing data problems

# Decision theory

Quick Example

Zed and Adrian and run a small bicycle shop called "Z to A Bicycles". They must order bicycles for the coming season. Orders for the bicycles must be placed in quantities of twenty (20). The cost per bicycle is 70 GBP if they order 20, 67 GBP if they order 40, 65 GBP if they order 60, and 64 GBP if they order 80. The bicycles will be sold for 100 GBP each. Any bicycles left over at the end of the season can be sold (for certain) at 45 GBP each. If Zed and Adrian run out of bicycles during the season, then they will suffer a loss of "goodwill" among their customers. They estimate this goodwill loss to be 5 GBP per customer who was unable to buy a bicycle. Zed and Adrian estimate that the demand for bicycles this season will be 10, 30, 50, or 70 bicycles with probabilities of 0.2, 0.4, 0.3, and 0.1 respectively.

# Notation

- $X, Y, Z$ Capital letters for random variables.
- $x, y, z$ Lower case letters for realisations of random variables.
- $\mathbb{E}_X(\cdot)$ Expectation with respect to the random variable $X$.
- $\boldsymbol{\phi} = \{\phi_1, \ldots, \phi_k\}$ Sometimes we will use bold symbols to denote a vector of parameters.

Lecture 1 - Exponential families

# Parametric families

$f(x;\theta)$, $\theta \in \Theta$, probability density of a random variable (rv) which could be discrete or continuous. $\theta$ can be 1-dimensional or of higher dimension. Equivalent notation: $f_\theta(x)$, $f(x \mid \theta)$, $f(x, \theta)$.

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Examples

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3. Regression:

$f(y;\theta) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}(y_i - \sum_{j=1}^p x_{ij}\beta_j)^2}, y \in \mathbb{R}^n, \sigma > 0,\ \beta \in \mathbb{R}^p.\ \theta = \{\beta,\sigma\}$.

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# Exponential families of distributions

**Definition 1** GJJ 2.6, DRC 2.3
A rv $X$ belongs to a $k$-parameter exponential family if its probability density function (pdf) can be written as

$$f(x;\theta) = \exp\left\{\sum_{j=1}^{k} A_j(\theta)B_j(x) + C(x) + D(\theta)\right\}$$
$$= \omega(x)\exp\left\{\sum_{j=1}^{k} A_j(\theta)B_j(x)\right\}\psi(\theta),$$

where $x \in \chi$, $\theta \in \Theta$, $A_1(\theta), \ldots, A_k(\theta), D(\theta)$ are functions of $\theta$ alone and $B_1(x), B_2(x), \ldots, B_k(x), C(x)$ are well behaved functions of $x$ alone.

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$\psi(\theta)$ is a normalising factor.

Exponential families are widely used in practice - for example in generalised linear models (see BS1a).

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So $A(\theta) = \log\theta$, $B(x) = x$, $C(x) = -\log x!$, $D(\theta) = -\theta$.

# Examples of 1-parameter Exponential families

Binomial, Poisson, Normal, Exponential.

| Distn | $f(x;\theta)$ | $A(\theta)$ | $B(x)$ | $C(x)$ | $D(\theta)$ |
|---|---|---|---|---|---|
| $\mathrm{Bin}(n,p)$ | $\binom{n}{x}p^x(1-p)^{n-x}$ | $\log\frac{p}{(1-p)}$ | $x$ | $\log\binom{n}{x}$ | $n\log(1-p)$ |

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| $N(\mu,1)$ | $\frac{1}{\sqrt{2\pi}}\exp\{-\frac{(x-\mu)^2}{2}\}$ | $\mu$ | $x$ | $-x^2/2$ | $\frac{1}{2}(\mu^2-\log(2\pi))$ |

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| Exp$(\theta)$ | $\theta e^{-\theta x}$ | $-\theta$ | $x$ | $0$ | $\log\theta$ |

Others : negative binomial, Pareto (with known minimum), Weibull (with known shape), Laplace (with known mean), Log-normal, inverse Gaussian, beta, Dirichlet, Wishart. Exercise: check these distributions

# Example 2 : a 2-parameter family (Gamma)

If $X \sim \text{Gamma}(\alpha, \beta)$ then let $\theta = (\alpha, \beta)$ so

$$f(x;\theta) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}$$

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\end{aligned}
$$

And we have
$A_1(\theta) = \alpha - 1$, $B_1(x) = \log x$,
$A_2(\theta) = -\beta$, $B_2(x) = x$.

