Simulation - Lectures - Part I

Julien Berestycki -(adapted from François Caron’s slides)

Part A Simulation and Statistical Programming

Hilary Term 2018
Lectures on Simulation (Prof. J. Berestycki):
Mondays 2-3pm Weeks 1-8.
LG.01, 24-29 St Giles’.

Computer Lab on Statistical Programming (Prof. R. Evans):
Wednesday 9-11am Weeks 2,3,5-8.
LG.02, 24-29 St Giles’.

Departmental problem classes:
Mon. 3.30 pm or Tue. 2pm - Weeks 3,5,7,8.
LG.??, 24-29 St Giles’.

Hand in problem sheet solutions by
Friday 12am of week before at the reception of 24-29 St Giles’.

Webpage:
http://www.stats.ox.ac.uk/~berestyc/teaching/A12.html

This course builds upon the notes and slides of Geoff Nicholls, Arnaud Doucet, Yee Whye Teh and Matti Vihola.
Outline

Introduction

Monte Carlo integration

Random variable generation
  Inversion Method
  Transformation Methods
  Rejection Sampling
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Introduction

Monte Carlo integration

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Monte Carlo Simulation Methods

- Computational tools for the simulation of random variables and the approximation of integrals/expectations.
- These simulation methods, aka Monte Carlo methods, are used in many fields including statistical physics, computational chemistry, statistical inference, genetics, finance etc.
- The Metropolis algorithm was named the top algorithm of the 20th century by a committee of mathematicians, computer scientists & physicists.
- With the dramatic increase of computational power, Monte Carlo methods are increasingly used.
Objectives of the Course

- Introduce the main tools for the simulation of random variables and the approximation of multidimensional integrals:
  - Integration by Monte Carlo,
  - inversion method,
  - transformation method,
  - rejection sampling,
  - importance sampling,
  - Markov chain Monte Carlo including Metropolis-Hastings.

- Understand the theoretical foundations and convergence properties of these methods.

- Learn to derive and implement specific algorithms for given random variables.
Computing Expectations

- Let $X$ be either
  - a discrete random variable (r.v.) taking values in a countable or finite set $\Omega$, with p.m.f. $f_X$
  - or a continuous r.v. taking values in $\Omega = \mathbb{R}^d$, with p.d.f. $f_X$
- Assume you are interested in computing

$$\theta = \mathbb{E}(\phi(X))$$

$$= \left\{ \begin{array}{ll}
\sum_{x \in \Omega} \phi(x)f_X(x) & \text{if } X \text{ is discrete} \\
\int_{\Omega} \phi(x)f_X(x)dx & \text{if } X \text{ is continuous}
\end{array} \right.$$ 

where $\phi : \Omega \rightarrow \mathbb{R}$.
- It is impossible to compute $\theta$ exactly in most realistic applications.
- Even if it is possible (for $\Omega$ finite) the number of elements may be so huge that it is practically impossible
- Example: $\Omega = \mathbb{R}^d$, $X \sim \mathcal{N}(\mu, \Sigma)$ and $\phi(x) = \mathbb{I}\left(\sum_{k=1}^{d} x_k^2 \geq \alpha\right)$.
- Example: $\Omega = \mathbb{R}^d$, $X \sim \mathcal{N}(\mu, \Sigma)$ and $\phi(x) = \mathbb{I}\left(x_1 < 0, \ldots, x_d < 0\right)$. 
Example: Queuing Systems

- Customers arrive at a shop and queue to be served. Their requests require varying amount of time.
- The manager cares about customer satisfaction and not excessively exceeding the 9am-5pm working day of his employees.
- Mathematically we could set up stochastic models for the arrival process of customers and for the service time based on past experience.
- **Question**: If the shop assistants continue to deal with all customers in the shop at 5pm, what is the probability that they will have served all the customers by 5.30pm?
- If we call $X \in \mathbb{N}$ the number of customers in the shop at 5.30pm then the probability of interest is

  \[ P(X = 0) = \mathbb{E}(\mathbb{I}(X = 0)). \]

- For realistic models, we typically do not know analytically the distribution of $X$. 
Example: Particle in a Random Medium

- A particle \((X_t)_{t=1,2,\ldots}\) evolves according to a stochastic model on \(\Omega = \mathbb{R}^d\).
- At each time step \(t\), it is absorbed with probability \(1 - G(X_t)\) where \(G : \Omega \to [0, 1]\).
- **Question**: What is the probability that the particle has not yet been absorbed at time \(T\)?
- The probability of interest is
  \[
  \mathbb{P}(\text{not absorbed at time } T) = \mathbb{E}[G(X_1)G(X_2) \cdots G(X_T)].
  \]
- For realistic models, we cannot compute this probability.
Example: Ising Model

- The Ising model serves to model the behavior of a magnet and is the best known/most researched model in statistical physics.
- The magnetism of a material is modelled by the collective contribution of dipole moments of many atomic spins.
- Consider a simple 2D-Ising model on a finite lattice $G = \{1, 2, \ldots, m\} \times \{1, 2, \ldots, m\}$ where each site $\sigma = (i, j)$ hosts a particle with a +1 or -1 spin modeled as a r.v. $X_\sigma$.
- The distribution of $X = \{X_\sigma\}_{\sigma \in G}$ on $\{-1, 1\}^{m^2}$ is given by
  \[
  \pi(x) = \frac{\exp(-\beta U(x))}{Z_\beta}
  \]
  where $\beta > 0$ is the inverse temperature and the potential energy is
  \[
  U(x) = -J \sum_{\sigma \sim \sigma'} x_\sigma x_{\sigma'}
  \]
- Physicists are interested in computing $\mathbb{E}[U(X)]$ and $Z_\beta$. 
Example: Ising Model

Sample from an Ising model for $m = 250$. 
Bayesian Inference

- Suppose $(X, Y)$ are both continuous r.v. with a joint density $f_{X,Y}(x, y)$.
- Think of $Y$ as data, and $X$ as unknown parameters of interest.
- We have
  
  $$f_{X,Y}(x, y) = f_X(x) f_{Y|X}(y|x)$$

  where, in many statistics problems, $f_X(x)$ can be thought of as a prior and $f_{Y|X}(y|x)$ as a likelihood function for a given $Y = y$.
- Using Bayes’ rule, we have
  
  $$f_{X|Y}(x|y) = \frac{f_X(x) f_{Y|X}(y|x)}{f_Y(y)}.$$

- For most problems of interest, $f_{X|Y}(x|y)$ does not admit an analytic expression and we cannot compute
  
  $$\mathbb{E}(\phi(X)|Y = y) = \int \phi(x) f_{X|Y}(x|y) dx.$$
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Monte Carlo Integration

Definition (Monte Carlo method)

Let $X$ be either a discrete r.v. taking values in a countable or finite set $\Omega$, with p.m.f. $f_X$, or a continuous r.v. taking values in $\Omega = \mathbb{R}^d$, with p.d.f. $f_X$. Consider

$$\theta = \mathbb{E}(\phi(X)) = \begin{cases} \sum_{x \in \Omega} \phi(x) f_X(x) & \text{if } X \text{ is discrete} \\ \int_{\Omega} \phi(x) f_X(x) dx & \text{if } X \text{ is continuous} \end{cases}$$

where $\phi : \Omega \to \mathbb{R}$. Let $X_1, ..., X_n$ be i.i.d. r.v. with p.d.f. (or p.m.f.) $f_X$. Then

$$\hat{\theta}_n = \frac{1}{n} \sum_{i=1}^{n} \phi(X_i),$$

is called the **Monte Carlo estimator** of the expectation $\theta$.

- Monte Carlo methods can be thought of as a stochastic way to approximate integrals.
Monte Carlo Integration

Algorithm 1 Monte Carlo Algorithm

- Simulate independent $X_1, \ldots, X_n$ with p.m.f. or p.d.f. $f_X$
- Return $\hat{\theta}_n = \frac{1}{n} \sum_{i=1}^{n} \phi(X_i)$. 
Consider the $2 \times 2$ square, say $S \subseteq \mathbb{R}^2$ with inscribed disk $D$ of radius 1.
Computing Pi with Monte Carlo Methods

- We have
  \[
  \frac{\int \int_D dx_1 dx_2}{\int \int_S dx_1 dx_2} = \frac{\pi}{4}.
  \]

- How could you estimate this quantity through simulation?
  \[
  \frac{\int \int_D dx_1 dx_2}{\int \int_S dx_1 dx_2} = \int \int_S \mathbb{I}((x_1, x_2) \in D) \frac{1}{4} dx_1 dx_2
  \]
  \[
  = E [\phi(X_1, X_2)] = \theta
  \]

  where the expectation is w.r.t. the uniform distribution on \( S \) and
  \[
  \phi(X_1, X_2) = \mathbb{I}((X_1, X_2) \in D).
  \]

- To sample uniformly on \( S = (-1, 1) \times (-1, 1) \) then simply use
  \[
  X_1 = 2U_1 - 1, \quad X_2 = 2U_2 - 1
  \]

  where \( U_1, U_2 \sim \mathcal{U}(0, 1) \).
Computing Pi with Monte Carlo Methods

```r
n <- 1000
x <- array(0, c(2,1000))
t <- array(0, c(1,1000))

for (i in 1:1000) {
    # generate point in square
    x[1,i] <- 2*runif(1)-1
    x[2,i] <- 2*runif(1)-1

    # compute phi(x); test whether in disk
    if (x[1,i]*x[1,i] + x[2,i]*x[2,i] <= 1) {
        t[i] <- 1
    } else {
        t[i] <- 0
    }
}

print(sum(t)/n*4)
```
A $2 \times 2$ square $S$ with inscribed disk $D$ of radius 1 and Monte Carlo samples.
\( \hat{\theta}_n - \theta \) as a function of the number of samples \( n \).
\( \hat{\theta}_n - \theta \) as a function of the number of samples \( n \), 100 independent realizations.
Applications

- **Toy example**: simulate a large number $n$ of independent r.v. $X_i \sim \mathcal{N}(\mu, \Sigma)$ and

  \[
  \hat{\theta}_n = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I} \left( \sum_{k=1}^{d} X_{k,i}^2 \geq \alpha \right).
  \]

- **Queuing**: simulate a large number $n$ of days using your stochastic models for the arrival process of customers and for the service time and compute

  \[
  \hat{\theta}_n = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I} (X_i = 0)
  \]

  where $X_i$ is the number of customers in the shop at 5.30pm for $i$th sample.