# Some other 2-parameter Exponential families

| Distribution | $f(x;\theta)$ | $A(\theta)$ | $B(x)$ | $C(x)$ | $D(\theta)$ |
|---|---|---|---|---|---|
| $N(\mu,\sigma^2)$ | $\frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$ | $A_1(\theta) = -1/2\sigma^2$ | $B_1(x) = x^2$ | 0 | $-\frac{1}{2}\log(2\pi\sigma^2)$ |
| | | $A_2(\theta) = \mu/\sigma^2$ | $B_2(x) = x$ | 0 | $-\frac{1}{2}\mu^2/\sigma^2$ |
| Gamma | $\frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}$ | $A_1(\theta) = \alpha - 1$ | $B_1(x) = \log x$ | 0 | $-\log\left[\Gamma(\alpha)\beta^{-\alpha}\right]$ |
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# Exponential family canonical form

Let $\phi_j = A_j(\theta)$, $j = 1, \ldots, k$

$$f(x;\phi) = \exp\left\{\sum_{j=1}^{k} \phi_j B_j(x) + C(x) + D(\theta)\right\}$$

$$= \omega(x)\psi(\theta) \exp\left\{\sum_{j=1}^{k} \phi_j B_j(x) + C(x)\right\}.$$

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Because $\psi(\theta) = \left(\int \omega(x)\exp\left\{\sum_{j=1}^{k} \phi_j B_j(x)\right\} dx\right)^{-1}$ depends only on $\phi$
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$\Phi := \{\phi : \int \omega(x)\exp\left\{\sum_{j=1}^{k} \phi_j B_j(x)\right\} dx < \infty\}$ is natural parameter space.

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Exercise: show $\mathbb{E}[\log X] = \psi_0(\alpha) - \log(\beta)$ where $\psi_0$ is the digamma function, and $\Gamma'(\alpha) = \Gamma(\alpha)\psi_0(\alpha)$.

# Cumulant Generating Function

In a scalar canonical exponential family ($k = 1$)

$$\begin{aligned} \mathbb{E}_X[e^{sB(X)}] &= \int_{\mathbb{R}} \exp\left\{(\phi + s)B(x) + C(x) + D(\phi)\right\} dx \\ &= \exp\{D(\phi) - D(\phi + s)\} \end{aligned}$$

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This is the cumulant generating function (defined as the log of the mgf) for the cumulants of $B(X)$ i.e.

$$\log(M_{B(X)}(s)) = \sum_{r=1}^{\infty} \kappa_r s^r / r!$$

where $\kappa_1 = \mathbb{E}(B(X))$ and $\kappa_2 = V(B(X))$ Exercise : prove this

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Therefore

$$\kappa_1 = \frac{\partial}{\partial s} \log(M_X(s))\Big|_{s=0} = n \frac{e^{\phi}}{1+e^{\phi}} = np$$

# Example 5 : Skew-logistic distribution

Consider the real valued random variable $X$ with pdf

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$$\Rightarrow \log(M_X(s)) = D(\phi) - D(\phi+s) = \log(-\phi) - \log(-(\phi+s))$$

$$\begin{aligned} \mathbb{E}(\log(1+e^{-x})) &= \left.\frac{-1}{\phi+s}\right|_{s=0} = \frac{1}{\theta} \\ \mathrm{Var}(\log(1+e^{-x})) &= \left.\frac{1}{(\phi+s)^2}\right|_{s=0} = \frac{1}{\theta^2} \end{aligned}$$

These results are harder to derive directly.

# Family preserved under transformations

A smooth invertible transformation of a rv from the Exponential family is also within the Exponential family. If $X \to Y$, $Y = Y(X)$ then

$$\begin{aligned} f_Y(y;\theta) &= f_X(x(y);\theta)|\partial X/\partial Y| \\ &= \exp\left\{\sum_{j=1}^{k} A_j(\theta)B_j(x(y)) + C(x(y)) + D(\theta)\right\}|\partial X/\partial Y|, \end{aligned}$$

The Jacobian depends only on $y$ and so the natural observation $B(x(y))$, the natural parameter $A(\theta)$, and $D(\theta)$ do not change, while

$$C(X) \to C(X(Y)) + \log|\partial X/\partial Y|.$$

- The family is minimal if no linear constraint on the $B_i$ or the $\phi_i$.

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- If $\theta = (\theta_1, \theta_2, \ldots, \theta_d)$ and $d < k$ the family is said to be curved and linear when $d = k$. We refer to a $(k, d)$ curved exponential family.
- If the family is minimal and the parameter space contains a $d$-dimensional open rectangle the family is said to be full rank.
- Example 6 $(X_1, X_2)$ independent, normal, unit variance, means $(\theta, c/\theta)$, $c$ known.

$$\log f(x; \theta) = x_1\theta + cx_2/\theta - \theta^2/2 - c^2\theta^{-2}/2 + ...$$

is a $(2, 1)$ curved exponential family.

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[Part of the "regularity conditions" which we cite later.]

# Linear relations among $A$'s do not generate curvature

Take a $k$-parameter exponential family and impose $A_1 = aA_2 + b$ for constants $a$ and $b$.

$$\sum_{i=1}^{k} A_i B_i + C + D = \sum_{i=3}^{k} A_i B_i + A_2(aB_1 + B_2) + (C + bB_1) + D$$

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This gives a $k-1$ parameter EF, not a $(k, k-1)$-CEF.

**Theorem**

*The natural/canonical parameter space of a linear $k$-dimensional exponential family is convex and contains a $k$-dimensional open interval.*

# Examples not in an exponential family

1. Uniform on $[0, \theta]$, $\theta > 0$.

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Other examples include the F-distribution, hypergeometric distribution and logistic distribution.