- **Particle in Random Medium**: simulate a large number $n$ of particle paths $(X_{1,i}, X_{2,i}, \ldots, X_{T,i})$ where $i = 1, \ldots, n$ and compute

  \[
  \hat{\theta}_n = \frac{1}{n} \sum_{i=1}^{n} G(X_{1,i}) G(X_{2,i}) \cdots G(X_{T,i})
  \]
Monte Carlo Integration: Properties

- **Proposition**: Assume $\theta = \mathbb{E}(\phi(X))$ exists. Then the Monte Carlo estimator $\hat{\theta}_n$ has the following properties
  - **Unbiasedness**
    $$\mathbb{E}\left(\hat{\theta}_n\right) = \theta$$
  - **Strong consistency**
    $$\hat{\theta}_n \rightarrow \theta \text{ almost surely as } n \rightarrow \infty$$

- **Proof**: We have
  $$\mathbb{E}\left(\hat{\theta}_n\right) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left(\phi(X_i)\right) = \theta.$$  

Strong consistency is a consequence of the strong law of large numbers applied to $Y_i = \phi(X_i)$ which is applicable as $\theta = \mathbb{E}(\phi(X))$ is assumed to exist.
Monte Carlo Integration: Central Limit Theorem

- **Proposition:** Assume $\theta = \mathbb{E}(\phi(X))$ and $\sigma^2 = \mathbb{V}(\phi(X))$ exist then

\[
\mathbb{E}\left( (\hat{\theta}_n - \theta)^2 \right) = \mathbb{V}(\hat{\theta}_n) = \frac{\sigma^2}{n}
\]

and

\[
\frac{\sqrt{n}}{\sigma} \left( \hat{\theta}_n - \theta \right) \xrightarrow{d} \mathcal{N}(0, 1).
\]

- **Proof.** We have $\mathbb{E}\left( (\hat{\theta}_n - \theta)^2 \right) = \mathbb{V}(\hat{\theta}_n)$ as $\mathbb{E}(\hat{\theta}_n) = \theta$ and

\[
\mathbb{V}(\hat{\theta}_n) = \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{V}(\phi(X_i)) = \frac{\sigma^2}{n}.
\]

The CLT applied to $Y_i = \phi(X_i)$ tells us that

\[
\frac{Y_1 + \cdots + Y_n - n\theta}{\sigma \sqrt{n}} \xrightarrow{d} \mathcal{N}(0, 1)
\]

so the result follows as $\hat{\theta}_n = \frac{1}{n} (Y_1 + \cdots + Y_n)$. 

Monte Carlo Integration: Variance Estimation

**Proposition:** Assume \( \sigma^2 = \text{Var}(\phi(X)) \) exists then

\[
S^2_{\phi(X)} = \frac{1}{n-1} \sum_{i=1}^{n} \left( \phi(X_i) - \hat{\theta}_n \right)^2
\]

is an unbiased sample variance estimator of \( \sigma^2 \).

**Proof.** Let \( Y_i = \phi(X_i) \) then we have

\[
\mathbb{E} \left( S^2_{\phi(X)} \right) = \frac{1}{n-1} \sum_{i=1}^{n} \mathbb{E} \left( (Y_i - \bar{Y})^2 \right)
\]

\[
= \frac{1}{n-1} \mathbb{E} \left( \sum_{i=1}^{n} Y_i^2 - n\bar{Y}^2 \right)
\]

\[
= \frac{n}{n-1} \left( \mathbb{V}(Y) + \theta^2 \right) - n \left( \mathbb{V}(\bar{Y}) + \theta^2 \right)
\]

\[
= \frac{n}{n-1} \mathbb{V}(Y) - \frac{n}{n-1} \left( \mathbb{V}(\bar{Y}) + \theta^2 \right)
\]

\[
= \mathbb{V}(Y) = \mathbb{V}(\phi(X)) .
\]

where \( Y = \phi(X) \) and \( \bar{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i \).
How Good is The Estimator?

▶ Chebyshev’s inequality yields the bound

$$
\mathbb{P}\left(\left|\hat{\theta}_n - \theta\right| > c \frac{\sigma}{\sqrt{n}}\right) \leq \frac{\mathbb{V}(\hat{\theta}_n)}{c^2 \sigma^2 / n} = \frac{1}{c^2}.
$$

▶ Another estimate follows from the CLT for large \(n\)

$$
\frac{\sqrt{n}}{\sigma} \left(\hat{\theta}_n - \theta\right) \overset{d}{\approx} \mathcal{N}(0, 1) \Rightarrow \mathbb{P}\left(\left|\hat{\theta}_n - \theta\right| > c \frac{\sigma}{\sqrt{n}}\right) \approx 2 \left(1 - \Phi(c)\right).
$$

▶ Hence by choosing \(c = c_\alpha\) s.t. \(2 \left(1 - \Phi(c_\alpha)\right) = \alpha\), an approximate \((1 - \alpha)100\%\)-CI for \(\theta\) is

$$
\left(\hat{\theta}_n \pm c_\alpha \frac{\sigma}{\sqrt{n}}\right) \approx \left(\hat{\theta}_n \pm c_\alpha \frac{S_{\Phi}(X)}{\sqrt{n}}\right).
$$
Monte Carlo Integration

- Whatever being $\Omega$; e.g. $\Omega = \mathbb{R}$ or $\Omega = \mathbb{R}^{1000}$, the error is still in $\sigma/\sqrt{n}$.
- This is in contrast with deterministic methods. The error in a product trapezoidal rule in $d$ dimensions is $O(n^{-2/d})$ for twice continuously differentiable integrands.
- It is sometimes said erroneously that it beats the curse of dimensionality but this is generally not true as $\sigma^2$ typically depends of $\text{dim}(\Omega)$. 
The aim of the game is to be able to generate complicated random variables and stochastic models.

Henceforth, we will assume that we have access to a sequence of independent random variables \((U_i, i \geq 1)\) that are uniformly distributed on \((0, 1)\); i.e. \(U_i \sim U[0, 1]\).

In R, the command \(u \leftarrow \text{runif}(100)\) return 100 realizations of uniform r.v. in \((0, 1)\).

Strictly speaking, we only have access to pseudo-random (deterministic) numbers.

The behaviour of modern random number generators (constructed on number theory) resembles mathematical random numbers in many respects: standard statistical tests for uniformity, independence, etc. do not show significant deviations.
If you like this book, I highly recommend that you read it in the original binary. As with most translations, conversion from binary to decimal frequently causes a loss of information and, unfortunately, it's the most significant digits that are lost in the conversion.

Or this somewhat more subtle nerd-joke, by BJ from Waltford, England:

For a supposedly serious reference work the omission of an index is a major impediment. I hope this will be corrected in the next edition.

...or from Fuat C. Baran:

A great read. Captivating. I couldn’t put it down. I would have given it five stars, but sadly there were too many distracting typos. For example: 46453 13987. Hopefully they will correct them in the next edition.
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A function $F : \mathbb{R} \to [0, 1]$ is a cumulative distribution function (cdf) if
- $F$ is increasing; i.e. if $x \leq y$ then $F(x) \leq F(y)$
- $F$ is right continuous; i.e. $F(x + \epsilon) \to F(x)$ as $\epsilon \to 0$ ($\epsilon > 0$)
- $F(x) \to 0$ as $x \to -\infty$ and $F(x) \to 1$ as $x \to +\infty$.

A random variable $X \in \mathbb{R}$ has cdf $F$ if $\mathbb{P}(X \leq x) = F(x)$ for all $x \in \mathbb{R}$.

If $F$ is differentiable on $\mathbb{R}$, with derivative $f$, then $X$ is continuously distributed with probability density function (pdf) $f$. 

Proposition. Let $F$ be a continuous and strictly increasing cdf on $\mathbb{R}$, with inverse $F^{-1} : [0, 1] \to \mathbb{R}$. Let $U \sim U[0, 1]$ then $X = F^{-1}(U)$ has cdf $F$.

Proof. We have

$$
P(X \leq x) = P(F^{-1}(U) \leq x) = P(U \leq F(x)) = F(x).
$$

Proposition. Let $F$ be a cdf on $\mathbb{R}$ and define its generalized inverse $F^{-1} : [0, 1] \to \mathbb{R}$,

$$
F^{-1}(u) = \inf \{ x \in \mathbb{R}; F(x) \geq u \}.
$$

Let $U \sim U[0, 1]$ then $X = F^{-1}(U)$ has cdf $F$. 

Part A Simulation. HT 2018. J. Berestycki. 33 / 66
Illustration of the Inversion Method

Top: pdf of a Gaussian r.v., bottom: associated cdf.
Examples

- **Weibull distribution.** Let $\alpha, \lambda > 0$ then the Weibull cdf is given by

  $$F(x) = 1 - \exp (-\lambda x^\alpha), \; x \geq 0.$$  

  We calculate

  $$u = F(x) \Leftrightarrow \log (1 - u) = -\lambda x^\alpha$$

  $$\Leftrightarrow x = \left( -\frac{\log (1 - u)}{\lambda} \right)^{1/\alpha}.$$  

- As $(1 - U) \sim \mathcal{U}[0, 1]$ when $U \sim \mathcal{U}[0, 1]$ we can use

  $$X = \left( -\frac{\log U}{\lambda} \right)^{1/\alpha}.$$
Examples

- **Cauchy distribution.** It has pdf and cdf

\[ f(x) = \frac{1}{\pi (1 + x^2)}, \quad F(x) = \frac{1}{2} + \frac{\arctan x}{\pi} \]

We have

\[ u = F(x) \iff u = \frac{1}{2} + \frac{\arctan x}{\pi} \]
\[ \iff x = \tan \left( \pi \left( u - \frac{1}{2} \right) \right) \]

- **Logistic distribution.** It has pdf and cdf

\[ f(x) = \frac{\exp(-x)}{(1 + \exp(-x))^2}, \quad F(x) = \frac{1}{1 + \exp(-x)} \]
\[ \iff x = \log \left( \frac{u}{1 - u} \right). \]

- **Practice:** Derive an algorithm to simulate from an Exponential random variable with rate \( \lambda > 0 \).
If $X$ is a discrete $\mathbb{N}$-r.v. with $\mathbb{P}(X = n) = p(n)$, we get $F(x) = \sum_{j=0}^{\lfloor x \rfloor} p(j)$ and $F^{-1}(u)$ is $x \in \mathbb{N}$ such that

$$\sum_{j=0}^{x-1} p(j) < u \leq \sum_{j=0}^{x} p(j)$$

with the LHS $= 0$ if $x = 0$.

Note: the mapping at the values $F(n)$ are irrelevant.

Note: the same method is applicable to any discrete valued r.v. $X$, $\mathbb{P}(X = x_n) = p(n)$. 
Illustration of the Inversion Method: Discrete case
Example: Geometric Distribution

- If $0 < p < 1$ and $q = 1 - p$ and we want to simulate $X \sim \text{Geom}(p)$ then
  \[ p(x) = pq^{x-1}, F(x) = 1 - q^x \quad x = 1, 2, 3, \ldots \]

- The smallest $x \in \mathbb{N}$ giving $F(x) \geq u$ is the smallest $x \geq 1$ satisfying
  \[ x \geq \log(1 - u) / \log(q) \]
  and this is given by
  \[ x = F^{-1}(u) = \left\lceil \frac{\log(1 - u)}{\log(q)} \right\rceil \]
  where $\lceil x \rceil$ rounds up and we could replace $1 - u$ with $u$. 
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Transformation Methods

- Suppose we have a random variable $Y \sim Q$, $Y \in \Omega_Q$, which we can simulate (eg, by inversion) and some other variable $X \sim P$, $X \in \Omega_P$, which we wish to simulate.
- Suppose we can find a function $\varphi : \Omega_Q \to \Omega_P$ with the property that $X = \varphi(Y)$.
- Then we can simulate from $X$ by first simulating $Y \sim Q$, and then set $X = \varphi(Y)$.
- Inversion is a special case of this idea.
- We may generalize this idea to take functions of collections of variables with different distributions.
Transformation Methods

Example: Let $Y_i, i = 1, 2, \ldots, \alpha$, be iid variables with $Y_i \sim \text{Exp}(1)$ and $X = \beta^{-1} \sum_{i=1}^{\alpha} Y_i$ then $X \sim \text{Gamma}(\alpha, \beta)$.

Proof: The MGF of the random variable $X$ is

$$
\mathbb{E} \left( e^{tX} \right) = \prod_{i=1}^{\alpha} \mathbb{E} \left( e^{\beta^{-1}tY_i} \right) = (1 - t/\beta)^{-\alpha}
$$

which is the MGF of a $\text{Gamma}(\alpha, \beta)$ variate. Incidentally, the $\text{Gamma}(\alpha, \beta)$ density is $f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$ for $x > 0$.

Practice: A generalized gamma variable $Z$ with parameters $a > 0, b > 0, \sigma > 0$ has density

$$
f_Z(z) = \frac{\sigma b^a}{\Gamma(a/\sigma)} z^{a-1} e^{-(bz)^\sigma}.
$$

Derive an algorithm to simulate from $Z$. 
Transformation Methods: Box-Muller Algorithm

- For continuous random variables, a tool is the transformation/change of variables formula for pdf.

- Proposition. If \( R^2 \sim \text{Exp}(\frac{1}{2}) \) and \( \Theta \sim \mathcal{U}[0, 2\pi] \) are independent then \( X = R \cos \Theta, \ Y = R \sin \Theta \) are independent with \( X \sim \mathcal{N}(0, 1), \ Y \sim \mathcal{N}(0, 1) \).

Proof: We have \( f_{R^2, \Theta}(r^2, \theta) = \frac{1}{2} \exp \left( -\frac{r^2}{2} \right) \frac{1}{2\pi} \) and

\[
f_{X,Y}(x,y) = f_{R^2,\Theta}(r^2, \theta) \left| \det \frac{\partial (r^2, \theta)}{\partial (x, y)} \right|
\]

where

\[
\left| \det \frac{\partial (r^2, \theta)}{\partial (x, y)} \right|^{-1} = \left| \det \begin{pmatrix} \frac{\partial x}{\partial r^2} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r^2} & \frac{\partial y}{\partial \theta} \end{pmatrix} \right| = \left| \det \begin{pmatrix} \cos \theta & -r \sin \theta \\ \frac{\sin \theta}{2r} & r \cos \theta \end{pmatrix} \right| = \frac{1}{2}.
\]
Let $U_1 \sim \mathcal{U}[0, 1]$ and $U_2 \sim \mathcal{U}[0, 1]$ then

$$R^2 = -2 \log(U_1) \sim \text{Exp} \left( \frac{1}{2} \right)$$

$$\Theta = 2\pi U_2 \sim \mathcal{U}[0, 2\pi]$$

and

$$X = R \cos \Theta \sim \mathcal{N}(0, 1)$$

$$Y = R \sin \Theta \sim \mathcal{N}(0, 1),$$

This still requires evaluating $\log, \cos$ and $\sin$. 
Let consider $X \in \mathbb{R}^d$, $X \sim N(\mu, \Sigma)$ where $\mu$ is the mean and $\Sigma$ is the (positive definite) covariance matrix.

$$f_X(x) = (2\pi)^{-d/2} |\det \Sigma|^{-1/2} \exp\left(-\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu)\right).$$

**Proposition.** Let $Z = (Z_1, \ldots, Z_d)$ be a collection of $d$ independent standard normal random variables. Let $L$ be a real $d \times d$ matrix satisfying

$$LL^T = \Sigma,$$

and

$$X = LZ + \mu.$$

Then

$$X \sim \mathcal{N}(\mu, \Sigma).$$
Simulating Multivariate Normal

Proof. We have \( f_Z(z) = (2\pi)^{d/2} \exp \left(-\frac{1}{2}z^Tz\right) \). The joint density of the new variables is

\[
f_X(x) = f_Z(z) \left|\det \frac{\partial z}{\partial x}\right|
\]

where \( \frac{\partial z}{\partial x} = \frac{1}{L} \) and \( \det(L) = \det(L^T) \) so \( \det(L^2) = \det(\Sigma) \), and \( \det(L^{-1}) = 1/\det(L) \) so \( \det(L^{-1}) = \det(\Sigma)^{-1/2} \). Also

\[
z^Tz = (x - \mu)^T (L^{-1})^T L^{-1} (x - \mu)
\]

= \( (x - \mu)^T \Sigma^{-1} (x - \mu) \).

If \( \Sigma = VDV^T \) is the eigendecomposition of \( \Sigma \), we can pick \( L = VD^{1/2} \).

Cholesky factorization \( \Sigma = LL^T \) where \( L \) is a lower triangular matrix.

See numerical analysis.
Rejection Sampling

- Let $X$ be a continuous r.v. on $\Omega$ with pdf $f_X$

- Consider a continuous r.v variable $U > 0$ such that the conditional pdf of $U$ given $X = x$ is

$$f_{U|X}(u|x) = \begin{cases} \frac{1}{f_X(x)} & \text{if } u < f_X(x) \\ 0 & \text{otherwise} \end{cases}$$

- The joint pdf of $(X, U)$ is

$$f_{X,U}(x,u) = f_X(x) \times f_{U|X}(u|x)$$

$$= f_X(x) \times \frac{1}{f_X(x)} \mathbb{I}(0 < u < f_X(x))$$

$$= \mathbb{I}(0 < u < f_X(x))$$

- Uniform distribution on the set $A = \{(x,u)|0 < u < f_X(x), x \in \Omega\}$
Rejection Sampling

Theorem (Fundamental Theorem of simulation)

Let $X$ be a rv on $\Omega$ with pdf or pmf $f_X$. Simulating $X$ is equivalent to simulating

$$(X, U) \sim \text{Unif}(\{(x, u)|x \in \Omega, 0 < u < f_X(x)\})$$
Rejection Sampling

- Direct sampling of $(X, U)$ uniformly over the set $A$ is in general challenging
- Let $S \supseteq A$ be a bigger set such that simulating uniform rv on $S$ is easy
- Rejection sampling technique:
  1. Simulate $(Y, V) \sim \text{Unif}(S)$, with simulated values $y$ and $v$
  2. if $(y, v) \in A$ then stop and return $X = y, U = v$,
  3. otherwise go back to 1.
- The resulting rv $(X, U)$ is uniformly distributed on $A$
- $X$ is marginally distributed from $f_X$
Example: Beta density

- Let $X \sim \text{Beta}(5, 5)$ be a continuous rv with pdf

$$f_X(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta)} x^{\alpha-1} (1 - x)^{\beta-1}, \ 0 < x < 1$$

where $\alpha = \beta = 5$.

- $f_X(x)$ is upper bounded by 3 on $[0, 1]$.  

![Graph of Beta density function](image)
Example: Beta density

Let \( S = \{(y, v) | y \in [0, 1], v \in [0, 3]\} \)

1. Simulate \( Y \sim \mathcal{U}([0, 1]) \) and \( V \sim \mathcal{U}([0, 3]) \), with simulated values \( y \) and \( v \)
2. If \( v < f_X(x) \), return \( X = x \)
3. Otherwise go back to Step 1.

Only requires simulating uniform random variables and evaluating the pdf pointwise
Rejection Sampling

► Consider $X$ a random variable on $\Omega$ with a pdf/pmf $f(x)$, a target distribution

► We want to sample from $f$ using a proposal pdf/pmf $q$ which we can sample.

► Proposition. Suppose we can find a constant $M$ such that $f(x)/q(x) \leq M$ for all $x \in \Omega$.

► The following ‘Rejection’ algorithm returns $X \sim f$.

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**Algorithm 2** Rejection sampling

**Step 1** - Simulate $Y \sim q$ and $U \sim \mathcal{U}[0,1]$, with simulated value $y$ and $u$ respectively.

**Step 2** - If $u \leq f(y)/q(y)/M$ then stop and return $X = y$,

**Step 3** - otherwise go back to Step 1.
Illustrations

- $f(x)$ is the pdf of a Beta(5, 5) rv
- Proposal density $q$ is the pdf of a uniform rv on $[0, 1]$
Illustrations

- $X \in \mathbb{R}$ with multimodal pdf
- Proposal density $q$ is the pdf of a standardized normal
Rejection Sampling: Proof for discrete rv

- We have

\[ \Pr(X = x) = \sum_{n=1}^{\infty} \Pr(\text{reject } n - 1 \text{ times, draw } Y = x \text{ and accept it}) \]

\[ = \sum_{n=1}^{\infty} \Pr(\text{reject } Y)^{n-1} \Pr(\text{draw } Y = x \text{ and accept it}) \]

- We have

\[ \Pr(\text{draw } Y = x \text{ and accept it}) \]

\[ = \Pr(\text{draw } Y = x) \Pr(\text{accept } Y|Y = x) \]

\[ = q(x) \Pr \left( U \leq \frac{f(Y)}{q(Y)} / M \bigg| Y = x \right) \]

\[ = \frac{f(x)}{M} \]
The probability of having a rejection is

$$\Pr (\text{reject } Y) = \sum_{x \in \Omega} \Pr (\text{draw } Y = x \text{ and reject it})$$

$$= \sum_{x \in \Omega} q(x) \Pr \left( U \geq \frac{f(Y)}{q(Y)/M} \bigg| Y = x \right)$$

$$= \sum_{x \in \Omega} q(x) \left( 1 - \frac{f(x)}{q(x)M} \right) = 1 - \frac{1}{M}$$

Hence we have

$$\Pr (X = x) = \sum_{n=1}^{\infty} \Pr (\text{reject } Y)^{n-1} \Pr (\text{draw } Y = x \text{ and accept it})$$

$$= \sum_{n=1}^{\infty} \left( 1 - \frac{1}{M} \right)^{n-1} \frac{f(x)}{M} = f(x).$$

Note the number of accept/reject trials has a geometric distribution of success probability $1/M$, so the mean number of trials is $M$. 
Here is an alternative proof given for a continuous scalar variable $X$, the rejection algorithm still works but $f, q$ are now pdfs.

We accept the proposal $Y$ whenever $(U, Y) \sim f_{U,Y}$ where $f_{U,Y}(u, y) = q(y)\mathbb{I}_{(0,1)}(u)$ satisfies $U \leq f(Y)/(Mq(Y))$.

We have

\[
\Pr(X \leq x) = \Pr(Y \leq x | U \leq f(Y)/Mq(Y)) \\
= \frac{\Pr(Y \leq x, U \leq f(Y)/Mq(Y))}{\Pr(U \leq f(Y)/Mq(Y))} \\
= \frac{\int_{-\infty}^{x} \int_{0}^{f(y)/Mq(y)} f_{U,Y}(u, y) dudy}{\int_{-\infty}^{\infty} \int_{0}^{f(y)/Mq(y)} f_{U,Y}(u, y) dudy} \\
= \frac{\int_{-\infty}^{\infty} \int_{0}^{f(y)/Mq(y)} q(y) dudy}{\int_{-\infty}^{\infty} \int_{0}^{f(y)/Mq(y)} q(y) dudy} = \int_{-\infty}^{x} f(y) dy.
\]
Example: Beta Density

- Assume you have for $\alpha, \beta \geq 1$

$$f(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \ 0 < x < 1$$

which is upper bounded on $[0, 1]$.

- We propose to use as a proposal $q(x) = \mathbb{I}_{(0,1)}(x)$ the uniform density on $[0, 1]$.

- We need to find a bound $M$ s.t. $f(x)/Mq(x) = f(x)/M \leq 1$. The smallest $M$ is $M = \max_{0<x<1} f(x)$ and we obtain by solving for $f'(x) = 0$

$$M = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{\alpha - 1}{\alpha + \beta - 2}\right)^{\alpha-1} \left(\frac{\beta - 1}{\alpha + \beta - 2}\right)^{\beta-1}$$

which gives

$$\frac{f(y)}{Mq(y)} = \frac{y^{\alpha-1}(1-y)^{\beta-1}}{M'}.$$
Dealing with Unknown Normalising Constants

- In most practical scenarios, we only know \( f(x) \) and \( q(x) \) up to some normalising constants; i.e.

\[
f(x) = \frac{\tilde{f}(x)}{Z_f} \quad \text{and} \quad q(x) = \frac{\tilde{q}(x)}{Z_q}
\]

where \( \tilde{f}(x) \), \( \tilde{q}(x) \) are known but \( Z_f = \int_{\Omega} \tilde{f}(x) \, dx \), \( Z_q = \int_{\Omega} \tilde{q}(x) \, dx \) are unknown/expensive to compute.

- Rejection can still be used: Indeed \( f(x)/q(x) \leq M \) for all \( x \in \Omega \) iff \( \tilde{f}(x)/\tilde{q}(x) \leq \tilde{M} \), with \( \tilde{M} = Z_f M / Z_q \).

- Practically, this means we can ignore the normalising constants from the start: if we can find \( \tilde{M} \) to bound \( \tilde{f}(x)/\tilde{q}(x) \) then it is correct to accept with probability \( \frac{\tilde{f}(x)}{\tilde{M} \tilde{q}(x)} \) in the rejection algorithm. In this case the mean number \( N \) of accept/reject trials will equal \( Z_q \tilde{M} / Z_f \) (that is, \( M \) again).
Simulating Gamma Random Variables

- We want to simulate a random variable $X \sim \text{Gamma}(\alpha, \beta)$ which works for any $\alpha \geq 1$ (not just integers);

$$f(x) = \frac{x^{\alpha-1} \exp(-\beta x)}{Z_f} \text{ for } x > 0, \quad Z_f = \frac{\Gamma(\alpha)}{\beta^\alpha}$$

so $\tilde{f}(x) = x^{\alpha-1} \exp(-\beta x)$ will do as our unnormalised target.

- When $\alpha = a$ is a positive integer we can simulate $X \sim \text{Gamma}(a, \beta)$ by adding $a$ independent $\text{Exp}(\beta)$ variables, $Y_i \sim \text{Exp}(\beta)$, $X = \sum_{i=1}^{a} Y_i$.

- Hence we can sample densities 'close' in shape to $\text{Gamma}(\alpha, \beta)$ since we can sample $\text{Gamma}([\alpha], \beta)$. Perhaps this, or something like it, would make an envelope/proposal density?
Let $a = \lfloor \alpha \rfloor$ and let’s try to use $\text{Gamma}(a, b)$ as the envelope, so $Y \sim \text{Gamma}(a, b)$ for integer $a \geq 1$ and some $b > 0$. The density of $Y$ is

$$q(x) = \frac{x^{a-1} \exp(-bx)}{Z_q} \quad \text{for } x > 0, \quad Z_q = \Gamma(a)/b^a$$

so $\tilde{q}(x) = x^{a-1} \exp(-bx)$ will do as our unnormalised envelope function.

We have to check whether the ratio $\tilde{f}(x)/\tilde{q}(x)$ is bounded over $\mathbb{R}_+$ where

$$\tilde{f}(x)/\tilde{q}(x) = x^{\alpha-a} \exp(-(\beta-b)x).$$

Consider (a) $x \to 0$ and (b) $x \to \infty$. For (a) we need $a \leq \alpha$ so $a = \lfloor \alpha \rfloor$ is indeed fine. For (b) we need $b < \beta$ (not $b = \beta$ since we need the exponential to kill off the growth of $x^{\alpha-a}$).
Given that we have chosen \( a = \lfloor \alpha \rfloor \) and \( b < \beta \) for the ratio to be bounded, we now compute the bound.

\[
\frac{d}{dx} \left( \frac{\tilde{f}(x)}{\tilde{q}(x)} \right) = 0 \text{ at } x = \frac{(\alpha - a)}{(\beta - b)} \text{ (and this must be a maximum at } x \geq 0 \text{ under our conditions on } a \text{ and } b), \text{ so } \frac{\tilde{f}(x)}{\tilde{q}(x)} \leq \tilde{M} \text{ for all } x \geq 0 \text{ if}
\]

\[
\tilde{M} = \left( \frac{\alpha - a}{\beta - b} \right)^{\alpha - a} \exp(- (\alpha - a)).
\]

Accept \( Y \) at step 2 of Rejection Sampler if \( U \leq \frac{\tilde{f}(Y)}{\tilde{M} \tilde{q}(Y)} \) where \( \frac{\tilde{f}(Y)}{\tilde{M} \tilde{q}(Y)} = Y^{\alpha - a} \exp(- (\beta - b)Y)/\tilde{M} \).
Simulating Gamma Random Variables: Best choice of $b$

- Any $0 < b < \beta$ will do, but is there a best choice of $b$?
- Idea: choose $b$ to minimize the expected number of simulations of $Y$ per sample $X$ output.
- Since the number $N$ of trials is Geometric, with success probability $Z_f / (\tilde{M} Z_q)$, the expected number of trials is $E(N) = Z_q \tilde{M} / Z_f$. Now $Z_f = \Gamma(\alpha) \beta^{-\alpha}$ where $\Gamma$ is the Gamma function related to the factorial.
- Practice: Show that the optimal $b$ solves $\frac{d}{db} (b^{-a} (\beta - b)^{-\alpha + a}) = 0$ so deduce that $b = \beta (a / \alpha)$ is the optimal choice.
Simulating Normal Random Variables

Let \( f(x) = (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}x^2\right) \) and \( q(x) = 1/\pi/(1 + x^2) \). We have

\[
\frac{\tilde{f}(x)}{\tilde{q}(x)} = (1 + x^2) \exp\left(-\frac{1}{2}x^2\right) \leq 2/\sqrt{e} = \tilde{M}
\]

which is attained at \( \pm 1 \).

Hence the probability of acceptance is

\[
\mathbb{P}\left( U \leq \frac{\tilde{f}(Y)}{\tilde{M}\tilde{q}(Y)} \right) = \frac{Z_f}{\tilde{M} Z_q} = \frac{\sqrt{2\pi}}{2/\sqrt{e}\pi} = \sqrt{\frac{e}{2\pi}} \approx 0.66
\]

and the mean number of trials to success is approximately \( 1/0.66 \approx 1.52 \).
Rejection Sampling in High Dimension

- Consider

\[ \tilde{f}(x_1, \ldots, x_d) = \exp \left( -\frac{1}{2} \sum_{k=1}^{d} x_k^2 \right) \]

and

\[ \tilde{q}(x_1, \ldots, x_d) = \exp \left( -\frac{1}{2\sigma^2} \sum_{k=1}^{d} x_k^2 \right) \]

- For \( \sigma > 1 \), we have

\[ \frac{\tilde{f}(x_1, \ldots, x_d)}{\tilde{q}(x_1, \ldots, x_d)} = \exp \left( -\frac{1}{2} \left( 1 - \sigma^{-2} \right) \sum_{k=1}^{d} x_k^2 \right) \leq 1 = \tilde{M}. \]

- The acceptance probability of a proposal for \( \sigma > 1 \) is

\[ \mathbb{P} \left( U \leq \frac{\tilde{f}(X_1, \ldots, X_d)}{\tilde{M} \tilde{q}(X_1, \ldots, X_d)} \right) = \frac{Z_f}{\tilde{M} Z_q} = \sigma^{-d}. \]

- The acceptance probability goes exponentially fast to zero with \( d \).